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<td>Author(s)</td>
<td>Ngoc, Pham Huu Anh; Naito, Toshiki</td>
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Robust $\mathcal{D}$-stability of linear difference equations

Abstract

We study robustness of $\mathcal{D}$-stability of linear difference equations under multi-perturbation and affine perturbation of coefficient matrices via the concept of $\mathcal{D}$-stability radius. Some explicit formulae are derived for these $\mathcal{D}$-stability radii. The obtained results include the corresponding ones established earlier in [3], [4], [9], [10] as particular cases.

1 Introduction and Preliminaries

Let $\mathcal{D} := D(\alpha, r)$ be a open disk centered at $(\alpha, 0)$ with radius $r$ in the complex plane. A linear discrete-time (time-invariant) system is called $\mathcal{D}$-stable if its characteristic equation has only roots in $\mathcal{D}$. In this paper, we study the robustness of $\mathcal{D}$-stability of linear discrete-time systems of the form

$$x(k + 1) = A_\nu x(k) + A_{\nu-1} x(k - 1) + \ldots + A_0 x(k - \nu), \quad k \in \mathbb{N}, k \geq \nu \quad (1)$$

under parameter perturbation of the coefficient matrices via the concept of $\mathcal{D}$-stability radius. It is important to note that the problems of computing of $D(0, 1)$-stability radii (or simpler, stability radii) of linear discrete-time systems have been studied during the last twenty years by many mathematical researchers, see e.g. [2]-[5], [9]-[11]. In particular, the problems of computing of stability radii of linear discrete-time systems of the form (1) under single perturbations, affine perturbations

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1This author is supported by the Japan Society for Promotion of Science (JSPS), ID No. P 05049. Corresponding author: Pham Huu Anh Ngoc, Email: phanhtngoc@yahoo.com.
and multi-perturbations have just been studied in the recent time, see [5], [9], [10]. It is also worth noticing that (robust) $\mathcal{D}$-stability problems of linear discrete-time systems have been received much attention from researchers for a long time. Some sufficient conditions for the (robust) $\mathcal{D}$-stability of the system (1) under parameter perturbations were proposed in [1], [6], [8], [13]-[15]. However, to the best of our knowledge, there is not any formula for the $\mathcal{D}$-stability radii of the system (1) under multi-perturbations or affine-perturbations in the case of $\mathcal{D} = \mathcal{D}(\alpha, r)$. In the present paper, using our recent new results on the problems of computing of stability radii (see e.g. [10]), we can compute the $\mathcal{D}(\alpha, r)$-stability radii of the system (1) under multi-perturbations and affine perturbations. The obtained results are the extensions of the corresponding results of [3], [4], [9], [10].

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $n, l, q$ be positive integers. Inequalities between real matrices or vectors will be understood componentwise. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{R}_{+}^{l \times q}$. If $P \in \mathbb{K}^{l \times q}$ we define $|P| = (|p_{ij}|)$. For any matrix $A \in \mathbb{K}^{n \times n}$ the spectral radius of $A$ is denoted by $\rho(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \}$, where $\sigma(A)$ is the set of all eigenvalues of $A$. A norm $\| \cdot \|$ on $\mathbb{K}^{n}$ is said to be monotonic if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{K}^{n}$. Every $p$-norm on $\mathbb{K}^{n}$, $1 \leq p \leq \infty$, is monotonic. Throughout the paper, the norm $\|M\|$ of a matrix $M \in \mathbb{K}^{l \times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1} \|My\|$, where $\mathbb{K}^{q}$ and $\mathbb{K}^{l}$ are provided with some monotonic vector norms.

2 \quad $\mathcal{D}$-stability radii of linear discrete-time systems

Let $\mathcal{D} = \mathcal{D}(\alpha, r)$ be the open disk centered at $(\alpha, 0)$ with radius $r$ in the complex plane. Consider a dynamical system described by a linear discrete-time system of the form

$$x(k + 1) = Ax(k), \quad k \in \mathbb{N},$$

where $A \in \mathbb{R}^{n \times n}$ is a given matrix. The system (2) is called $\mathcal{D}$-stable if $\sigma(A) \subset \mathcal{D}$.

It is important to note that, the system (2) is asymptotically stable in the Lyapunov' s sense in the case of $\mathcal{D} = \mathcal{D}(0, 1)$ and is strong stable in the case of $\mathcal{D} = \mathcal{D}(0, r), 0 < r < 1$. We now assume that the system (2) is $\mathcal{D}$-stable and the system matrix $A$ is subjected to one of the following perturbation types

$$A \rightarrow A + \sum_{i=1}^{N} D_{i} \Delta_{i} E_{i}, \quad \text{(multi-perturbation)},$$

$$A \rightarrow A + \sum_{i=1}^{N} \delta_{i} B_{i}, \quad \text{(affine perturbation)}.$$
Here $D_i \in \mathbb{R}^{n \times l}$, $E_i \in \mathbb{R}^{q_i \times n}$, $B_i \in \mathbb{R}^{q_i \times n}$, $i \in \mathcal{N} := \{1, 2, ..., N\}$ are given matrices defining the structure of perturbations and $\Delta_i \in \mathbb{K}^{l_i \times q_i}$, $\delta_i \in \mathbb{K}$ ($i \in \mathcal{N}$) unknown disturbance matrices and scalars, respectively. For class of multi-perturbations of the form (3), we always assume that the linear space $\Delta_{\mathbb{K}} = \mathbb{K}^{l_1 \times q_1} \times \ldots \times \mathbb{K}^{l_N \times q_N}$ of all perturbation families $\Delta = (\Delta_1, ..., \Delta_N)$, for $\Delta_i \in \mathbb{K}^{l_i \times q_i}$, is endowed with the norm $\gamma(\Delta) = \gamma(\Delta_1, ..., \Delta_N) = \sum_{i=1}^{N} ||\Delta_i||$, where the norms $||\Delta_i||$ are operator norms on $\mathbb{K}^{l_i \times q_i}$, induced by given monotonic vector norms on the spaces $\mathbb{K}^l, \mathbb{K}^q, i \in \mathcal{N}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).

**Definition 2.1.** Let the linear discrete time system (2) be $D-$stable.

(a) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to multi-perturbations of the form (3) are defined, respectively, by

\[ r_{\mathbb{C}}(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf\{\gamma(\Delta) : \Delta \in \Delta_{\mathbb{C}}, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D\}, \]
\[ r_{\mathbb{R}}(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf\{\gamma(\Delta) : \Delta \in \Delta_{\mathbb{R}}, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D\}. \]

(b) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to affine perturbations of the form (4) are defined, respectively, by

\[ r_{\mathbb{C}}(A, (B_i)_{i \in \mathcal{N}}; D) = \inf\{\max_{i \in \mathcal{N}} |\delta_i| : \delta_i \in \mathbb{C}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D\}, \]
\[ r_{\mathbb{R}}(A, (B_i)_{i \in \mathcal{N}}; D) = \inf\{\max_{i \in \mathcal{N}} |\delta_i| : \delta_i \in \mathbb{R}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D\}. \]

As noted in Introduction, the problems of computing of the stability radii (i.e. $D(0, 1)$-stability radii) of the system (2) have been studied during the last twenty years and have got the full results, see e.g. [3], [12], [4], [10]. We list here the interesting results for the class of positive systems (i.e. $A$ is a nonnegative matrix).

**Theorem 2.2.** [4] Let the system (2) be $D(0, 1)-$stable and positive. Suppose the system matrix $A$ is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}^{q \times n}$, $i \in \mathcal{N}$. Then

\[ r_{\mathbb{C}}(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = r_{\mathbb{R}}(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\rho(\sum_{i=1}^{N} B_i (I_n - A)^{-1})}. \]

**Theorem 2.3.** [10] Let the system (2) be $D(0, 1)-$stable and positive. Assume that the matrix $A$ is subjected to parameter multi-perturbations (3). If $D_i = D \in \mathbb{R}^{n \times l}$ and $E_i \in \mathbb{R}^{q \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}^{q \times n}$ and $D_i \in \mathbb{R}^{n \times l}$ for every $i \in \mathcal{N}$, then

\[ r_{\mathbb{C}}(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = r_{\mathbb{R}}(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\max_{i \in \mathcal{N}} ||E_i (I_n - A)^{-1} D_i||}. \]

The following theorem extends the above results to the general case of $D = D(\alpha, r)$.

**Theorem 2.4.** Let the system (2) be $D(\alpha, r)-$stable and $A \geq \alpha I_n$. (i) If the matrix $A$ is subjected to multi-perturbations (3), where $D_i = D \in \mathbb{R}^{n \times l}$ and $E_i \in \mathbb{R}^{q \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}^{q \times n}$ and $D_i \in \mathbb{R}^{n \times l}$ for every $i \in \mathcal{N}$, then
\[ r_C(A, (D_i)_{i \in \mathcal{N}}; D(\alpha, r)) = r_R(A, (D_i)_{i \in \mathcal{N}}; D(\alpha, r)) = \max_{i \in \mathcal{N}} \|E_i((\alpha+r)I_n-A)^{-1}D_i\| \].

(ii) If the matrix $A$ is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}^{n \times n}$, $i \in \mathcal{N}$, then $r_C(A, (B_i)_{i \in \mathcal{N}}; D(\alpha, r)) = r_R(A, (B_i)_{i \in \mathcal{N}}; D(\alpha, r)) = \frac{1}{\rho(\sum_{i=1}^{\nu} B_i((\alpha+r)I_n-A)^{-1})}.

Proof. The proof is based on Theorems 2.2, 2.3 and the fact that the system $x(k+1) = Ax(k), k \in \mathbb{N}$ is $D(\alpha, r)$ stable if and only if the system $x(k+1) = (A - \alpha I_n)x(k), k \in \mathbb{N}$ is $D(0,1)$-stable. For sake of space, it is omitted here.

The following is an extension of the main result of [7].

**Corollary 2.5.** Let $P(z) := I_n z^{\nu+1} - A_{\nu} z^\nu - \ldots - A_0$ be a given polynomial matrix. Assume that $|\alpha| < r, |\alpha| + r \leq 1$ and $\|A_0A_1\ldots A_\nu\|_{\infty} < (r - |\alpha|)^{\nu+1}$. Then all the roots of the equation $\det P(z) = 0$ lie inside the disk $D(\alpha, r)$.

### 3 \(D\)-stability radii of linear discrete time-delay systems

Consider a dynamical system described by a linear discrete time-delay system of the form (1), where $A_i \in \mathbb{R}^{n \times n}, i \in \mathcal{V} := \{0, 1, 2, \ldots, \nu\}$, are given matrices. For the linear discrete time-delay system (1), we consider the stable region $\mathcal{D} = D(\alpha, r), |\alpha| < r, r + |\alpha| \leq 1$, see e.g. [8], [13], [14]. We associate the system (1) with the following polynomial matrix $P(z) := (z^{\nu+1}I_n - A_{\nu} z^\nu - A_{\nu-1} z^{\nu-1} - \ldots - A_0), z \in \mathbb{C}$. Denote by $\sigma((A_i)_{i \in \mathcal{V}}) := \{z \in \mathbb{C} : \det P(z) = 0\}$ the set of all roots of the characteristic equation of the linear discrete time-delay system (1). Then $\sigma((A_i)_{i \in \mathcal{V}})$ is called the spectral set of the linear discrete time-delay system (1) and $\rho((A_i)_{i \in \mathcal{V}}) := \max \{|s| : s \in \sigma((A_i)_{i \in \mathcal{V}})\}$ is called spectral radius of the linear discrete time-delay system (1). Recall that the system (1) is said to be $\mathcal{D}$-stable if $\sigma((A_i)_{i \in \mathcal{V}}) \subset \mathcal{D}$. We now assume that the system (1) is $\mathcal{D}$-stable and the coefficient matrices $A_i, i \in \mathcal{V}$ are subjected to parameter perturbations

\[ A_i \rightarrow A_i + \sum_{j=1}^{\nu} D_{ij} \Delta_{ij} E_{ij}, \quad \text{multi-perturbation} \]  

\[ A_i \rightarrow A_i + \sum_{j=1}^{\nu} \delta_{ij} B_{ij}, \quad \text{affine-perturbation} \]

where $D_{ij} \in \mathbb{R}^{n \times n}, E_{ij} \in \mathbb{R}^{q_j \times n}, (i \in \mathcal{V}, j \in \mathcal{N} := \{1, 2, \ldots, N\})$; $B_{ij} \in \mathbb{R}^{n \times n}$, $(i \in \mathcal{V}, j \in \mathcal{N})$ are given matrices defining the structure of perturbations and $\Delta_{ij} \in \mathbb{K}^{l_{ij} \times n}, (i \in \mathcal{V}, j \in \mathcal{N})$; $\delta_{ij} \in \mathbb{K}, (i \in \mathcal{V}, j \in \mathcal{N})$ are perturbation matrices, perturbation scalars, respectively. For the class of multi-perturbations of the form
of (5), we define \( \tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_\nu) \), where \( \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in \mathbb{K}^{l_i \times q_i}, i \in \nu \). Then the size of each perturbation \( \tilde{\Delta} \) is measured by \( \gamma(\tilde{\Delta}) := \sum_{i=0}^{\nu} \sum_{j=1}^{N} \|\Delta_{ij}\| \). With the class of affine perturbations of the form (6), we denote \( \delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu1}, \ldots, \delta_{\nuN})) \in \mathbb{K}^{\nu \times N} \) and the size of each perturbation \( \delta \) is measured by \( \gamma(\delta) = \max_{i \in \nu, j \in N} |\delta_{ij}| \).

**Definition 3.1.** Let the linear discrete time-delay system (1) be \( D \)-stable.

(a) The complex, real \( D(\alpha, r) \)-stability radius of the system (1) with respect to multi-perturbations of the form (5) is defined, respectively, by

\[
\begin{align*}
\rho_{C}^D(D) &= \inf \{ \gamma(\tilde{\Delta}) : \tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_\nu), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}), \\
& \quad \in \mathbb{C}^{l_i \times q_i} \times \cdots \times \mathbb{C}^{l_N \times q_N}, i \in \nu, \sigma \left( \left( A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij} \right)_{i \in \nu} \right) \not\subset D \}, \\
\rho_{R}^D(D) &= \inf \{ \gamma(\tilde{\Delta}) : \tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_\nu), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}), \\
& \quad \in \mathbb{R}^{l_i \times q_i} \times \cdots \times \mathbb{R}^{l_N \times q_N}, i \in \nu, \sigma \left( \left( A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij} \right)_{i \in \nu} \right) \not\subset D \}.
\end{align*}
\]

(b) The complex, real \( D(\alpha, r) \)-stability radius of the system (1) with respect to affine perturbations of the form (6) is defined, respectively, by

\[
\begin{align*}
\rho_{C}^D(D) &= \inf \{ \gamma(\delta) : \delta \in \mathbb{C}^{(\nu+1)N}, \sigma \left( \left( A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij} \right)_{i \in \nu} \right) \not\subset D \}, \\
\rho_{R}^D(D) &= \inf \{ \gamma(\delta) : \delta \in \mathbb{R}^{(\nu+1)N}, \sigma \left( \left( A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij} \right)_{i \in \nu} \right) \not\subset D \}.
\end{align*}
\]

In particular case of \( D = D(0, 1) \), the problems of computing the stability radii of the linear discrete-time systems (1) under single perturbations, affine perturbations and multi-perturbations have been done recently by ourselves (see [5], [9], [10]). We summarize here some existing results of these problems. Recall that the system (1) is positive if and only if system matrices \( A_0, A_1, \ldots, A_\nu \) are nonnegative.

**Theorem 3.2.** [9] Suppose the linear discrete time-delay system (1) is \( D(0,1) \)-stable, positive and the system matrices \( A_i, i \in \nu \) are subjected to affine perturbations of the form (6) where \( B_{ij} \in \mathbb{R}^{l_i \times q_j} \), \( i \in \nu, j \in N \). Then,

\[
\rho_{C}^D(D(0,1)) = \rho_{R}^D(D(0,1)) = \frac{1}{\rho(P^{-1}B)} \cdot B := \sum_{j=1}^{N} B_{0j} + \sum_{j=1}^{N} B_{1j} + \ldots + \sum_{j=1}^{N} B_{\nu j}.
\]

**Remark 3.3.** In the proof of Theorem 3.2, we showed that the real perturbation \( \delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu1}, \ldots, \delta_{\nuN})) \in \mathbb{R}^{(\nu+1)N}, \delta_{ij} = \frac{1}{\rho(P^{-1}B)}, (i \in \nu, j \in N) \) is a minimal size destabilizing perturbation. This fact will be used in the sequel.

**Theorem 3.4.** [10] Let the linear discrete time-delay system (1) be positive, \( D(0,1) \)-stable. Assume that the system matrices \( A_i, i \in m \) are subjected to multi-perturbations of the form (5) where \( D_{ij} := D \in \mathbb{R}^{l_i \times l_j}, E_{ij} \in \mathbb{R}^{q_i \times q_j} \) for all \( i \in \nu, j \in N \) or \( E_{ij} := E \in \mathbb{R}^{l_i \times l_j}, D_{ij} \in \mathbb{R}^{q_i \times q_j} \) for all \( i \in \nu, j \in N \). Then,

\[
\rho_{C}^D(D(0,1)) = \rho_{R}^D(D(0,1)) = \frac{1}{\max \{|E|, P^{-1}D| : E \in \nu, j \in N \}}.
\]
Theorem 3.5. Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the coefficient matrices $A_i, i \in \overline{\nu}$ are subjected to the multi-perturbations (5), where $D_{ij} := D \in \mathbb{R}_{+}^{n \times l}, E_{ij} \in \mathbb{R}_{+}^{l_{ij} \times n} (i \in \overline{\nu}, j \in \mathbb{N})$ or $D_{ij} \in \mathbb{R}_{+}^{n \times l}, E_{ij} := E \in \mathbb{R}_{+}^{l_{ij} \times n} (i \in \overline{\nu}, j \in \mathbb{N})$. If $\alpha \leq 0$ and $A_0, A_1, ..., A_{\nu-1}, (A_{\nu} - \alpha I_n) \in \mathbb{R}_{+}^{n \times n}$, then

$$r_c^m(D(\alpha, r)) = r_c^m(D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in \mathbb{N}} \|((\alpha + r)^i E_{ij} P(\alpha + r)^{-1} D_{ij})\|}.$$

Proof. Assume $D_{ij} := D \in \mathbb{R}_{+}^{n \times l}, E_{ij} \in \mathbb{R}_{+}^{l_{ij} \times n} (i \in \overline{\nu}, j \in \mathbb{N})$. Consider the companion matrix of the polynomial matrix $P(z) = (z^{\nu+1} I_n - A_\nu z^\nu - A_{\nu-1} z^{\nu-1} - \ldots - A_0)$:

$$A_c := \begin{bmatrix}
0 & I_n & 0 & \ldots & 0 & 0 \\
0 & 0 & I_n & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 & I_n \\
A_0 & A_1 & \ldots & \ldots & \ldots & A_{\nu}
\end{bmatrix} \in \mathbb{R}^{(\nu+1)n \times (\nu+1)n},$$

and similarly $A_c(\tilde{\Delta})$ for the perturbed polynomial matrix $P_{\tilde{\Delta}}(z) := z^{\nu+1} I_n - \sum_{i=0}^{\nu} (A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij}) z^i$, where $\tilde{\Delta} := (\Delta_0, \Delta_1, ..., \Delta_{\nu})$, $\Delta_i := (\Delta_{i0}, \Delta_{i1}, ..., \Delta_{i\nu}) \in \mathbb{R}^{l_{ij} \times q_{ij}}$, $i \in \overline{\nu}$. Then the matrix $A_c(\tilde{\Delta})$ can be represented by the following form

$$A_c(\tilde{\Delta}) = A_c + \sum_{j=1}^{N} \tilde{D}_{ij} \Delta_{ij} E_{ij} + \sum_{j=1}^{N} \tilde{D}_{ij} \tilde{\Delta}_{ij} + \ldots + \sum_{j=1}^{N} \tilde{D}_{ij} \Delta_{ij} E_{ij},$$

where

$$\tilde{D}_{ij} = \tilde{D} := [0, ..., 0, D^T] \in \mathbb{R}^{(\nu+1)n \times l}, \tilde{E}_{ij} := [E_{ij}, 0, ..., 0] \in \mathbb{R}^{l_{ij} \times (\nu+1)n}.$$

for every $i \in \overline{\nu}$, $j \in \mathbb{N}$. It follows from the equality $\det P_{\tilde{\Delta}}(z) = \det (z I_{(\nu+1)n} - A_c(\tilde{\Delta}))$ that $\sigma(A_c + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{ij \in \mathbb{N}} = \sigma(A_c(\tilde{\Delta}))$. So, we get $r_c^m(D(\alpha, r)) = r_c(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{ij \in \mathbb{N}}, D(\alpha, r)) = r_c(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{ij \in \mathbb{N}}, D(\alpha, r)) = r_c(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{ij \in \mathbb{N}}, D(\alpha, r)) = r_c(A_c, (\tilde{D}_{ij}, \tilde{E}_{ij})_{ij \in \mathbb{N}}, D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in \mathbb{N}} \|(\tilde{E}_{ij}((\alpha+r)^i I_{(\nu+1)n}-A_c)^{-1} \tilde{D}_{ij})\|}$.

On the other hand, it is easy to check that

$$(z I_{(\nu+1)n} - A_c)^{-1} \tilde{D} = \begin{pmatrix}
P(z)^{-1} \\
z P(z)^{-1} \\
\vdots \\
z^\nu P(z)^{-1}
\end{pmatrix}.$$
Therefore, \( r_{\mathbb{C}}^{m}(D(\alpha, r)) = r_{\mathbb{R}}^{m}(D(\alpha, r)) = \max_{i, j \in \mathbb{N}} \frac{1}{\|\alpha^{i+j} E_{ij} P(\alpha+r)^{-1} B_{ij}\|}. \) The proof of the case of \( D_{ij} \in \mathbb{R}_{+}^{n \times n}, E_{ij} := E \in \mathbb{R}_{+}^{q \times n} (i \in \mathbb{N}, j \in \mathbb{N}), \) can be done by a similar way. This completes our proof. \( \square \)

We now turn to the problem of computing the complex, real \( D \)-stability radius under affine perturbations (6). For every \( i \in \mathbb{N}, \) let us define

\[ A_{i}^{*} := \frac{1}{r_{i+1}} \left( C_{i}^{v-i} \alpha^{v-i} A_{v} + C_{i-1}^{v-1-i} \alpha^{v-1-i} A_{v-1} + \ldots + A_{i} - C_{i+1}^{v+1-i} \alpha^{v+1-i} I_{n} \right), \] (7)

where \( C_{i}^{v} := \frac{u!}{v!(u-v)!}, u, v \in \mathbb{N}, u \geq v. \) The following theorem is an extension of Theorem 3.2 to the general case of \( D = D(\alpha, r). \)

**Theorem 3.6.** Let the linear discrete time-delay system (1) be \( D(\alpha, r) \)-stable. Suppose the system matrices \( A_{i} \), \( i \in \mathbb{N}, \) are subjected to affine perturbations (6), where \( B_{ij} \in \mathbb{R}_{+}^{n \times n} (i \in \mathbb{N}, j \in \mathbb{N}). \) If either \( \alpha \leq 0 \) and \( A_{0}, A_{1}, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}_{+}^{n \times n}, \) or \( \alpha > 0 \) and \( A_{i}^{*} \in \mathbb{R}_{+}^{n \times n}, i \in \mathbb{N}, \) then \( r_{\mathbb{C}}^{m}(D(\alpha, r)) = r_{\mathbb{R}}^{m}(D(\alpha, r)) = \frac{1}{\rho(P(\alpha+r)^{-1} B)}, \) where \( B := \sum_{i=0}^{\nu} \left( \sum_{j=1}^{N} B_{ij} \right) (\alpha+r)^{i}. \)

**Proof.** In the case of \( \alpha \leq 0 \) and \( A_{0}, A_{1}, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}_{+}^{n \times n}, \) the proof is similar to that of Theorem 3.5, based on the result of Theorem 2.4(i). Then, we have \( r_{\mathbb{C}}^{m}(D(\alpha, r)) = r_{\mathbb{R}}^{m}(D(\alpha, r)) = \frac{1}{\rho(P(\alpha+r)^{-1} B)}. \) We now assume that \( \alpha > 0 \) and \( A_{i}^{*} \in \mathbb{R}_{+}^{n \times n}, i \in \mathbb{N}. \) Denote by \( P^{*}(z) := z^{\nu+1} I_{n} - A_{\nu} z^\nu - \ldots - A_{0}^{*}. \) Let \( s \in \mathbb{C}, |s - \alpha| \geq r \) satisfy \( \det P(s) = 0. \) Setting \( z = \frac{\alpha + r}{r}, |z| \geq 1, \) by a direct computation, we have \( \det P^{*}(z) = 0 \) if and only if \( \det P^{*}(\alpha) = 0. \) So the discrete time-delay system (1) is \( D(\alpha, r) \)-stable if and only if the following discrete time-delay system

\[ x(k+1) = A_{\nu}^{*} x(k) + A_{\nu-1}^{*} x(k-1) + \ldots + A_{0}^{*} x(k-\nu), \quad k \in \mathbb{N}, \quad k \geq \nu, \] (8)

is \( D(0, 1) \)-stable. Similarly, the perturbed system

\[ x(k+1) = (A_{\nu} + \sum_{j=1}^{N} \delta_{\nu j} B_{\nu j}) x(k) + \ldots + (A_{0} + \sum_{j=1}^{n} \delta_{0 j} B_{0 j}) x(k-\nu), \quad k \in \mathbb{N}, \quad k \geq \nu, \] (9)

is \( D(\alpha, r) \)-stable if and only if the following discrete time-delay system is \( D(0, 1) \)-stable

\[ x(k+1) = (A_{\nu}^{*} + B_{\nu}^{*}) x(k) + \ldots + (A_{0}^{*} + B_{0}^{*}) x(k-\nu), \quad k \in \mathbb{N}, \quad k \geq \nu. \] (10)

Here, \( B_{ij}^{*} := (\sum_{j=1}^{N} \delta_{ij} \left( \frac{1}{r_{i+1}} C_{i}^{v-i} \alpha^{v-i} B_{ij} + \sum_{j=1}^{N} \delta_{(i+1)j} \left( \frac{1}{r_{i+1}} C_{i}^{v+1-i} \alpha^{v+1-i} B_{(i+1)j} + \ldots + \frac{1}{r_{i+1}} B_{ij} \right) \right) i \in \mathbb{N}, (i \in \mathbb{N}, j \in \mathbb{N}), \) we have

\[
\frac{1}{r_{i+1}} C_{i}^{v-i} \alpha^{v-i} B_{ij}, \quad \frac{1}{r_{i+1}} C_{i}^{v+1-i} \alpha^{v+1-i} B_{(i+1)j}, \ldots, \frac{1}{r_{i+1}} B_{ij} \in \mathbb{R}_{+}^{n \times n}, \quad i \in \mathbb{N}, j \in \mathbb{N}.
\]
It follows from Theorem 3.2 that the system (10) is $D(0,1)$-stable for every $\delta$ satisfying $\max_{i \in \nu, j \in \mathbb{N}} |\delta_{ij}| < \frac{1}{\rho(P^{*}(1)^{-1}G)}$, where

$$G := \sum_{i=0}^{\nu} \left( \sum_{j=1}^{N} \frac{1}{\nu+1-i} C_{\nu+1-i}^i \alpha^{\nu+1-1-i} B_{\nu j} + \sum_{j=1}^{N} \frac{1}{\nu+1-i} C_{\nu}^i \alpha^{\nu+1-i} B_{(\nu-1)j} + \ldots + \sum_{j=1}^{N} \frac{1}{\nu+1-i} B_{ij} \right). \tag{11}$$

Hence, the perturbed system (9) is $D(\alpha, r)$-stable for every complex perturbation $\delta$ such that $\max_{i \in \nu, j \in \mathbb{N}} |\delta_{ij}| < \frac{1}{\rho(P^{*}(1)^{-1}G)}$. By the definition of the complex $D(\alpha, r)$-stability radius of the system (1) under affine perturbations of the form (6), we get $r_{\mathbb{C}}^a(D(\alpha, r)) \geq \frac{1}{\rho(P^{*}(1)^{-1}G)}$. On the other hand, taking Remark 3.3 into account, the system (10) is not $D(0,1)$-stable if $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu 1}, \ldots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}$; $\delta_{ij} = \frac{1}{\rho(P^{*}(1)^{-1}G)} (i \in \nu, j \in N)$. Then the perturbed system (9) is not $D(\alpha, r)$-stable if

$$\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu 1}, \ldots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}; \delta_{ij} = \frac{1}{\rho(P^{*}(1)^{-1}G)} (i \in \nu, j \in N).$$

We derive that $r_{\mathbb{R}}^a(D(\alpha, r)) \leq \frac{1}{\rho(P^{*}(1)^{-1}G)}$. So we get the following inequalities

$$\frac{1}{\rho(P^{*}(1)^{-1}G)} \leq r_{\mathbb{C}}^a(D(\alpha, r)) \leq r_{\mathbb{R}}^a(D(\alpha, r)) \leq \frac{1}{\rho(P^{*}(1)^{-1}G)}.$$

Therefore $r_{\mathbb{C}}^a(D(\alpha, r)) = r_{\mathbb{R}}^a(D(\alpha, r)) = \frac{1}{\rho(P^{*}(1)^{-1}G)}$. Finally, by a direct computation, we get $P^{*}(1)^{-1}G = P(\alpha + r)^{-1}B$. This completes our proof. $\square$

References


