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A topic of nonlinear Schrödinger equation

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1 Introduction

We consider a wave function defined by the following Fourier transformation

$$u_{m}(x,t)=\int_{-\infty}^{\infty}S_{m}(k)\,e^{i\{kx-\omega(k)t\}}dk,$$

where $i=\sqrt{-1}$, k is a frequency number, $S_{m}(k)$ is a spectrum function and $\omega(k)$ is an angular frequency. From the definition we can see that $u_{m}(x,t)$ is a mixture of some waves with different frequencies each other on some bandwidth controlled by the spectrum function $S_{m}(k)$. When $S_{m}(k)$ is a delta function $\delta_{k_{0}}(\cdot)$ concentrated on a frequency k_{0} the wave function $u_{m}(x,t)$ is called the (purely) monochromatic wave $u_{1}(x,t)$, i.e.

$$\begin{aligned} u_{1}(x,t) &= \int_{-\infty}^{\infty}\delta_{k_{0}}(k)\,e^{i\{kx-\omega(k)t\}}dk \\ &= \cos\{k_{0}x-\omega(k)t\}+i\sin\{k_{0}x-\omega(k)t\}. \end{aligned}$$

On the other hand $u_{m}(x,t)$ is called a nearly monochromatic wave function if $S_{m}(k)$ is a unimodal function with a small compact support. As to some application of nearly monochromatic waves, see for example [6]. In this note we focus on the envelope function defined by

$$A_{m}(x,t)=\frac{u_{m}(x,t)}{u_{1}(x,t)},$$

and show that the envelope function $A_{m}(x,t)$ satisfies Schrödinger equation under some conditions for the spectrum function $S_{m}(k)$ and the angular function $\omega(k)$. Furthermore we deal with the cases when the spectrum function is a bimodal function $S_{b}(k)$ with a compact support which constructs a bichromatic wave function $u_{b}(x,t)$ and the angular frequency is a two dimensional function $\omega(k,\cdot)$, respectively. In these cases we show that the envelope function satisfies a kind of nonlinear Schrödinger equation under some conditions for the spectrum function and the angular function. As for the details of the nearly monochromatic waves, see e.g. [5]. Further, for more applications of nearly monochromatic waves and bichromatic waves, see [1]-[6].

2 Profile of nearly monochromatic waves

For analyzing the envelope function $A_m(x,t)$ we first introduce a profile of $u_m(x,t)$ which means an approximation of $u_m(x,t)$ by replacing $\omega(k)$ or $\omega(k,\cdot)$ with the Taylor expansion of them as following.

Definition 1 *Suppose that an angular function $\omega(k) \in C^\infty$ can be represented by*

$$\omega(k) = \sum_{j=0}^{\infty} \frac{\omega^{(j)}(k_0)}{j!}(k-k_0)^j$$

from the Taylor expansion. The nth order profile of the envelope function of nearly monochromatic wave defined by

$$\tilde{A}_m^n(x,t) = \frac{\int_K S_m(k)\, e^{i\left\{kx - \sum_{j=0}^{n} \frac{\omega^{(j)}(k_0)}{j!}(k-k_0)^j t\right\}} dk}{u_1(x,t)},$$

where K is a compact support of $S_m(k)$.

Theorem 1 *The second order profile $\tilde{A}_m^2(x,t)$ satisfies the linear Schrrmödinger equation,*

$$i\left\{\frac{\partial \tilde{A}_m^2(x,t)}{\partial t} + \omega'(k_0)\frac{\partial \tilde{A}_m^2(x,t)}{\partial x}\right\} + \frac{1}{2!}\omega''(k_0)\frac{\partial^2 \tilde{A}_m^2(x,t)}{\partial x^2} = 0.$$

3 Bichromatic waves and nearly bichromatic waves

Put $S_2(k) = \frac{\delta_{k_0}(k)+\delta_{k_1}(k)}{2}$ for some frequencies k_0 and k_1 with $|k_1 - k_0| = O(\Delta)$ for sufficiently small positive constant Δ. Then the (purely) bichromatic wave is defined by

$$u_2(x,t) = \int_{-\infty}^{\infty} S_2(k)\, e^{i\{kx-\omega(k)t\}} dk = \frac{1}{2}\left[e^{i\{k_0x-\omega(k_0)t\}} + e^{i\{k_1x-\omega(k_1)t\}}\right].$$

Furthermore let $S_b(k)$ be a bimodal spectrum function with a compact support K and suppose that the length of the support is $|K| = O(\Delta)$. Let $u_b(x,t)$ be a nearly bichromatic wave defined by

$$u_b(x,t) = \int_K S_b(k)\, e^{i\{kx-\omega(k)t\}} dk$$

and $A_b(x,t)$ be the envelope function of $u_b(x,t)$ defined by

$$A_b(x,t) = \frac{u_b(x,t)}{u_2(x,t)}.$$

Furthermore $\tilde{A}_b^n(x,t)$ is the n-th order profile of $\tilde{A}_b(x,t)$ defined by

$$\tilde{A}_b^n(x,t)=\frac{\int_K S_b(k)\,e^{i\left\{kx-\sum_{j=0}^{n}\frac{\omega^{(j)}(k_0)}{j!}(k-k_0)^j t\right\}}dk}{u_2(x,t)}.$$

Theorem 2 *The second order profile of the envelope function of nearly bichromatic waves $\tilde{A}_b^2(x,t)$ satisfies the following Ginzburg-Landau type equation,*

$$\frac{\partial \tilde{A}_b^2(x,t)}{\partial t}+\left\{\omega'(k_0)+(k_1-k_0)\,\omega''(k_0)\frac{e^{ig(x,t)}}{1+e^{ig(x,t)}}\right\}\frac{\partial \tilde{A}_b^2(x,t)}{\partial x}$$

$$=\frac{i}{2!}\omega''(k_0)\frac{\partial^2\tilde{A}_b^2(x,t)}{\partial x^2}+\sum_{n=3}^{\infty}\frac{(k_1-k_0)^n}{n!}\omega^{(n)}(k)\frac{e^{ig(x,t)}}{1+e^{ig(x,t)}}\tilde{A}_b^2(x,t),$$

where

$$g(x,t)=(k_1-k_0)\,x-\left\{(k_1-k_0)\,\omega'(k_0)+\frac{1}{2!}(k_1-k_0)^2\omega''(k_0)+\cdots\right\}t.$$

4 Nearly monochromatic waves with $\omega(\xi,\varsigma)$

We next consider the wave equation given by

$$\hat{u}_m(x,t)=\int_K S_m(k)\,e^{i\left\{kx-\omega\left(k,\left|\hat{A}_m(x,t)\right|\right)t\right\}}dk,$$

where $\omega(\xi,\varsigma)$ is a two dimensional angular frequency function and

$$\hat{A}_m(x,t)=\frac{\hat{u}_m(x,t)}{u_1(x,t)}$$

is the envelope function of $\hat{u}_m(x,t)$. Since the above equation is a kind of an integral equation and it is difficult to obtain its exact solution, we give a relation between the integral equation and nonlinear Schrödiger equation to investigate the solution.

Theorem 3 *Assume the following conditions.*

(1) $\Delta>0$ is small enough.

(2) All partial derivatives of $\omega(\xi,\varsigma)$ less than third degree are uniformly bounded in a neighborhood of $(k_0,0)$.

(3) $S_m(k)$ is bounded and its bound is independent of .

Then we have for $0\le t\le Const.\Delta$, as $\Delta\to 0$

$$i\left\{\frac{\partial \hat{A}_m(x,t)}{\partial t}+\omega_\xi(k_0,0)\frac{\partial \hat{A}_m(x,t)}{\partial x}\right\}+\frac{1}{2!}\omega_{\xi\xi}(k_0)\frac{\partial^2\hat{A}_m(x,t)}{\partial x^2}$$

$$-\omega_\varsigma(k_0,0)\left|\hat{A}_m(x,t)\right|^2\hat{A}_m(x,t)=O\left(\Delta^4\right).$$

5 Nearly bichromatic waves with $\omega(\xi, \varsigma)$

We next consider the wave equation

$$\hat{u}_b(x,t) = \int_K S_b(k)\, e^{i\left\{kx - \omega\left(k, \left|\hat{A}_b(x,t)\right|\right)t\right\}} dk,$$

where $\hat{A}_b(x,t)$ is the envelope function of $\hat{u}_b(x,t)$ defined by

$$\hat{A}_b(x,t) = \frac{\hat{u}_b(x,t)}{u_2(x,t)}.$$

Similarly to $\hat{A}_m(x,t)$, the above wave equation means an integral equation and it is difficult to obtain the exact solution. From the next theorem we can see the exact equation as the solution of nonlinear Schrödinger equation.

Theorem 4 *Assume all assumptions of Theorem 3. If* $|k_1 - k_0| = O(\Delta^2)$, *then* $\hat{A}_b(x,t)$ *satisfies the same nonlinear Schrödinger equation in Theorem 3,*

$$i\left\{\frac{\partial \hat{A}_b(x,t)}{\partial t} + \omega_\xi(k_0,0)\frac{\partial \hat{A}_b(x,t)}{\partial x}\right\} + \frac{1}{2!}\omega_{\xi\xi}(k_0)\frac{\partial^2 \hat{A}_b(x,t)}{\partial x^2}$$
$$-\omega_\varsigma(k_0,0)\left|\hat{A}_b(x,t)\right|^2 \hat{A}_b(x,t) = O\left(\Delta^4\right),$$

for $0 \le t \le Const.\Delta$, *as* $\Delta \to 0$.

Theorem 5 *Assume all assumptions of Theorem 3. If* $|k_1 - k_0| = O(\Delta)$, *then* $\hat{A}_b(x,t)$ *satisfies the following nonlinear Schrödinger equation,*

$$i\left\{\frac{\partial \hat{A}_b(x,t)}{\partial t} + \left(\omega_\xi(k_0,0) + \frac{1}{2}\omega_{\xi\xi}(k_0,0)|k_1 - k_0|\right)\frac{\partial \hat{A}_b(x,t)}{\partial x}\right\}$$
$$+\frac{1}{2!}\omega_{\xi\xi}(k_0)\frac{\partial^2 \hat{A}_b(x,t)}{\partial x^2} - \omega_\varsigma(k_0,0)\left|\hat{A}_b(x,t)\right|^2 \hat{A}_b(x,t) = O\left(\Delta^4\right),$$

for $0 \le t \le Const.\Delta$, *as* $\Delta \to 0$.

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