Robust $\mathcal{D}$-stability of linear difference equations

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Abstract

We study robustness of $\mathcal{D}$-stability of linear difference equations under multi-
perturbation and affine perturbation of coefficient matrices via the concept of $\mathcal{D}$-stability radius. Some explicit formulae are derived for these $\mathcal{D}$-stability radii. The obtained results include the corresponding ones established earlier in [3], [4], [9], [10] as particular cases.

1 Introduction and Preliminaries

Let $\mathcal{D} := D(\alpha, r)$ be a open disk centered at $(\alpha, 0)$ with radius $r$ in the complex plane. A linear discrete-time (time-invariant) system is called $\mathcal{D}$-stable if its characteristic equation has only roots in $\mathcal{D}$. In this paper, we study the robustness of $\mathcal{D}$-stability of linear discrete-time systems of the form

$$x(k + 1) = A_\nu x(k) + A_{\nu-1} x(k - 1) + \cdots + A_0 x(k - \nu), \quad k \in \mathbb{N}, k \geq \nu \quad (1)$$

under parameter perturbation of the coefficient matrices via the concept of $\mathcal{D}$-
stability radius. It is important to note that the problems of computing of $D(0, 1)$-
stability radii (or simpler, stability radii) of linear discrete-time systems have been studied during the last twenty years by many mathematical researchers, see e.g. [2]-[5], [9]-[11]. In particular, the problems of computing of stability radii of linear discrete-time systems of the form (1) under single perturbations, affine perturbations

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and multi-perturbations have just been studied in the recent time, see [5], [9], [10]. It is also worth noticing that (robust) \( \mathcal{D} \)-stability problems of linear discrete-time systems have been received much attention from researchers for a long time. Some sufficient conditions for the (robust) \( \mathcal{D} \)-stability of the system (1) under parameter perturbations were proposed in [1], [6], [8], [13]-[15]. However, to the best of our knowledge, there is not any formula for the \( \mathcal{D} \)-stability radii of the system (1) under multi-perturbations or affine-perturbations in the case of \( \mathcal{D} = \mathcal{D}(\alpha, r) \). In the present paper, using our recent new results on the problems of computing stability radii (see e.g. [10]), we can compute the \( \mathcal{D}(\alpha, r) \)-stability radii of the system (1) under multi-perturbations and affine perturbations. The obtained results are the extensions of the corresponding results of [3], [4], [9], [10].

Let \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{R} \) and \( n, l, q \) be positive integers. Inequalities between real matrices or vectors will be understood componentwise. The set of all nonnegative \( l \times q \)-matrices is denoted by \( \mathbb{R}^{l \times q}_{+} \). If \( P \in \mathbb{R}^{l \times q} \) we define \( |P| = (|p_{ij}|) \). For any matrix \( A \in \mathbb{K}^{n \times n} \) the spectral radius of \( A \) is denoted by \( \rho(A) = \max \{ |\lambda| : \lambda \in \sigma(A) \} \), where \( \sigma(A) \) is the set of all eigenvalues of \( A \). A norm \( || \cdot || \) on \( \mathbb{K}^{n} \) is said to be monotonic if \( |x| \leq |y| \) implies \( ||x|| \leq ||y|| \) for all \( x, y \in \mathbb{K}^{n} \). Any \( p \)-norm on \( \mathbb{K}^{n} \), \( 1 \leq p \leq \infty \), is monotonic. Throughout the paper, the norm \( ||M|| \) of a matrix \( M \in \mathbb{K}^{l \times q} \) is always understood as the operator norm defined by \( ||M|| = \max_{||y||=1} ||My|| \), where \( \mathbb{K}^{q} \) and \( \mathbb{K}^{l} \) are provided with some monotonic vector norms.

## 2 \( \mathcal{D} \)-stability radii of linear discrete-time systems

Let \( \mathcal{D} = \mathcal{D}(\alpha, r) \) be the open disk centered at \( (\alpha, 0) \) with radius \( r \) in the complex plane. Consider a dynamical system described by a linear discrete-time system of the form

\[
x(k+1) = Ax(k), \quad k \in \mathbb{N},
\]

(2)

where \( A \in \mathbb{R}^{n \times n} \) is a given matrix. The system (2) is called \( \mathcal{D} \)-stable if \( \sigma(A) \subset \mathcal{D} \).

It is important to note that, the system (2) is asymptotically stable in the Lyapunov' s sense in the case of \( \mathcal{D} = \mathcal{D}(0,1) \) and is strong stable in the case of \( \mathcal{D} = \mathcal{D}(0,r), 0 < r < 1 \). We now assume that the system (2) is \( \mathcal{D} \)-stable and the system matrix \( A \) is subjected to one of the following perturbation types

\[
A \rightarrow A + \sum_{i=1}^{N} D_{i} \Delta_{i} E_{i}, \quad \text{(multi-perturbation),}
\]

(3)

\[
A \rightarrow A + \sum_{i=1}^{N} \delta_{i} B_{i}, \quad \text{(affine perturbation).}
\]

(4)
Here $D_i \in \mathbb{R}^{n \times l_i}, E_i \in \mathbb{R}^{q_i \times n}, B_i \in \mathbb{R}^{q_i \times n}, i \in \mathcal{N} := \{1, 2, \ldots, N\}$ are given matrices defining the structure of perturbations and $\Delta_i \in \mathbb{K}^{l_i \times q_i}, \delta_i \in \mathbb{K} (i \in \mathcal{N})$ unknown disturbance matrices and scalars, respectively. For class of multi-perturbations of the form (3), we always assume that the linear space $\Delta_{\mathcal{K}} = \mathbb{K}^{l_1 \times q_1} \times \ldots \times \mathbb{K}^{l_N \times q_N}$ of all perturbation families $\Delta = (\Delta_1, \ldots, \Delta_N)$, where $\Delta_i \in \mathbb{K}^{l_i \times q_i}$, is endowed with the norm

$$\gamma(\Delta) = \gamma(\Delta_1, \ldots, \Delta_N) = \sum_{i=1}^{N} ||\Delta_i||,$$

where the norms $||\Delta_i||$ are operator norms on $\mathbb{K}^{l_i \times q_i}$, induced by given monotonic vector norms on the spaces $\mathbb{K}^{l_i}, \mathbb{K}^{q_i}, i \in \mathcal{N} (\mathbb{K} = \mathbb{R}, \mathbb{C})$.

**Definition 2.1.** Let the linear discrete time system (2) be $D-$stable.

(a) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to multi-perturbations of the form (3) are defined, respectively, by

$$r_C(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf\{\gamma(\Delta) : \Delta \in \Delta_C, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D\},$$

$$r_R(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf\{\gamma(\Delta) : \Delta \in \Delta_R, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D\}.$$

(b) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to affine perturbations of the form (4) are defined, respectively, by

$$r_C(A, (B_i)_{i \in \mathcal{N}}; D) = \inf\{\max_{i \in \mathcal{N}} |\delta_i| : \delta_i \in \mathbb{C}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D\},$$

$$r_R(A, (B_i)_{i \in \mathcal{N}}; D) = \inf\{\max_{i \in \mathcal{N}} |\delta_i| : \delta_i \in \mathbb{R}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D\}.$$

As noted in Introduction, the problems of computing of the stability radii (i.e. $D(0, 1)$-stability radii) of the system (2) have been studied during the last twenty years and have got the full results, see e.g. [3], [12], [4], [10]. We list here the interesting results for the class of positive systems (i.e. $A$ is a nonnegative matrix).

**Theorem 2.2.** [4] Let the system (2) be $D(0, 1)-$stable and positive. Suppose the system matrix $A$ is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}^{q \times n}, i \in \mathcal{N}$. Then

$$r_C(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = r_R(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\rho(\sum_{i=1}^{N} B_i (I_n - A)^{-1})}.$$

**Theorem 2.3.** [10] Let the system (2) be $D(0, 1)-$stable and positive. Assume that the matrix $A$ is subjected to parameter multi-perturbations (3). If $D_i = D \in \mathbb{R}^{q \times q}$ and $E_i \in \mathbb{R}^{q \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}^{q \times n}$ and $D_i \in \mathbb{R}^{q \times l_i}$ for every $i \in \mathcal{N}$, then

$$r_C(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = r_R(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\max_{i \in \mathcal{N}} ||E_i (I_n - A)^{-1} D_i||}.$$

The following theorem extends the above results to the general case of $D = D(\alpha, r)$.

**Theorem 2.4.** Let the system (2) be $D(\alpha, r)-$stable and $A \geq \alpha I_n$. (i) If the matrix $A$ is subjected to multi-perturbations (3), where $D_i = D \in \mathbb{R}^{q \times q}$ and $E_i \in \mathbb{R}^{q \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}^{q \times n}$ and $D_i \in \mathbb{R}^{q \times l_i}$ for every $i \in \mathcal{N}$, then
\[ r_C(A, (D_i)_{i \in \mathbb{N}}; D(\alpha, r)) = r_\mathbb{R}(A, (D_i)_{i \in \mathbb{N}}; D(\alpha, r)) = \max_{i \in \mathbb{N}} \|E_i((\alpha+r)I_n-A)^{-1}D_i\|, \]

(ii) If the matrix \( A \) is subjected to affine-perturbations (4), where \( B_i \in \mathbb{R}^{n \times n}, i \in \mathbb{N}, \) then \( r_C(A, (B_i)_{i \in \mathbb{N}}; D(\alpha, r)) = r_\mathbb{R}(A, (B_i)_{i \in \mathbb{N}}; D(\alpha, r)) = \frac{1}{\rho(\sum_{i=1}^{N} B_i((\alpha+r)I_n-A)^{-1})}. \]

**Proof.** The proof is based on Theorems 2.2, 2.3 and the fact that the system \( x(k + 1) = Ax(k), k \in \mathbb{N} \) is \( D(\alpha, r) \)-stable if and only if the system \( x(k + 1) = (A - \alpha I_n)x(k), k \in \mathbb{N} \) is \( D(0, 1) \)-stable. For sake of space, it is omitted here.

The following is an extension of the main result of [7].

**Corollary 2.5.** Let \( P(z) := I_n z^{\nu+1} - A_\nu z^\nu - ... - A_0 \) be a given polynomial matrix. Assume that \(|\alpha| < r, |\alpha| + r \leq 1 \) and \( \|[A_0 A_1 ... A_\nu]\|_\infty < (r - |\alpha|)^{\nu+1} \). Then all the roots of the equation \( \det P(z) = 0 \) lie inside the disk \( D(\alpha, r) \).

### 3 \( D \)-stability radii of linear discrete time-delay systems

Consider a dynamical system described by a linear discrete time-delay system of the form (1), where \( A_i \in \mathbb{R}^{n \times n}, i \in \overline{\nu} := \{0, 1, 2, ..., \nu\}, \) are given matrices. For the linear discrete time-delay system (1), we consider the stable region \( D = D(\alpha, r), |\alpha| < r, r + |\alpha| \leq 1 \), see e.g. [8], [13], [14]. We associate the system (1) with the following polynomial matrix \( P(z) := (z^{\nu+1}I_n - A_\nu z^\nu - A_{\nu-1} z^{\nu-1} - ... - A_0) \), \( z \in \mathbb{C} \). Denote by \( \sigma((A_i)_{i \in \overline{\nu}}) := \{z \in \mathbb{C} : \det P(z) = 0\} \) the set of all roots of the characteristic equation of the linear discrete time-delay system (1). Then \( \sigma((A_i)_{i \in \overline{\nu}}) \) is called the spectral set of the linear discrete time-delay system (1) and \( \rho((A_i)_{i \in \overline{\nu}}) := \max \{|s| : s \in \sigma((A_i)_{i \in \overline{\nu}})\} \) is called spectral radius of the linear discrete time-delay system (1). Recall that the system (1) is said to be \( D \)-stable if \( \sigma((A_i)_{i \in \overline{\nu}}) \subset D \). We now assume that the system (1) is \( D \)-stable and the coefficient matrices \( A_i, i \in \overline{\nu} \) are subjected to parameter perturbations

\[
A_i \rightarrow A_i + \sum_{j=1}^{\nu} D_{ij} \Delta_{ij} E_{ij}, \quad \text{(multi-perturbation)}
\]

\[
A_i \rightarrow A_i + \sum_{j=1}^{\nu} \delta_{ij} B_{ij}, \quad \text{(affine-perturbation)}
\]

where \( D_{ij} \in \mathbb{R}^{n \times n}, E_{ij} \in \mathbb{R}^{q_i \times n}, \) \( (i \in \overline{\nu}, j \in \mathbb{N} := \{1, 2, ..., N\}) \); \( B_{ij} \in \mathbb{R}^{n \times n}, (i \in \overline{\nu}, j \in \mathbb{N}) \) are given matrices defining the structure of perturbations and \( \Delta_{ij} \in \mathbb{K}^{l_{ij} \times q_i}, (i \in \overline{\nu}, j \in \mathbb{N}) \); \( \delta_{ij} \in \mathbb{K}, (i \in \overline{\nu}, j \in \mathbb{N}) \) are perturbation matrices, perturbation scalars, respectively. For the class of multi-perturbations of the form
We define $\tilde{\Delta} := (\Delta_0, \alpha_1, \ldots, \alpha_{\nu})$, where $\alpha_i := (\alpha_{i1}, \alpha_{i2}, \ldots, \alpha_{iN}) \in K_{k_i \times q_i} \times \ldots \times K_{k_{\nu} \times q_{\nu}}, \alpha \in \nu$. Then the size of each perturbation $\tilde{\Delta}$ is measured by $\gamma(\tilde{\Delta}) := \sum_{i=0}^{\nu} \sum_{j=1}^{N} ||\Delta_{ij}||$. With the class of affine perturbations of the form (6), we denote $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu1}, \ldots, \delta_{\nuN})) \in K_{k_{\nu}N}$ and the size of each perturbation $\delta$ is measured by $\gamma(\delta) = \max_{i \in \nu, j \in N} |\delta_{ij}|$.

**Definition 3.1.** Let the linear discrete time-delay system (1) be $D$-stable.

(a) The complex, real $D(\alpha, \gamma)$-stability radius of the system (1) with respect to multi-perturbations of the form (5) is defined, respectively, by

$$r^{c}_{D}(\delta) = \inf \{ \gamma(\Delta) : \Delta := (\Delta_0, \Delta_1, \ldots, \Delta_{\nu}), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in C_{k_i \times q_i} \times \ldots \times C_{k_{\nu} \times q_{\nu}}, i \in \nu, \sigma \left( (A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \nu} \right) \not\subset D \}$$

$$r^{r}_{D}(\delta) = \inf \{ \gamma(\Delta) : \Delta := (\Delta_0, \Delta_1, \ldots, \Delta_{\nu}), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in R_{k_i \times q_i} \times \ldots \times R_{k_{\nu} \times q_{\nu}}, i \in \nu, \sigma \left( (A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \nu} \right) \not\subset D \}$$.

(b) The complex, real $D(\alpha, \gamma)$-stability radius of the system (1) with respect to affine perturbations of the form (6) is defined, respectively, by

$$r^{c}_{D}(\delta) = \inf \{ \gamma(\Delta) : \Delta := (\Delta_0, \Delta_1, \ldots, \Delta_{\nu}), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in C_{k_i \times q_i} \times \ldots \times C_{k_{\nu} \times q_{\nu}}, i \in \nu, \sigma \left( (A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \nu} \right) \not\subset D \}$$

$$r^{r}_{D}(\delta) = \inf \{ \gamma(\Delta) : \Delta := (\Delta_0, \Delta_1, \ldots, \Delta_{\nu}), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in R_{k_i \times q_i} \times \ldots \times R_{k_{\nu} \times q_{\nu}}, i \in \nu, \sigma \left( (A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \nu} \right) \not\subset D \}$$.

In particular case of $D = D(0,1)$, the problems of computing of the stability radii of the linear discrete-time systems (1) under single perturbations, affine perturbations and multi-perturbations have been done recently by ourselves (see [5], [9], [10]). We summarize here some existing results of these problems. Recall that the system (1) is positive if and only if system matrices $A_0, A_1, \ldots, A_{\nu}$ are nonnegative.

**Theorem 3.2.** [9] Suppose the linear discrete time-delay system (1) is $D(0,1)$-stable, positive and the system matrices $A_i, i \in \nu$ are subjected to affine perturbations of the form (6) where $B_{ij} \in R_{n \times n}, i \in \nu, j \in N$. Then, $r^{c}_{D}(D(0,1)) = r^{r}_{D}(D(0,1)) = \frac{1}{\rho(B)}$, where $B := \sum_{j=1}^{N} B_{0j} + \sum_{j=1}^{N} B_{1j} + \ldots + \sum_{j=1}^{N} B_{\nu j}$.

**Remark 3.3.** In the proof of Theorem 3.2, we showed that the real perturbation $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu1}, \ldots, \delta_{\nuN})) \in R^{(\nu+1)N}$, $\delta_{ij} = \frac{1}{\rho(B)}$, for $i \in \nu, j \in N$ is a minimal size destabilizing perturbation. This fact will be used in the sequel.

**Theorem 3.4.** [10] Let the linear discrete time-delay system (1) be positive, $D(0,1)$-stable. Assume that the system matrices $A_i, i \in m$ are subjected to multi-perturbations of the form (5) where $D_{ij} := D \in R_{n \times l}, E_{ij} \in R_{q_i \times n}$ for all $i \in \nu, j \in N$ or $E_{ij} := E \in R_{q_i \times l}, D_{ij} \in R_{n \times q_i}$ for all $i \in \nu, j \in N$. Then, $r^{c}_{D}(D(0,1)) = r^{r}_{D}(D(0,1)) = \frac{1}{\max ||B_{ij} P_{iN}^{-1} D_{ij}||_{i \in \nu, j \in N}}$. 
Theorem 3.5. Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the coefficient matrices $A_i, i \in \overline{\nu}$ are subjected to the multi-perturbations (5), where $D_{ij} := D \in \mathbb{R}_{+}^{n \times l}, E_{ij} \in \mathbb{R}_{+}^{q_{ij} \times n} (i \in \overline{\nu}, j \in \underline{N})$ or $D_{ij} \in \mathbb{R}_{+}^{n \times l}, E_{ij} := E \in \mathbb{R}_{+}^{q_{ij} \times n} (i \in \overline{\nu}, j \in \underline{N})$. If $\alpha \leq 0$ and $A_0, A_1, ..., A_{\nu-1}, (A_{\nu} - \alpha I_n) \in \mathbb{R}_{+}^{n \times n}$, then

$$r_{m}^{m}(D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in \underline{N}} \|(\alpha + r)^{i}E_{ij}P(\alpha + r)^{-1}D_{ij}\|}.$$ 

Proof. Assume $D_{ij} := D \in \mathbb{R}_{+}^{n \times l}, E_{ij} \in \mathbb{R}_{+}^{q_{ij} \times n} (i \in \overline{\nu}, j \in \underline{N})$. Consider the companion matrix of the polynomial matrix $P(z) := (z^{\nu+1}I_n - A_{\nu}z^{\nu} - A_{\nu-1}z^{\nu-1} - \ldots - A_0)$:

$$A_{c} := \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\
0 & 0 & I_n & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0 & I_n \\
A_0 & A_1 & \ldots & \ldots & \ldots & A_{\nu} \end{bmatrix} \in \mathbb{R}^{(\nu+1)n \times (\nu+1)n},$$

and similarly $A_{c}(\tilde{\Delta})$ for the perturbed polynomial matrix $P_{\Delta}(z) := z^{\nu+1}I_n - \sum_{i=0}^{\nu}(A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij}) z^i$, where $\tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_\nu), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in \mathbb{R}^{l \times q_i \times \ldots \times k_{iN}, i \in \overline{\nu}}$. Then the matrix $A_{c}(\tilde{\Delta})$ can be represented by the following form $A_{c}(\tilde{\Delta}) = A_{c} + \sum_{j=1}^{N} \tilde{D}_{ij} \Delta_{ij} \tilde{E}_{ij} + \sum_{j=1}^{N} \tilde{D}_{ij} \Delta_{ij} \tilde{E}_{ij} + \ldots + \sum_{j=1}^{N} \tilde{D}_{ij} \Delta_{ij} \tilde{E}_{ij},$ where

$$\tilde{D}_{ij} := D \in \mathbb{R}_{+}^{(\nu+1)n \times l}, \tilde{E}_{ij} := [0, \ldots, 0, E_{ij}] \in \mathbb{R}^{l_{ij} \times (\nu+1)n},$$

for every $i \in \overline{\nu}, j \in \underline{N}$. It follows from the equality $\det P_{\Delta}(z) = \det (zI_{(\nu+1)n} - A_{c}(\tilde{\Delta}))$ that $\sigma((A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \overline{\nu}}) = \sigma(A_{c}(\tilde{\Delta}))$. So, we get $r_{m}^{m}(D(\alpha, r)) = r_{C}(A_{c}, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i \in \overline{\nu}, j \in \underline{N}}; D(\alpha, r))$.

On the other hand, it is easy to check that

$$(zI_{(\nu+1)n} - A_{c})^{-1} \tilde{D} = \begin{bmatrix} P(z)^{-1} \\ zP(z)^{-1} \\ \vdots \\ z^{\nu}P(z)^{-1} \end{bmatrix}.$$
Therefore, \( r^n_{v}(D(\alpha, r)) = r^n_{v}(D(\alpha, r)) = \max_{i \in \overline{v}, j \in \overline{N}} \| (\alpha + r)^{i}E_{ij}P(\alpha + r)^{-1}B_{ij} \| \). The proof of the case of \( D_{ij} \in \mathbb{R}^{n \times n}_{+}, E_{ij} := E \in \mathbb{R}^{q \times n}_{+}(i \in \overline{v}, j \in \overline{N}) \), can be done by a similar way. This completes our proof. \( \square \)

We now turn to the problem of computing of the complex, real \( D \)-stability radius under affine perturbations (6). For every \( i \in \overline{v} \), let us define

\[
A^*_i := \frac{1}{r^{\nu+1-i}} \left( C_{\nu}^\nu \alpha^{\nu-i} A_{\nu} + C_{\nu-1}^{\nu-1-i} \alpha^{\nu-1-i} A_{\nu-1} + \ldots + A_i - C_{\nu+1}^{\nu+1-i} \alpha^{\nu+1-i} I_n \right),
\]

where \( C^\nu_u := \frac{u!}{(u-v)!v!} \), \( u, v \in \mathbb{N}, u \geq v \). The following theorem is an extension of Theorem 3.2 to the general case of \( D = D(\alpha, r) \).

**Theorem 3.6.** Let the linear discrete time-delay system (1) be \( D(\alpha, r) \)-stable. Suppose the system matrices \( A_i, i \in \overline{v} \) are subjected to affine perturbations (6), where \( B_{ij} \in \mathbb{R}^{n \times n}_{+}(i \in \overline{v}, j \in \overline{N}) \). If either \( \alpha \leq 0 \) and \( A_0, A_1, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_n) \in \mathbb{R}^{n \times n}_{+} \), or \( \alpha > 0 \) and \( A^*_i \in \mathbb{R}^{n \times n}_{+}, i \in \overline{v} \), then \( r^n_{v}(D(\alpha, r)) = r^n_{v}(D(\alpha, r)) = \frac{1}{\rho(P(\alpha + r)^{-1}B)} \), where \( B := \sum^\nu_{i=0} \left( \sum^\nu_{j=1} B_{ij} \right) (\alpha + r)^i \).

**Proof.** In the case of \( \alpha \leq 0 \) and \( A_0, A_1, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_n) \in \mathbb{R}^{n \times n}_{+} \), the proof is similar to that of Theorem 3.5, based on the result of Theorem 2.4(i). Then, we have \( r^n_{v}(D(\alpha, r)) = r^n_{v}(D(\alpha, r)) = \frac{1}{\rho(P(\alpha + r)^{-1}B)} \). We now assume that \( \alpha > 0 \) and \( A^*_i \in \mathbb{R}^{n \times n}_{+}, i \in \overline{v} \). Denote by \( P^*(z) := z^{\nu+1}I_n - A^*_\nu z^\nu - \ldots - A^*_0 \). Let \( s \in \mathbb{C}, |s - \alpha| \geq r \) satisfy \( \det P(s) = 0 \). Setting \( z = \frac{s - \alpha}{r} \), \( |z| \geq 1 \), by a direct computation, we have \( \det P(s) = 0 \) if and only if \( \det P^*(z) = 0 \). So the discrete time-delay system (1) is \( D(\alpha, r) \)-stable if and only if the following discrete time-delay system

\[
x(k + 1) = A^*_\nu x(k) + A^*_\nu x(k - 1) + \ldots + A^*_0 x(k - \nu), \quad k \in \mathbb{N}, k \geq \nu,
\]

is \( D(0, 1) \)-stable. Similarly, the perturbed system

\[
x(k + 1) = (A_0 + \sum^\nu_{j=1} \delta_{0j} B_{0j}) x(k) + \ldots + (A_0 + \sum^\nu_{j=1} \delta_{0j} B_{0j}) x(k - \nu), \quad k \in \mathbb{N}, k \geq \nu,
\]

is \( D(\alpha, r) \)-stable if and only if the following discrete time-delay system is \( D(0, 1) \)-stable

\[
x(k + 1) = (A_0^* + B^*_0) x(k) + \ldots + (A_0^* + B^*_0) x(k - \nu), \quad k \in \mathbb{N}, k \geq \nu.
\]

Here, \( B^*_i := (\sum^\nu_{j=1} \delta_{uj}(1/(r^{\nu+1-i}C_{\nu}^\nu \alpha^{\nu-i} B_{uj}) + \sum^\nu_{j=1} \delta_{uj}(1/(r^{\nu+1-i}C_{\nu-1}^{\nu-1-i} \alpha^{\nu-1-i} B_{(\nu-1)j}) + \ldots + \sum^\nu_{j=1} \delta_{uj}(1/(r^{\nu+1-i} B_{ij})) \), \( i \in \overline{v} \). Since \( B_{ij} \in \mathbb{R}^{n \times n}_{+}, (i \in \overline{v}, j \in \overline{N}) \), we have

\[
\frac{1}{r^{\nu+1-i}} C_{\nu}^\nu \alpha^{\nu-i} B_{uj}, \quad \frac{1}{r^{\nu+1-i}} C_{\nu-1}^{\nu-1-i} B_{(\nu-1)j}, \ldots, \frac{1}{r^{\nu+1-i}} B_{ij} \in \mathbb{R}^{n \times n}_{+}, \quad i \in \overline{v}, j \in \overline{N}.
\]
It follows from Theorem 3.2 that the system (10) is $D(0,1)$-stable for every $\delta$ satisfying $\max_{i \in \overline{\nu}, j \in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(\rho(1)^{-1}G)}$, where

$$G := \sum_{i=0}^{\nu} \left( \sum_{j=1}^{N} \frac{1}{\nu+1-i} C_{\nu-i}^\nu \alpha^{\nu-i} B_{\nu j} + \sum_{j=1}^{N} \frac{1}{\nu+1-i} C_{\nu-1}^{\nu-1-i} \alpha^{\nu-1-i} B_{(\nu-1)j} + \ldots + \sum_{j=1}^{N} \frac{1}{\nu+1-i} B_{ij} \right).$$

Hence, the perturbed system (9) is $D(\alpha, r)$-stable for every complex perturbation $\delta$ such that $\max_{i \in \overline{\nu}, j \in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(\rho(1)^{-1}G)}$. By the definition of the complex $D(\alpha, r)$-stability radius of the system (1) under affine perturbations of the form (6), we get $r_{\mathbb{C}}^a(D(\alpha, r)) \geq \frac{1}{\rho(\rho(1)^{-1}G)}$. On the other hand, taking Remark 3.3 into account, the system (10) is not $D(0,1)$-stable if $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu 1}, \ldots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}$. Then the perturbed system (9) is not $D(\alpha, r)$-stable if

$$\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu 1}, \ldots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}; \delta_{ij} = \frac{1}{\rho(\rho(1)^{-1}G)} (i \in \overline{\nu}, j \in \underline{N}).$$

We derive that $r_{\mathbb{R}}^a(D(\alpha, r)) \leq \frac{1}{\rho(\rho(1)^{-1}G)}$. So we get the following inequalities

$$\frac{1}{\rho(\rho(1)^{-1}G)} \leq r_{\mathbb{C}}^a(D(\alpha, r)) \leq r_{\mathbb{R}}^a(D(\alpha, r)) \leq \frac{1}{\rho(\rho(1)^{-1}G)}.$$

Therefore $r_{\mathbb{C}}^a(D(\alpha, r)) = r_{\mathbb{R}}^a(D(\alpha, r)) = \frac{1}{\rho(\rho(1)^{-1}G)}$. Finally, by a direct computation, we get $\rho(1)^{-1}G = P(\alpha + r)^{-1}B$. This completes our proof. 

\section*{References}


