Robust $\mathcal{D}$-stability of linear difference equations

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Abstract

We study robustness of $\mathcal{D}$-stability of linear difference equations under multi-perturbation and affine perturbation of coefficient matrices via the concept of $\mathcal{D}$-stability radius. Some explicit formulae are derived for these $\mathcal{D}$-stability radii. The obtained results include the corresponding ones established earlier in [3], [4], [9], [10] as particular cases.

1 Introduction and Preliminaries

Let $\mathcal{D} := D(\alpha, r)$ be a open disk centered at $(\alpha, 0)$ with radius $r$ in the complex plane. A linear discrete-time (time-invariant) system is called $\mathcal{D}$-stable if its characteristic equation has only roots in $\mathcal{D}$. In this paper, we study the robustness of $\mathcal{D}$-stability of linear discrete-time systems of the form

$$x(k + 1) = A_{\nu}x(k) + A_{\nu-1}x(k-1) + \cdots + A_{0}x(k-\nu), \quad k \in \mathbb{N}, k \geq \nu \quad (1)$$

under parameter perturbation of the coefficient matrices via the concept of $\mathcal{D}$-stability radius. It is important to note that the problems of computing of $D(0, 1)$-stability radii (or simpler, stability radii) of linear discrete-time systems have been studied during the last twenty years by many mathematical researchers, see e.g. [2]-[5], [9]-[11]. In particular, the problems of computing of stability radii of linear discrete-time systems of the form (1) under single perturbations, affine perturbations

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and multi-perturbations have just been studied in the recent time, see [5], [9], [10]. It is also worth noticing that (robust) \( \mathcal{D} \)-stability problems of linear discrete-time systems have been received much attention from researchers for a long time. Some sufficient conditions for the (robust) \( \mathcal{D} \)-stability of the system (1) under parameter perturbations were proposed in [1], [6], [8], [13]-[15]. However, to the best of our knowledge, there is not any formula for the \( \mathcal{D} \)-stability radii of the system (1) under multi-perturbations or affine-perturbations in the case of \( \mathcal{D} = D(\alpha, r) \). In the present paper, using our recent new results on the problems of computing of stability radii (see e.g. [10]), we can compute the \( D(\alpha, r) \)-stability radii of the system (1) under multi-perturbations and affine perturbations. The obtained results are the extensions of the corresponding results of [3], [4], [9], [10].

Let \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{R} \) and \( n, l, q \) be positive integers. Inequalities between real matrices or vectors will be understood componentwise. The set of all nonnegative \( l \times q \)-matrices is denoted by \( \mathbb{R}^{l \times q}_{+} \). If \( P \in \mathbb{K}^{l \times q} \) we define \( |P| = (|p_{ij}|) \). For any matrix \( A \in \mathbb{K}^{n \times n} \) the spectral radius of \( A \) is denoted by \( \rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\} \), where \( \sigma(A) \) is the set of all eigenvalues of \( A \). A norm \( || \cdot || \) on \( \mathbb{K}^{n} \) is said to be monotonic if \( |x| \leq |y| \) implies \( ||x|| \leq ||y|| \) for all \( x, y \in \mathbb{K}^{n} \). Every \( p \)-norm on \( \mathbb{K}^{n} \), \( 1 \leq p \leq \infty \), is monotonic. Throughout the paper, the norm \( ||M|| \) of a matrix \( M \in \mathbb{K}^{l \times q} \) is always understood as the operator norm defined by \( ||M|| = \max_{||y||=1} ||My|| \), where \( \mathbb{K}^{q} \) and \( \mathbb{K}^{l} \) are provided with some monotonic vector norms.

\section{\( \mathcal{D} \)-stability radii of linear discrete-time systems}

Let \( \mathcal{D} = D(\alpha, r) \) be the open disk centered at \( (\alpha, 0) \) with radius \( r \) in the complex plane. Consider a dynamical system described by a linear discrete-time system of the form

\[ x(k+1) = Ax(k), \quad k \in \mathbb{N}, \]

where \( A \in \mathbb{R}^{n \times n} \) is a given matrix. The system (2) is called \( \mathcal{D} \)-stable if \( \sigma(A) \subset \mathcal{D} \).

It is important to note that, the system (2) is asymptotically stable in the Lyapunov's sense in the case of \( \mathcal{D} = D(0, 1) \) and is strong stable in the case of \( \mathcal{D} = D(0, r), 0 < r < 1 \). We now assume that the system (2) is \( \mathcal{D} \)-stable and the system matrix \( A \) is subjected to one of the following perturbation types

\begin{align*}
A &\rightarrow A + \sum_{i=1}^{N} D_i \Delta_i E_i, \quad \text{(multi-perturbation),} \\
A &\rightarrow A + \sum_{i=1}^{N} \delta_i B_i, \quad \text{(affine perturbation).}
\end{align*}

\section{Extensions of the monograph}

In [5], we have proposed the following extensions of the monograph. Theorem 1.6 in [5] can be extended to the following theorem:

\begin{theorem}
Let \( A \) be a linear system matrix and \( \mathcal{D} \) be a disk centered at \( (\alpha, 0) \) with radius \( r < 1 \). If there exists a positive definite matrix \( P \in \mathbb{R}^{n \times n} \) such that

\[ AP + PA^\prime < -\rho(A) P \]

then the system (2) is \( \mathcal{D} \)-stable.
\end{theorem}
Here $D_i \in \mathbb{R}^{n \times l_i}, E_i \in \mathbb{R}^{q_i \times n}, B_i \in \mathbb{R}^{n \times n}, i \in \mathcal{N} := \{1, 2, ..., N\}$ are given matrices defining the structure of perturbations and $\Delta_i \in \mathbb{K}^{l_i \times q_i}, \delta_i \in \mathbb{K}$ ($i \in \mathcal{N}$) unknown disturbance matrices and scalars, respectively. For class of multi-perturbations of the form (3), we always assume that the linear space $\Delta_k = \mathbb{K}^{l_1 \times q_1} \times \cdots \times \mathbb{K}^{l_N \times q_N}$ of all perturbation families $\Delta = (\Delta_1, ..., \Delta_N)$, with $\Delta_i \in \mathbb{K}^{l_i \times q_i}$, is endowed with the norm

$$
\gamma(\Delta) = \gamma(\Delta_1, ..., \Delta_N) = \sum_{i=1}^{N} \| \Delta_i \|,
$$

where the norms $\| \Delta_i \|$ are operator norms on $\mathbb{K}^{l_i \times q_i}$, induced by given monotonic vector norms on the spaces $\mathbb{K}^{l_i}, \mathbb{K}^{q_i}, i \in \mathcal{N}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).

**Definition 2.1.** Let the linear discrete time system (2) be $D-$stable.

(a) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to multi-perturbations of the form (3) are defined, respectively, by

$$
r_C(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf \{ \gamma(\Delta) : \Delta \in \Delta_C, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D \},
$$

$$
r_R(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf \{ \gamma(\Delta) : \Delta \in \Delta_R, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D \}.
$$

(b) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to affine perturbations of the form (4) are defined, respectively, by

$$
r_C(A, (B_i)_{i \in \mathcal{N}}; D) = \inf \{ \max_{i \in \mathcal{N}} | \delta_i | : \delta_i \in \mathbb{C}, \delta_i \in \mathbb{R}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D \},
$$

$$
r_R(A, (B_i)_{i \in \mathcal{N}}; D) = \inf \{ \max_{i \in \mathcal{N}} | \delta_i | : \delta_i \in \mathbb{R}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D \}.
$$

As noted in Introduction, the problems of computing of the stability radii (i.e. $D(0, 1)$-stability radii) of the system (2) have been studied during the last twenty years and have got the full results, see e.g. [3], [12], [4], [10]. We list here the interesting results for the class of positive systems (i.e. $A$ is a nonnegative matrix).

**Theorem 2.2.** [4] Let the system (2) be $D(0, 1)-$stable and positive. Suppose the system matrix $A$ is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}_{+}^{n \times n}, i \in \mathcal{N}$. Then

$$
r_C(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = r_R(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\rho(\sum_{i=1}^{N} B_i (I_n - A)^{-1})}.
$$

**Theorem 2.3.** [10] Let the system (2) be $D(0, 1)-$stable and positive. Assume that the matrix $A$ is subjected to parameter multi-perturbations (3). If $D_i = D \in \mathbb{R}_{+}^{n \times l_i}$ and $E_i \in \mathbb{R}_{+}^{q_i \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}_{+}^{q \times n}$ and $D_i \in \mathbb{R}_{+}^{n \times l_i}$ for every $i \in \mathcal{N}$, then $r_C(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = r_R(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\max_{i \in \mathcal{N}} \| E_i (I_n - A)^{-1} D_i \|}$.

The following theorem extends the above results to the general case of $D = D(\alpha, r)$.

**Theorem 2.4.** Let the system (2) be $D(\alpha, r)-$stable and $A \geq \alpha I_n$. (i) If the matrix $A$ is subjected to multi-perturbations (3), where $D_i = D \in \mathbb{R}_{+}^{n \times l_i}$ and $E_i \in \mathbb{R}_{+}^{q_i \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}_{+}^{q \times n}$, and $D_i \in \mathbb{R}_{+}^{n \times l_i}$ for every $i \in \mathcal{N}$, then
\[ r_C(A, (D_i)_{i \in \mathbb{N}}; D(\alpha, r)) = r_R(A, (D_i)_{i \in \mathbb{N}}; D(\alpha, r)) = \max_{i \in \mathbb{N}} \|E_i((\alpha+r)I_{n}-A)^{-1}D_i\| \]  

(ii) If the matrix \( A \) is subjected to affine-perturbations (4), where \( B_i \in \mathbb{R}^{n \times n}, i \in \mathbb{N}, \) then \( r_C(A, (B_i)_{i \in \mathbb{N}}; D(\alpha, r)) = r_R(A, (B_i)_{i \in \mathbb{N}}; D(\alpha, r)) = \frac{1}{\rho(\sum_{i=1}^{n} B_i((\alpha+r)I_{n}-A)^{-1})}. \]

Proof. The proof is based on Theorems 2.2, 2.3 and the fact that the system \( x(k+1) = Ax(k), k \in \mathbb{N} \) is \( D(\alpha, r)- \)stable if and only if the system \( x(k+1) = (A - \alpha I_n)x(k), k \in \mathbb{N} \) is \( D(0,1)- \)stable. For sake of space, it is omitted here. \( \square \)

The following is an extension of the main result of [7].

Corollary 2.5. Let \( P(z) := I_n z^{\nu+1} - A_\nu z^\nu - \ldots - A_0 \) be a given polynomial matrix. Assume that \( |\alpha| < r, |\alpha| + r \leq 1 \) and \( ||A_0 A_1 \ldots A_\nu||_\infty < (r - |\alpha|)^{\nu+1}. \) Then all the roots of the equation \( \det P(z) = 0 \) lie inside the disk \( D(\alpha, r). \)

3 \( D \)-stability radii of linear discrete time-delay systems

Consider a dynamical system described by a linear discrete time-delay system of the form (1), where \( A_i \in \mathbb{R}^{n \times n}, i \in \mathbb{D} := \{0, 1, 2, \ldots, \nu\}, \) are given matrices. For the linear discrete time-delay system (1), we consider the stable region \( \mathcal{D} = D(\alpha, r), |\alpha| < r, r + |\alpha| \leq 1, \) see e.g. [8], [13], [14]. We associate the system (1) with the following polynomial matrix \( P(z) := (z^{\nu+1}I_n - A_\nu z^\nu - A_{\nu-1} z^{\nu-1} - \ldots - A_0), z \in \mathbb{C}. \) Denote by \( \sigma((A_i)_{i \in \mathbb{D}}) := \{z \in \mathbb{C} : \det P(z) = 0\} \) the set of all roots of the characteristic equation of the linear discrete time-delay system (1). Then \( \sigma((A_i)_{i \in \mathbb{D}}) \) is called the spectral set of the linear discrete time-delay system (1) and \( \rho((A_i)_{i \in \mathbb{D}}) := \max \{|s| : s \in \sigma((A_i)_{i \in \mathbb{D}})\} \) is called spectral radius of the linear discrete time-delay system (1). Recall that the system (1) is said to be \( D \)-stable if \( \sigma((A_i)_{i \in \mathbb{D}}) \subset \mathcal{D}. \) We now assume that the system (1) is \( D \)-stable and the coefficient matrices \( A_i, i \in \mathbb{D} \) are subjected to parameter perturbations

\[
A_i \to A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij}, \quad \text{(multi-perturbation)}
\]

\[
A_i \to A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij}, \quad \text{(affine-perturbation)}
\]

where \( D_{ij} \in \mathbb{R}^{n \times n}, E_{ij} \in \mathbb{R}^{n \times n}, (i \in \mathbb{D}, j \in \mathbb{N} := \{1, 2, \ldots, N\}) \); \( B_{ij} \in \mathbb{R}^{n \times n}, (i \in \mathbb{D}, j \in \mathbb{N}) \) are given matrices defining the structure of perturbations and \( \Delta_{ij} \in \mathbb{K}^{n \times n}, (i \in \mathbb{D}, j \in \mathbb{N}) \); \( \delta_{ij} \in \mathbb{K}, (i \in \mathbb{D}, j \in \mathbb{N}) \) are perturbation matrices, perturbation scalars, respectively. For the class of multi-perturbations of the form
We define \( \tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_v) \), where \( \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in K^{l_{i1} \times q_{i1}} \times \ldots \times K^{l_{iN} \times q_{iN}}, i \in \mathcal{V} \). Then the size of each perturbation \( \tilde{\Delta} \) is measured by \( \gamma(\tilde{\Delta}) := \sum_{i=0}^{v} \sum_{j=1}^{N} ||\Delta_{ij}|| \). With the class of affine perturbations of the form (6), we denote \( \delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{v1}, \ldots, \delta_{vN})) \in K^{vN} \) and the size of each perturbation \( \delta \) is measured by \( \gamma(\delta) = \max_{i \in \mathcal{V}, j \in \mathcal{N}} |\delta_{ij}| \).

**Definition 3.1.** Let the linear discrete time-delay system (1) be \( \mathcal{D} \)-stable.

(a) The complex, real \( D(\alpha, r) \)-stability radius of the system (1) with respect to multi-perturbations of the form (5) is defined, respectively, by

\[
\begin{align*}
\rho_c^c(\mathcal{D}) &= \inf \{ \gamma(\tilde{\Delta}) : \tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_v), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in C^{l_{i1} \times q_{i1}} \times \ldots \times C^{l_{iN} \times q_{iN}}, i \in \mathcal{V}, \sigma((A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \mathcal{V}}) \not\subset \mathcal{D} \}, \\
\rho_r^r(\mathcal{D}) &= \inf \{ \gamma(\Delta) : \Delta := (\Delta_0, \Delta_1, \ldots, \Delta_v), \Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in R^{l_{i1} \times q_{i1}} \times \ldots \times R^{l_{iN} \times q_{iN}}, i \in \mathcal{V}, \sigma((A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \mathcal{V}}) \not\subset \mathcal{D} \}. 
\end{align*}
\]

(b) The complex, real \( D(\alpha, r) \)-stability radius of the system (1) with respect to affine perturbations of the form (6) is defined, respectively, by

\[
\begin{align*}
\rho_c^a(\mathcal{D}) &= \inf \{ \gamma(\delta) : \delta \in C^{(v+1)N}, \sigma((A_i + \sum_{j=1}^{N} \Delta_{ij} B_{ij})_{i \in \mathcal{V}}) \not\subset \mathcal{D} \}, \\
\rho_r^a(\mathcal{D}) &= \inf \{ \gamma(\delta) : \delta \in R^{(v+1)N}, \sigma((A_i + \sum_{j=1}^{N} \Delta_{ij} B_{ij})_{i \in \mathcal{V}}) \not\subset \mathcal{D} \}.
\end{align*}
\]

In particular case of \( \mathcal{D} = D(0,1) \), the problems of computing of the stability radii of the linear discrete-time systems (1) under single perturbations, affine perturbations and multi-perturbations have been done recently by ourselves (see [5], [9], [10]). We summarize here some existing results of these problems. Recall that the system (1) is positive if and only if system matrices \( A_0, A_1, \ldots, A_v \) are nonnegative.

**Theorem 3.2.** [9] Suppose the linear discrete time-delay system (1) is \( D(0,1) \)-stable, positive and the system matrices \( A_i, i \in \mathcal{V} \) are subjected to affine perturbations of the form (6) where \( B_{ij} \in R^{l_{i1} \times q_{j1}}, i \in \mathcal{V}, j \in \mathcal{N} \). Then, \( \rho_c^a(D(0,1)) = \rho_r^a(D(0,1)) = \frac{1}{\rho(P(1)^{-1}B)} \), where \( B := \sum_{i=1}^{N} B_{0i} + \sum_{i=1}^{N} B_{1i} + \ldots + \sum_{i=1}^{N} B_{vi} \).

**Remark 3.3.** In the proof of Theorem 3.2, we showed that the real perturbation \( \delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{v1}, \ldots, \delta_{vN})) \in R^{(v+1)N}; \delta_{ij} = \frac{1}{\rho(P(1)^{-1}B)}, (i \in \mathcal{V}, j \in \mathcal{N}) \) is a minimal size destabilizing perturbation. This fact will be used in the sequel.

**Theorem 3.4.** [10] Let the linear discrete time-delay system (1) be positive, \( D(0,1) \)-stable. Assume that the system matrices \( A_i, i \in \mathcal{V} \) are subjected to the multi-perturbations of the form (5) where \( D_{ij} := D \in R^{l_{i1} \times q_{j1}}, E_{ij} \in R^{q_{j1} \times q_{j1}} \) for all \( i \in \mathcal{V}, j \in \mathcal{N} \) or \( E_{ij} := E \in R^{q_{j1} \times q_{j1}}, D_{ij} \in R^{l_{i1} \times q_{j1}} \) for all \( i \in \mathcal{V}, j \in \mathcal{N} \). Then, \( \rho_c^m(D(0,1)) = \rho_r^m(D(0,1)) = \max \{|E_{ij} P(1)^{-1} D_{ij}| : i \in \mathcal{V}, j \in \mathcal{N}\} \).
Theorem 3.5. Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the coefficient matrices $A_i, i \in \overline{\nu}$ are subjected to the multi-perturbations (5), where $D_{ij} := D \in \mathbb{R}_{+}^{q_{x}l_{j}}, E_{ij} \in \mathbb{R}_{+}^{q_{y}x_{n}}(i \in \overline{\nu}, j \in \overline{N})$ or $D_{ij} \in \mathbb{R}_{+}^{q_{x}l_{j}}, E_{ij} := E \in \mathbb{R}_{+}^{q_{y}x_{n}}(i \in \overline{\nu}, j \in \overline{N})$. If $\alpha \leq 0$ and $A_0, A_1, ..., A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}_{+}^{n_{x}n}$, then

$$r^{m}_{C}(D(\alpha, r)) = r^{m}_{R}(D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in \overline{N}} \|(\alpha + r)^{i}E_{ij}P(\alpha + r)^{-1}D_{ij}\|}.$$ 

Proof. Assume $D_{ij} := D \in \mathbb{R}_{+}^{q_{x}l_{j}}, E_{ij} \in \mathbb{R}_{+}^{q_{y}x_{n}}(i \in \overline{\nu}, j \in \overline{N})$. Consider the companion matrix of the polynomial matrix $P(z) = (z^{\nu+1}I_{n} - A_{\nu}z^{\nu} - A_{\nu-1}z^{\nu-1} - \ldots - A_{0})$:

$$A_{c} := \begin{bmatrix} 0 & I_{n} & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_{n} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & I_{n} \\ A_{0} & A_{1} & \ldots & \ldots & \ldots & A_{\nu} \end{bmatrix} \in \mathbb{R}^{(\nu+1)n \times (\nu+1)n},$$

and similarly $A_{c}(\Delta)$ for the perturbed polynomial matrix $P_{\Delta}(z) := z^{\nu+1}I_{n} - \sum_{i=0}^{\nu} (A_{i} + \sum_{j=1}^{\nu} D_{ij} \Delta_{ij} E_{ij})z^{i}$, where $\Delta := (\Delta_{0}, \Delta_{1}, \ldots, \Delta_{\nu})$, $\Delta_{i} := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in \mathbb{R}^{l_{i} \times q_{i}}, i \in \overline{\nu}$. Then the matrix $A_{c}(\Delta)$ can be represented by the following form $A_{c}(\Delta) = A_{c} + \sum_{j=1}^{\nu} \tilde{D}_{0j}\tilde{E}_{0j} + \sum_{j=1}^{\nu} \tilde{D}_{\nu j}\tilde{E}_{\nu j} + \ldots + \sum_{j=1}^{\nu} \tilde{D}_{\nu j}\tilde{E}_{\nu j}$, where

$$\tilde{D}_{ij} = \tilde{D} = [0, ..., 0, D^{T}] \in (\nu+1)n \times l, \tilde{E}_{0j} := [E_{0j}, 0, \ldots, 0] \in \mathbb{R}^{q_{ij} \times (\nu+1)n},$$

for every $i \in \overline{\nu}, j \in \overline{N}$. It follows from the equality $\det P_{\Delta}(z) = \det ((zI_{(\nu+1)n} - A_{c}(\Delta)))$ that $\sigma((A_{i} + \sum_{j=1}^{\nu} D_{ij} \Delta_{ij} E_{ij})_{i \in \overline{\nu}}) = \sigma(A_{c}(\Delta))$. So, we get $r^{m}_{C}(D(\alpha, r)) = r_{C}(A_{c}, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i \in \overline{\nu}, j \in \overline{N}}; D(\alpha, r)) = r_{C}(A_{c}, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i \in \overline{\nu}, j \in \overline{N}}; D(\alpha, r))$. By the assumption $\alpha \leq 0$, $A_{0}, A_{1}, ..., A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}_{+}^{n_{x}n}, D_{ij} \in \mathbb{R}_{+}^{q_{x}l_{j}}, E_{ij} \in \mathbb{R}_{+}^{q_{y}x_{n}}(i \in \overline{\nu}, j \in \overline{N})$, we have $A_{c} \geq \alpha I_{(\nu+1)n}$ and $\tilde{D} \in \mathbb{R}_{+}^{(\nu+1)n \times l}, \tilde{E}_{ij} \in \mathbb{R}_{+}^{q_{ij} \times (\nu+1)n}(i \in \overline{\nu}, j \in \overline{N})$. Hence, from Theorem 2.4, we get $r_{C}(A_{c}, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i \in \overline{\nu}, j \in \overline{N}}; D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in \overline{N}} \|(\tilde{E}_{ij}(\alpha+r)I_{(\nu+1)n} - A_{c})^{-1}D_{ij}\|}$. On the other hand, it is easy to check that

$$(zI_{(\nu+1)n} - A_{c})^{-1} \tilde{D} = \begin{pmatrix} P(z)^{-1} \\ zP(z)^{-1} \\ \vdots \\ z^{\nu}P(z)^{-1} \end{pmatrix}.$$
Therefore, $r_{c}^{n}(D(\alpha, r)) = r_{\mathbb{R}}^{n}(D(\alpha, r)) = \max_{i \in \mathbb{R}, j \in \mathbb{N}} \| (\alpha+r)^{j} \hat{A}_{i} \|$. The proof of the case of $D_{ij} \in \mathbb{R}_{+}^{n \times n}$, $E_{ij} := E \in \mathbb{R}_{+}^{q \times n} (i \in \mathbb{N}, j \in \mathbb{N})$, can be done by a similar way. This completes our proof.

We now turn to the problem of computing of the complex, real $D$-stability radius under affine perturbations (6). For every $i \in \mathbb{N}$, let us define

$$A_{i}^{*} := \frac{1}{r^{\nu+1-i}} \left( C_{\nu}^{v-i} A_{\nu} + C_{\nu-1}^{v-1-i} A_{\nu-1} + \ldots + A_{i} - C_{\nu+1}^{v+1-i} A_{n} \right),$$

(7)

where $C_{\nu}^{v} := \frac{u^{v}}{v!}$, $u, v \in \mathbb{N}, u \geq v$. The following theorem is an extension of Theorem 3.2 to the general case of $D = D(\alpha, r)$.

**Theorem 3.6.** Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the system matrices $A_{i}, i \in \mathbb{N}$ are subjected to affine perturbations (6), where $B_{ij} \in \mathbb{R}_{+}^{n \times n}$ ($i \in \mathbb{N}, j \in \mathbb{N}$). If either $\alpha \leq 0$ and $A_{0}, A_{1}, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}_{+}^{n \times n}$, or $\alpha > 0$ and $A_{i}^{*} \in \mathbb{R}_{+}^{n \times n}, i \in \mathbb{N}$, then $r_{c}^{n}(D(\alpha, r)) = r_{\mathbb{R}}^{n}(D(\alpha, r)) = \frac{1}{\rho(P(\alpha+r)^{-1}B)}$, where $B := \sum_{i=0}^{\nu} \left( \sum_{j=1}^{N} B_{ij} \right) (\alpha+r)^{i}$.

**Proof.** In the case of $\alpha \leq 0$ and $A_{0}, A_{1}, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}_{+}^{n \times n}$, the proof is similar to that of Theorem 3.5, based on the result of Theorem 2.4(i). Then, we have $r_{c}^{n}(D(\alpha, r)) = r_{\mathbb{R}}^{n}(D(\alpha, r)) = \frac{1}{\rho(P(\alpha+r)^{-1}B)}$. We now assume that $\alpha > 0$ and $A_{i}^{*} \in \mathbb{R}_{+}^{n \times n}, i \in \mathbb{N}$. Denote by $P^{*}(z) := z^{\nu+1} I_{n} - A_{\nu}^{*} z^{\nu} - \ldots - A_{0}^{*}$. Let $s \in \mathbb{C}$, $|s - \alpha| \geq r$ satisfy $\det P(s) = 0$. Setting $z = \frac{s - \alpha}{r}$, $|z| \geq 1$, by a direct computation, we have $\det P(s) = 0$ if and only if $\det P^{*}(z) = 0$. So the discrete time-delay system (1) is $D(\alpha, r)$-stable if and only if the following discrete time-delay system

$$x(k+1) = A_{\nu}^{*} x(k) + A_{\nu-1}^{*} x(k-1) + \ldots + A_{0}^{*} x(k-\nu), \quad k \in \mathbb{N}, \quad k \geq \nu,$$

(8)

is $D(0,1)$-stable. Similarly, the perturbed system

$$x(k+1) = (A_{\nu} + \sum_{j=1}^{N} B_{\nu j}) x(k) + \ldots + (A_{0} + \sum_{j=1}^{n} \delta_{0j} B_{0j}) x(k-\nu), \quad k \in \mathbb{N}, \quad k \geq \nu,$$

(9)

is $D(\alpha, r)$-stable if and only if the following discrete time-delay system is $D(0,1)$-stable

$$x(k+1) = (A_{\nu}^{*} + B_{\nu}^{*}) x(k) + \ldots + (A_{0}^{*} + B_{0}^{*}) x(k-\nu), \quad k \in \mathbb{N}, \quad k \geq \nu.$$

(10)

Here, $B_{ij}^{*} := \left( \sum_{j=1}^{N} \delta_{ij} (\frac{1}{r^{\nu+1-i}} C_{\nu}^{v-i} B_{\nu j} + \sum_{j=1}^{n} \delta_{0j} B_{0j} (\frac{1}{r^{\nu+1-i}} C_{\nu-1}^{v-i} A_{\nu-1} + \ldots + \sum_{j=1}^{N} \delta_{ij} A_{\nu} (\frac{1}{r^{\nu+1-i}})) B_{ij} \right), i \in \mathbb{N}$. Since $B_{ij} \in \mathbb{R}_{+}^{n \times n}, (i \in \mathbb{N}, j \in \mathbb{N})$, we have

$$\frac{1}{r^{\nu+1-i}} C_{\nu}^{v-i} A_{\nu} B_{\nu j} + \frac{1}{r^{\nu+1-i}} C_{\nu-1}^{v-1-i} A_{\nu-1} B_{(\nu-1)j} + \ldots + \frac{1}{r^{\nu+1-i}} B_{ij} \in \mathbb{R}_{+}^{n \times n}, \quad i \in \mathbb{N}, j \in \mathbb{N}.$$
It follows from Theorem 3.2 that the system (10) is $D(0, 1)$-stable for every $\delta$ satisfying $\max_{i \in \overline{\nu}, j \in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(P^*(1)^{-1}G)}$, where

$$G := \sum_{i=0}^{\nu} \left( \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} C_{\nu}^{\nu-i} \alpha^{\nu-i} B_{\nu j} + \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} C_{\nu-1}^{\nu-1-i} \alpha^{\nu-1-i} B_{(\nu-1)j} + \cdots + \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} B_{ij} \right).$$ (11)

Therefore

$$r_{\mathbb{C}}^a(D(\alpha, r)) = \frac{1}{\rho(P^*(1)^{-1}G)}.$$

Finally, by a direct computation, we get $P^*(1)^{-1}G = P(\alpha+r)^{-1}B$. This completes our proof. $\square$

References


