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# On the construction of weak solution to a free-boundary problem modelling the vibration of film near obstacle

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**Abstract** The motion of thin film with an obstacle is treated numerically. This amounts to the analysis of a wave operator of degenerate type. The discrete Morse flow of hyperbolic type is applied to construct approximate solution. The possibility of constructing weak solution in one dimension by adding a higher-integrable term is investigated.

## 1 Introduction

In this paper we treat an obstacle problem related to a degenerate hyperbolic equation, to be specific, we would like to analyse the motion of a rubber film with an obstacle where the reflection constant is zero. In [1], a similar problem is studied but the method there relies on the assumption of nonzero reflection rate and is therefore essentailly different from the one presented here. For the analysis of one-dimensional case, see [3]. For numerical results, we refer to the original paper [5].

## 2 Formulation of the problem

The shape of the rubber film is described by the graph of a scalar function $u : \Omega \times [0,\infty) \to \mathbf{R}$, where $\Omega$ is a domain in $\mathbf{R}^n$. The obstacle is a plane fixed at the zero level set of $u$.

The mathematical problem reads: Find function $u : \Omega \times [0,\infty) \to \boldsymbol{R}$ satisfying the following degenerate hyperbolic equation:

$$\begin{cases} \chi_{u>0} u_{tt} = \Delta u & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \\ u_t(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

under suitable boundary conditions. Here, $\chi_E$ is the characteristic function of set $E$.

In [5], equation (2.1) is derived and justified. In short, for the energy-conserving case, we consider the Lagrangian

$$J(u) = \int_0^T \int_\Omega (|\nabla u|^2 - (u_t)^2)\chi_{u>0} dxdt.$$

and show that equations obtained by its variation correspond well to (2.1).

# 3 Minimizing method

We introduce the following functionals for $m \geq 2$

$$J_m(u) = \int_{\Omega\cap\{\{u>0\}\cup\{u_{m-1}>0\}\}} \frac{|u - 2u_{m-1} + u_{m-2}|^2}{2h^2} dx + \frac{1}{2}\int_\Omega |\nabla u|^2 dx. \quad (3.1)$$

We will determine a sequence $\{u_m\}$ in $\mathcal{K} = \{u \in H^1(\Omega; \boldsymbol{R}); u = u_0 \text{ on } \partial\Omega\}$ by induction as follows: For given $u_0$ and $u_1 = u_0 + hv_0$ and for $m = 2, 3, \ldots$ find $\tilde{u}_m$ as the minimizer of $J_m$ in $\mathcal{K}$. Then set $u_m := \max(\tilde{u}_m, 0)$.

**Remark.** *If there is no intersection in the integration domain in $J_m$, by the minimizing process we obtain formally the weak form of the discretized wave equation. Therefore, it makes no difficulty to establish weak solution. However, if we add the set $\{u > 0\}$, which expresses the fact that the solution cannot go under zero, we obtain a free boundary problem. It is not known how to introduce weak solution, we even do not get any kind of energy estimate for the approximate solutions. In order to obtain an estimate we have added the set $\{u_{m-1} > 0\}$ (see Proposition 4.1). This may cause the negativity of minimizers and that is why we adjust them by taking $u_m := \max(\tilde{u}_m, 0)$.*

The following two results are also taken from [5].

**Theorem 3.1** *If $J_m(u_0) < \infty$, then there exists a minimizer $\tilde{u}_m$ of $J_m$.*

**Theorem 3.2** *For all $\tilde{\Omega} \subset\subset \Omega$, there exists a positive constant $\delta$ $(0 < \delta < 1)$ independent of $m$, such that the minimizers $u_m$ belong to $C^\delta(\tilde{\Omega})$.*

Using the above theorem, we can choose the support of test functions in the set $\{u > 0\}$. Then the first variation formula of $J_m(u)$ is

$$\int_\Omega \left( \frac{u - 2u_{m-1} + u_{m-2}}{h^2}\phi + \nabla u \nabla \phi \right) dx = 0$$
$$\forall \phi \in C_0^\infty(\Omega \cap \{u > 0\})$$
$$u \equiv 0 \qquad \text{outside the set } \{u > 0\}.$$

Now we interpolate the minimizers $\{u_m\}$ in time and define the approximate weak solution. We define $\bar{u}^h$ and $u^h$ on $\Omega \times (0, \infty)$ by

$$\bar{u}^h(x,t) = u_m(x),$$
$$u^h(x,t) = \frac{t-(m-1)h}{h} u_m(x) + \frac{mh - t}{h} u_{m-1}(x),$$

for $(x,t) \in \Omega \times ((m-1)h, mh]$, $n \in \boldsymbol{N}$. We define the approximate solution as follows.

**Definition 3.1** *We call the solution of the following equation* an approximate solution *to the rubber film problem:*

$$\int_h^T \int_\Omega \left( \frac{u_t^h(t) - u_t^h(t-h)}{h}\phi + \nabla \bar{u}^h \nabla \phi \right) dx dt = 0,$$
$$\forall \phi \in C_0^\infty([0,T) \times \Omega \cap \{u^h > 0\}), \qquad (3.2)$$
$$u^h \equiv 0 \quad \text{in} \quad (h,T) \times \Omega \setminus \{u^h > 0\}.$$

*Further, we require that it satisfy the initial conditions $u^h(0) = u_0$ and $u^h(h) = u_0 + hv_0$.*

If one can pass to the limit as $h \to 0$, then the above approximate solutions are expected to converge to the solution of the following type of equation.

**Definition 3.2** *We call $u$ a weak solution to (2.1), if $u$ satisfies the following:*

$$\int_0^T \int_\Omega (-u_t\phi_t + \nabla u \nabla \phi)\, dx\, dt - \int_\Omega v_0 \phi(x,0)\, dx = 0$$
$$\forall \phi \in C_0^\infty(\Omega \times [0,T) \cap \{u > 0\}),$$
$$u \equiv 0 \qquad \textit{outside of } \{u > 0\}$$

*and $u(0) = u_0$ in the sense of traces.*

# 4 Energy estimate

We shall derive an energy estimate for the minimizers of $J_m$, $m = 2, 3, \ldots$ .

**Proposition 4.1** *We have for* $m = 2, 3, \ldots$

$$\left\|\frac{u_m - u_{m-1}}{h}\right\|^2_{L^2(\Omega)} + \|\nabla u_m\|^2_{L^2(\Omega)} \leq \|v_0\|^2_{L^2(\Omega)} + \|\nabla u_1\|^2_{L^2(\Omega)}.$$

**Proof.** Choose $\lambda$ arbitrary ($0 < \lambda < 1$). By the minimality property we have $J_m(\tilde{u}_m) \leq J_m((1-\lambda)\tilde{u}_m + \lambda u_{m-1})$, thus,

$$\lim_{\lambda \to 0+} \frac{1}{\lambda}(J_m(\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m)) - J_m(\tilde{u}_m)) \geq 0. \tag{4.1}$$

By $A_m$ we denote the set $\{\tilde{u}_m > 0\} \cup \{u_{m-1} > 0\}$. We investigate the behaviour of individual terms in (4.1). For the gradient term we get

$$\begin{aligned}
\lim_{\lambda \to 0+} \frac{1}{2\lambda} \int_\Omega & |\nabla(\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m))|^2 - |\nabla \tilde{u}_m|^2 \, dx \\
&= \int_\Omega \nabla \tilde{u}_m \cdot \nabla(u_{m-1} - \tilde{u}_m) \, dx \\
&\leq \frac{1}{2} \int_\Omega |\nabla u_{m-1}|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla \tilde{u}_m|^2 \, dx \\
&\leq \frac{1}{2} \int_\Omega |\nabla u_{m-1}|^2 \, dx - \frac{1}{2} \int_\Omega |\nabla u_m|^2 \, dx.
\end{aligned}$$

Taking into account that $\{\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) > 0\} \subset A_m$, we find

$$\begin{aligned}
I(\lambda) := \int_\Omega & \frac{1}{2h^2} |\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) - 2u_{m-1} + u_{m-2}|^2 \\
& \cdot \chi_{\{\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) > 0\} \cup \{u_{m-1} > 0\}} dx \\
& - \int_\Omega \frac{1}{2h^2} |\tilde{u}_m - 2u_{m-1} + u_{m-2}|^2 \chi_{\Omega \cap A_m} dx \\
\leq \int_{\Omega \cap A_m} & \frac{1}{2h^2} (|\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) - 2u_{m-1} + u_{m-2}|^2 \\
& - |\tilde{u}_m - 2u_{m-1} + u_{m-2}|^2) dx.
\end{aligned}$$

Here, we have omitted a term of the form

$$|\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) - 2u_{m-1} + u_{m-2}|^2 \cdot \underbrace{\left(\chi_{\{\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) > 0\} \cup \{u_{m-1} > 0\}} - \chi_{\{\tilde{u}_m > 0\} \cup \{u_{m-1} > 0\}}\right)}_{H} .$$

Since it holds

$$x \notin \{\tilde{u}_m > 0\} \cup \{u_{m-1} > 0\} \Rightarrow x \notin \{\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m) > 0\} \cup \{u_{m-1} > 0\},$$

we conclude that $H$ is nonpositive and thus the whole term can be neglected. Then we have

$$\begin{aligned}\lim_{\lambda\to 0+} I(\lambda) &\leq \frac{1}{h^2}\int_{\Omega\cap A_m} (u_{m-1} - \tilde{u}_m)(\tilde{u}_m - u_{m-1} - (u_{m-1} - u_{m-2}))\,dx \\ &\leq \frac{1}{2h^2}\Big( - \|\tilde{u}_m - u_{m-1}\|^2_{L^2(\Omega\cap A_m)} + \|u_{m-1} - u_{m-2}\|^2_{L^2(\Omega\cap A_m)}\Big).\end{aligned}$$

The inequality is preserved even if we replace $\tilde{u}_m$ by $u_m = \max(\tilde{u}_m, 0)$. Noting that the sets $\{\tilde{u}_m > 0\}$ and $\{u_m > 0\}$ are the same, we can make the same replacement also in the integration domain. Hence, we obtain

$$\begin{aligned}\lim_{\lambda\to 0+} I(\lambda) &\leq \frac{1}{2h^2}\int_{\Omega\cap\{\{u_m>0\}\cup\{u_{m-1}>0\}\}} (u_{m-1} - u_{m-2})^2 - (u_m - u_{m-1})^2\,dx \\ &\leq \frac{1}{2h^2}\int_{\Omega} (u_{m-1} - u_{m-2})^2 - (u_m - u_{m-1})^2\,dx.\end{aligned}$$

Using the above estimates, we arrive at

$$\begin{aligned}0 &\leq \lim_{\lambda\to 0+} \frac{2}{\lambda}(J_m(\tilde{u}_m + \lambda(u_{m-1} - \tilde{u}_m)) - J_m(\tilde{u}_m)) \\ &\leq \frac{1}{h^2}\left(\|u_{m-1} - u_{m-2}\|^2 - \|u_m - u_{m-1}\|^2\right) + \left(\|\nabla u_{m-1}\|^2 - \|\nabla u_m\|^2\right).\end{aligned}$$

Summing up we obtain the assertion. □

# 5 Weak solution

The energy estimate derived in the previous section allows us to extract a weakly convergent subsequence from the approximate solutions. However, we do not get uniform convergence which is necessary to pass to the limit as $h \to 0$ in (3.2).

We can get the uniform convergence by adding a certain term into the original equation:

$$\chi_{u>0} u_{tt} = \Delta u - u_t^\gamma, \qquad \gamma > 2. \tag{5.1}$$

We employ the same method where in the functional (3.1) a new term of the form

$$\frac{h}{\gamma}\int_{\Omega}\left|\frac{u-u_{m-1}}{h}\right|^{\gamma}\,dx$$

appears. We prove the following

**Theorem 5.1** *There exists a subsequence $\{u^h\}_{h\to 0+}$ of the approximate weak solutions which converges to a weak solution of (5.1).*

**Proof.** We give only the idea of the proof. First, we prove uniform higher integrability of a subsequence of approximate solutions by use of Gehring's theory. To this end, we need an energy estimate and a Caccioppoli inequality.

The energy estimate is calculated as in Proposition 4.1. We get

$$\|u_t^h(t)\|^2_{L^2(\Omega)} + \|\nabla \bar{u}^h(t)\|^2_{L^2(\Omega)} + \int_0^t\int_{\Omega}|u_t^h|^{\gamma}\,dx\,dt \leq \text{ const.} \tag{5.2}$$

To deduce Caccioppoli inequality, we have to consider two cases: 'inside the set $\{u_m > 0\}$' and 'near the boundary $\partial\{u_m > 0\}$'.

In the first case we note that $J_m(\tilde{u}_m) \leq J_m(\psi)$ for $\psi = \tilde{u}_m - \varepsilon(\tilde{u}_m - U)\eta_m^2$. Here $\varepsilon > 0$, $\eta_m$ is a standard cut-off function on $B_{2R}(x_0)$ with $B_{2R} \subset \{u_m > 0\}$ and $U$ is a mean value to be defined later. By variation of $J_m$ we get

$$\begin{aligned}\int_{\Omega}|\nabla\tilde{u}_m|^2\eta_m^2 &\leq -\frac{1}{h^2}\int_{\Omega\cap\{\tilde{u}_m>0\}}(\tilde{u}_m - 2u_{m-1} + u_{m-2})(\tilde{u}_m - U)\eta_m^2 \\ &-2\int_{\Omega}\nabla\tilde{u}_m\nabla\eta_m\eta_m(\tilde{u}_m - U) - \int_{\Omega}\left|\frac{\tilde{u}_m - u_{m-1}}{h}\right|^{\gamma-2}\frac{\tilde{u}_m - u_{m-1}}{h}(\tilde{u}_m - U)\eta_m^2.\end{aligned} \tag{5.3}$$

Here it is worth noting that we could get rid of the characteristic function by

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega}\frac{|\tilde{u}_m - 2u_{m-1} + u_{m-2} - \varepsilon(\tilde{u}_m - U)\eta_m^2|^2}{2h^2\varepsilon}\chi_{\{\tilde{u}_m - \varepsilon(\tilde{u}_m - U)\eta_m^2 > 0\}\cup\{u_{m-1}>0\}} \\ &\quad -|\tilde{u}_m - 2u_{m-1} + u_{m-2}|^2\chi_{\{\tilde{u}_m>0\}\cup\{u_{m-1}>0\}}\,dx \\ &= -\frac{1}{h^2}\int_{\Omega\cap(\{\tilde{u}_m>0\}\cup\{u_{m-1}>0\})}(\tilde{u}_m - 2u_{m-1} + u_{m-2})(\tilde{u}_m - U)\eta_m^2\,dx \\ &+\lim\frac{1}{2h^2\varepsilon}\int_{\Omega}|\tilde{u}_m - 2u_{m-1} + u_{m-2} - \varepsilon(\tilde{u}_m - U)\eta_m^2|^2 \\ &\quad\cdot\left(\chi_{\{\tilde{u}_m - \varepsilon(\tilde{u}_m - U)\eta_m^2>0\}\cup\{u_{m-1}>0\}} - \chi_{\{\tilde{u}_m>0\}\cup\{u_{m-1}>0\}}\right)\,dx \\ &\leq -\frac{1}{h^2}\int_{\Omega\cap(\{\tilde{u}_m>0\}\cup\{u_{m-1}>0\})}(\tilde{u}_m - 2u_{m-1} + u_{m-2})(\tilde{u}_m - U)\eta_m^2\,dx,\end{aligned}$$

since the term in brackets is nonpositive. Rewriting in the $u^h$-notation, selecting $\eta_m$ appropriately for each $m$, summing with respect to $m$ and making further technical rearrangements we are supposed to get roughly

$$\begin{aligned}\int_{Q_R} |\nabla u^h|^2\, dz \le c \int_{Q_{2R}} |u_t^h|^2\, dz \\ + \frac{c}{R^2} \int_{Q_{2R}} |u^h - U|^2\, dz + c \int_{Q_{2R}} |u_t^h|^{\gamma-1} |u^h - U|\, dz,\end{aligned} \tag{5.4}$$

where $z = (x,t)$, $Q_R = (ih - R, ih + R) \times Q'_R(x_0)$ with $i \in \mathbf{N}$ and $Q'_R$ a standard $n$-cube.

If we set

$$U = \frac{1}{|Q'_{2R}|} \int_{Q'_{2R}} u^h\, dx,$$

the second term on the right-hand side of (5.4) can be estimated using Sobolev-Poincaré inequality (separately on each interval of the time-partition)

$$\begin{aligned}\frac{c}{R^2} \int_{Q_{2R}} |u^h - U|^2\, dz &\le \frac{c}{R^2} \left( \int_{Q_{2R}} |\nabla u^h|^q\, dz \right)^{2/q} \\ &\le cR^{n+1} \left( ⨍_{Q_{2R}} |\nabla u^h|^q\, dz \right)^{2/q},\end{aligned}$$

where $q = \frac{2(n+1)}{n+3} < 2$ and the symbol $⨍$ stands for the mean value. For the last term we have by the energy estimate and Sobolev-Poincaré inequality

$$\begin{aligned}&\int_{Q_{2R}} |u_t^h|^{\gamma-1} |u^h - U|\, dz \\ &\le \left( \int_{Q_{2R}} |u^h - U|^{2^*}\, dz \right)^{\frac{1}{2^*}} \cdot \left( \int_{Q_{2R}} |u_t^h|^{\frac{(\gamma-1)2^*}{2^*-1}}\, dz \right)^{1-\frac{1}{2^*}} \\ &\le \left\{ \int \left( \int_{Q'_{2R}} |\nabla u^h|^2\, dx \right)^{\frac{2^*}{2}} dt \right\}^{\frac{1}{2^*}} \cdot \left( \int_{Q_{2R}} |u_t^h|^{\frac{(\gamma-1)2^*}{2^*-1}}\, dz \right)^{1-\frac{1}{2^*}} \\ &\le \left( \int_{Q_{2R}} |\nabla u^h|^2\, dz \right)^{\frac{1}{2^*}} \cdot \left( \int_{Q_{2R}} |u_t^h|^{\frac{(\gamma-1)2^*}{2^*-1}}\, dz \right)^{1-\frac{1}{2^*}} \\ &\le \theta \int_{Q_{2R}} |\nabla u^h|^2\, dz + c(\theta) \int_{Q_{2R}} |u_t^h|^{\frac{(\gamma-1)2^*}{2^*-1}}\, dz.\end{aligned}$$

However, here $2^* = \frac{2n}{n-2}$ and therefore, this estimate does not hold for the case $n = 1$ we are most interested in. For $n = 1$ we proceed as follows

$$\begin{aligned}
\int_{Q_{2R}} |u_t^h|^{\gamma-1}|u^h - U|\,dz &\le \left(\int_{Q_{2R}} |u^h - U|^{\beta'}\,dz\right)^{1/\beta'} \left(\int_{Q_{2R}} |u_t^h|^{(\gamma-1)\beta}\,dz\right)^{1/\beta} \\
&\le \left\{\int \left(\int_{Q'_{2R}} |\nabla u^h|\,dx\right)^{\beta'} dt\right\}^{\frac{1}{\beta'}} \cdot \left(\int_{Q_{2R}} |u_t^h|^{(\gamma-1)\beta}\,dz\right)^{1/\beta} \\
&\le \left\{\int \left(\int_{Q'_{2R}} |\nabla u^h|^2\,dx\right)^{\frac{\beta'}{2}} R^{\frac{\beta'}{2}}\,dt\right\}^{\frac{1}{\beta'}} \cdot \left(\int_{Q_{2R}} |u_t^h|^{(\gamma-1)\beta}\,dz\right)^{1/\beta} \\
&\le R^{\frac{1}{2}} \left(\int_{Q_{2R}} |\nabla u^h|^2\,dz\right)^{\frac{1}{\beta'}} \cdot \left(\int_{Q_{2R}} |u_t^h|^{(\gamma-1)\beta}\,dz\right)^{1/\beta} \\
&\le \theta \int_{Q_{2R}} |\nabla u^h|^2\,dz + cR^{\frac{\beta}{2}} \int_{Q_{2R}} |u_t^h|^{(\gamma-1)\beta}\,dz.
\end{aligned}$$

Choosing $\beta = \frac{2}{\gamma-1}$, i.e., $\beta' = \frac{2}{3-\gamma}$, $\gamma \in (2,3)$, we get

$$\int_{Q_{2R}} |u_t^h|^{\gamma-1}|u^h - U|\,dz \le cR^{\frac{1}{\gamma-1}} \int_{Q_{2R}} |u_t^h|^2\,dz + \theta \int_{Q_{2R}} |\nabla u^h|^2\,dz.$$

Altogether we have

$$⨍_{Q_R} |\nabla u^h|^2\,dx \le c\left(⨍_{Q_{2R}} |\nabla u^h|^q\right)^{2/q} + c ⨍_{Q_{2R}} |u_t^h|^2\,dx + \frac{1}{2} ⨍_{Q_{2R}} |\nabla \bar{u}^h|^2\,dx.$$

Thus, we can apply Gehring's theory for time-discretized equations from [2] and prove higher integrability $\nabla u^h \in L^{2+\delta}$ with $\delta > 0$ independent of $h$. Higher integrability of $u_t^h$ follows from (5.2).

We must consider also the 'boundary' case. In this case let us select the test function $\psi = \tilde{u}_m - \varepsilon u_m \eta_m^2$, $u_m = \max\{\tilde{u}_m, 0\}$. Then we get in a similar way the same estimate as in (5.3) only with the change that $\tilde{u}_m - U$ is replaced by $u_m$. The derivation of the last term goes on without problems since $\chi_{\{\tilde{u}_m - \varepsilon u_m \eta_m^2 > 0\}} - \chi_{\{\tilde{u}_m > 0\}} \le 0$ as before.

As we have proven the continuity of $u_m$ and $x_0$ lies on the free boundary, there is a sufficiently large area where $u_m = 0$ in $B_{2R}$. Therefore, we are again able to apply Sobolev-Poincaré inequality and get the same result.

Now, in the one-dimensional case, from the Sobolev imbedding theorem we get a uniform bound on the Hölder norms of a subsequence of $u^h$ and we are able to pass to the limit in the approximate equation. $\square$

# 6 Conclusion

We have formulated a hyperbolic free boundary problem describing the interaction of a film and an obstacle and we have suggested its numerical solution. Several properties of the approximate solutions are shown. We also found out that by adding a higher integrable term, it is possible, using Gehring's theory, to construct weak solution and prove its regularity.

# References


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