Title
Robust $\mathcal{D}$-stability of linear difference equations (Dynamics of functional equations and numerical simulation)

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Abstract

We study robustness of $\mathcal{D}$-stability of linear difference equations under multi-perturbation and affine perturbation of coefficient matrices via the concept of $\mathcal{D}$-stability radius. Some explicit formulae are derived for these $\mathcal{D}$-stability radii. The obtained results include the corresponding ones established earlier in [3], [4], [9], [10] as particular cases.

1 Introduction and Preliminaries

Let $\mathcal{D} := D(\alpha, r)$ be a open disk centered at $(\alpha, 0)$ with radius $r$ in the complex plane. A linear discrete-time (time-invariant) system is called $\mathcal{D}$-stable if its characteristic equation has only roots in $\mathcal{D}$. In this paper, we study the robustness of $\mathcal{D}$-stability of linear discrete-time systems of the form

$$x(k + 1) = A_\nu x(k) + A_{\nu-1}x(k-1) + \cdots + A_0x(k-\nu), \quad k \in \mathbb{N}, k \geq \nu$$

(1)

under parameter perturbation of the coefficient matrices via the concept of $\mathcal{D}$-stability radius. It is important to note that the problems of computing of $D(0,1)$-stability radii (or simpler, stability radii) of linear discrete-time systems have been studied during the last twenty years by many mathematical researchers, see e.g. [2]-[5], [9]-[11]. In particular, the problems of computing of stability radii of linear discrete-time systems of the form (1) under single perturbations, affine perturbations

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and multi-perturbations have just been studied in the recent time, see [5], [9], [10]. It is also worth noticing that (robust) $\mathcal{D}$-stability problems of linear discrete-time systems have been received much attention from researchers for a long time. Some sufficient conditions for the (robust) $\mathcal{D}$-stability of the system (1) under parameter perturbations were proposed in [1], [6], [8], [13]-[15]. However, to the best of our knowledge, there is not any formula for the $\mathcal{D}$-stability radii of the system (1) under multi-perturbations or affine-perturbations in the case of $\mathcal{D} = D(\alpha, r)$. In the present paper, using our recent new results on the problems of computing of stability radii (see e.g. [10]), we can compute the $D(\alpha, r)$-stability radii of the system (1) under multi-perturbations and affine perturbations. The obtained results are the extensions of the corresponding results of [3], [4], [9], [10].

Let $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ and $n, l, q$ be positive integers. Inequalities between real matrices or vectors will be understood componentwise. The set of all nonnegative $l \times q$-matrices is denoted by $\mathbb{K}_{+}^{l \times q}$. If $P \in \mathbb{K}_{+}^{l \times q}$ we define $|P| = (|p_{ij}|)$. For any matrix $A \in \mathbb{K}^{n \times n}$ the spectral radius of $A$ is denoted by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, where $\sigma(A)$ is the set of all eigenvalues of $A$. A norm $\| \cdot \|$ on $\mathbb{K}^{n}$ is said to be monotonic if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for all $x, y \in \mathbb{K}^{n}$. Every $p$-norm on $\mathbb{K}^{n}$, $1 \leq p \leq \infty$, is monotonic. Throughout the paper, the norm $\|M\|$ of a matrix $M \in \mathbb{K}^{l \times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1} \|My\|$, where $\mathbb{K}^{q}$ and $\mathbb{K}^{l}$ are provided with some monotonic vector norms.

## 2 $\mathcal{D}$-stability radii of linear discrete-time systems

Let $\mathcal{D} = D(\alpha, r)$ be the open disk centered at $(\alpha, 0)$ with radius $r$ in the complex plane. Consider a dynamical system described by a linear discrete-time system of the form

$$x(k+1) = Ax(k), \quad k \in \mathbb{N},$$

where $A \in \mathbb{R}^{n \times n}$ is a given matrix. The system (2) is called $\mathcal{D}$-stable if $\sigma(A) \subset \mathcal{D}$.

It is important to note that, the system (2) is asymptotically stable in the Lyapunov’s sense in the case of $\mathcal{D} = D(0, 1)$ and is strong stable in the case of $\mathcal{D} = D(0, r), 0 < r < 1$. We now assume that the system (2) is $\mathcal{D}$-stable and the system matrix $A$ is subjected to one of the following perturbation types

$$A \rightarrow A + \sum_{i=1}^{N} D_{i} \Delta_{i} E_{i}, \quad \text{(multi-perturbation)},$$

$$A \rightarrow A + \sum_{i=1}^{N} \delta_{i} B_{i}, \quad \text{(affine perturbation)}.$$
Here $D_i \in \mathbb{R}^{n \times l_i}, E_i \in \mathbb{R}^{q_i \times n}, B_i \in \mathbb{R}^{q_i \times n}, i \in \mathcal{N} := \{1, 2, \ldots, N\}$ are given matrices defining the structure of perturbations and $\Delta_i \in \mathbb{K}^{l_i \times q_i}, \delta_i \in \mathbb{K}$ ($i \in \mathcal{N}$) unknown disturbance matrices and scalars, respectively. For class of multi-perturbations of the form (3), we always assume that the linear space $\Delta_{\mathbb{K}} = \mathbb{K}^{l_1 \times q_1} \times \cdots \times \mathbb{K}^{l_N \times q_N}$ of all perturbation families $\Delta = (\Delta_1, \ldots, \Delta_N)$, with $\Delta_i \in \mathbb{K}^{l_i \times q_i}$, is endowed with the norm $\gamma(\Delta) = \gamma(\Delta_1, \ldots, \Delta_N) = \sum_{i=1}^{N}||\Delta_i||$, where the norms $||\Delta_i||$ are operator norms on $\mathbb{K}^{l_i \times q_i}$, induced by given monotonic vector norms on the spaces $\mathbb{K}^{l_i}, \mathbb{K}^{q_i}, i \in \mathcal{N}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$).

**Definition 2.1.** Let the linear discrete time system (2) be $D-$stable.

(a) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to multi-perturbations of the form (3) are defined, respectively, by

$$r_{C}(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf \{\gamma(\Delta) : \Delta \in \Delta_{\mathbb{C}}, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D\},$$

$$r_{R}(A, (D_i, E_i)_{i \in \mathcal{N}}; D) = \inf \{\gamma(\Delta) : \Delta \in \Delta_{\mathbb{R}}, \sigma(A + \sum_{i=1}^{N} D_i \Delta_i E_i) \not\subset D\}.$$  

(b) The complex, real $D(\alpha, r)$-stability radius of the system (2) with respect to affine perturbations of the form (4) are defined, respectively, by

$$r_{C}(A, (B_i)_{i \in \mathcal{N}}; D) = \inf \{\max_{i \in \mathcal{N}} |\delta_i| : \delta_i \in \mathbb{C}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D\},$$

$$r_{R}(A, (B_i)_{i \in \mathcal{N}}; D) = \inf \{\max_{i \in \mathcal{N}} |\delta_i| : \delta_i \in \mathbb{R}, i \in \mathcal{N}, \sigma(A + \sum_{i=1}^{N} \delta_i B_i) \not\subset D\}.$$  

As noted in Introduction, the problems of computing of the stability radii (i.e. $D(0, 1)$-stability radii) of the system (2) have been studied during the last twenty years and have got the full results, see e.g. [3], [12], [4], [10]. We list here the interesting results for the class of positive systems (i.e. A is a nonnegative matrix).

**Theorem 2.2.** [4] Let the system (2) be $D(0, 1)$-stable and positive. Suppose the system matrix A is subjected to affine-perturbations (4), where $B_i \in \mathbb{R}^{q_i \times n}, i \in \mathcal{N}$. Then

$$r_{C}(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = r_{R}(A, (B_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\rho(\sum_{i=1}^{N} B_i (I_n - A)^{-1})}.$$

**Theorem 2.3.** [10] Let the system (2) be $D(0, 1)$-stable and positive. Assume that the matrix A is subjected to parameter multi-perturbations (3). If $D_i = D \in \mathbb{R}^{n \times l}$ and $E_i \in \mathbb{R}^{q_i \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}^{q \times n}$ and $D_i \in \mathbb{R}^{n \times l_i}$ for every $i \in \mathcal{N}$, then

$$r_{C}(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = r_{R}(A, (D_i, E_i)_{i \in \mathcal{N}}; D(0, 1)) = \frac{1}{\max_{i \in \mathcal{N}} ||E_i (I_n - A)^{-1}D_i||}.$$  

The following theorem extends the above results to the general case of $D = D(\alpha, r)$.

**Theorem 2.4.** Let the system (2) be $D(\alpha, r)$-stable and $A \geq \alpha I_n$. (i) If the matrix A is subjected to multi-perturbations (3), where $D_i = D \in \mathbb{R}^{n \times l}$ and $E_i \in \mathbb{R}^{q_i \times n}$ for every $i \in \mathcal{N}$ or $E_i = E \in \mathbb{R}^{q \times n}$ and $D_i \in \mathbb{R}^{n \times l_i}$ for every $i \in \mathcal{N}$, then
\[ r_C(A, (D_i)_{i \in \mathbb{N}}; D(\alpha, r)) = r_{\mathbb{R}}(A, (D_i)_{i \in \mathbb{N}}; D(\alpha, r)) = \max_{i \in \mathbb{N}} \|E_i((\alpha+r)I_n-A)^{-1}D_i\| \]

(ii) If the matrix \( A \) is subjected to affine-perturbations (4), where \( B_i \in \mathbb{R}^{n \times n}, i \in \mathbb{N}, \) then \( r_C(A, (B_i)_{i \in \mathbb{N}}; D(\alpha, r)) = r_{\mathbb{R}}(A, (B_i)_{i \in \mathbb{N}}; D(\alpha, r)) = \frac{1}{\rho(\sum_{i=1}^{\nu} B_i((\alpha+r)I_n-A)^{-1})}. \)

**Proof.** The proof is based on Theorems 2.2, 2.3 and the fact that the system \( x(k+1) = Ax(k), k \in \mathbb{N} \) is \( D(\alpha, r) \)-stable if and only if the system \( x(k+1) = (A - \alpha I_n)x(k), k \in \mathbb{N} \) is \( D(0,1) \)-stable. For sake of space, it is omitted here. \( \square \)

The following is an extension of the main result of [7].

**Corollary 2.5.** Let \( P(z) := I_n z^{\nu+1} - A_\nu z^{\nu} - \ldots - A_0 \) be a given polynomial matrix. Assume that \( |\alpha| < r, |\alpha| + r \leq 1 \) and \( ||[A_0 A_1 \ldots A_\nu]||_{\infty} < (r - |\alpha|)^{\nu+1} \). Then all the roots of the equation \( \det P(z) = 0 \) lie inside the disk \( D(\alpha, r) \).

## 3 \( D \)-stability radii of linear discrete time-delay systems

Consider a dynamical system described by a linear discrete time-delay system of the form (1), where \( A_i \in \mathbb{R}^{n \times n}, i \in \mathbb{P} := \{0, 1, 2, \ldots, \nu\}, \) are given matrices. For the linear discrete time-delay system (1), we consider the stable region \( \mathbb{D} = D(\alpha, r), |\alpha| < r, r + |\alpha| \leq 1, \) see e.g. [8], [13], [14]. We associate the system (1) with the following polynomial matrix \( P(z) := (z^{\nu+1}I_n - A_\nu z^{\nu} - A_{\nu-1} z^{\nu-1} - \ldots - A_0), \ z \in \mathbb{C}. \) Denote by \( \sigma((A_i)_{i \in \mathbb{P}}) := \{z \in \mathbb{C} : \det P(z) = 0\} \) the set of all roots of the characteristic equation of the linear discrete time-delay system (1). Then \( \sigma((A_i)_{i \in \mathbb{P}}) \) is called the spectral set of the linear discrete time-delay system (1) and \( \rho((A_i)_{i \in \mathbb{P}}) := \max \{|s| : s \in \sigma((A_i)_{i \in \mathbb{P}})\} \) is called spectral radius of the linear discrete time-delay system (1). Recall that the system (1) is said to be \( D \)-stable if \( \sigma((A_i)_{i \in \mathbb{P}}) \subset \mathbb{D}. \) We now assume that the system (1) is \( D \)-stable and the coefficient matrices \( A_i, i \in \mathbb{P} \) are subjected to parameter perturbations

\[
A_i \rightarrow A_i + \sum_{j=1}^{\nu} D_{ij} \Delta_{ij} E_{ij}, \quad \text{(multi-perturbation)}
\]

\[
A_i \rightarrow A_i + \sum_{j=1}^{\nu} \delta_{ij} B_{ij}, \quad \text{(affine-perturbation)}
\]

where \( D_{ij} \in \mathbb{R}^{n \times l_{ij}}, E_{ij} \in \mathbb{R}^{q_{ij} \times n}, (i \in \mathbb{P}, j \in \mathbb{N} := \{1, 2, \ldots, N\}); B_{ij} \in \mathbb{R}^{n \times n}, (i \in \mathbb{P}, j \in \mathbb{N}) \) are given matrices defining the structure of perturbations and \( \Delta_{ij} \in \mathbb{K}^{l_{ij} \times q_{ij}}, (i \in \mathbb{P}, j \in \mathbb{N}); \delta_{ij} \in \mathbb{K}, (i \in \mathbb{P}, j \in \mathbb{N}) \) are perturbation matrices, perturbation scalars, respectively. For the class of multi-perturbations of the form
(5), we define $\tilde{\Delta} := (\Delta_0, \Delta_1, \ldots, \Delta_N)$, where $\Delta_i := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in K_{l_{i} x q_{i1}} \times \ldots \times K_{l_{iN} x q_{iN}}$, $i \in \nu$. Then the size of each perturbation $\Delta$ is measured by $\gamma(\Delta) := \sum_{i=0}^{\nu} \sum_{j=1}^{N} ||\Delta_{ij}||$. With the class of affine perturbations of the form (6), we denote $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu1}, \ldots, \delta_{\nuN})) \in K^{\nu N}$ and the size of each perturbation $\delta$ is measured by $\gamma(\delta) = \max_{i \in \nu, j \in \underline{N}} |\delta_{ij}|$.

Definition 3.1. Let the linear discrete time-delay system (1) be $\mathcal{D}$-stable.

(a) The complex, real $D(\alpha, r)$-stability radius of the system (1) with respect to multiperturbations of the form (5) is defined, respectively, by

$$r_{C}^{(\alpha)}(\mathcal{D}) = \inf \{ \gamma(\Delta) : \tilde{\Delta} \in C_{l_{0} x q_{0}} \times \ldots \times C_{l_{\nu} x q_{\nu}}, i \in \nu, \sigma \left( A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij} \right) \not\subset \mathcal{D} \},$$

$$r_{R}^{(\alpha)}(\mathcal{D}) = \inf \{ \gamma(\Delta) : \tilde{\Delta} \in R_{l_{0} x q_{0}} \times \ldots \times R_{l_{\nu} x q_{\nu}}, i \in \nu, \sigma \left( A_i + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij} \right) \not\subset \mathcal{D} \}.$$

(b) The complex, real $D(\alpha, r)$-stability radius of the system (1) with respect to affine perturbations of the form (6) is defined, respectively, by

$$r_{C}^{(\alpha)}(\mathcal{D}) = \inf \{ \gamma(\delta) : \delta \in C^{(\nu+1) N}, \sigma \left( A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij} \right) \not\subset \mathcal{D} \},$$

$$r_{R}^{(\alpha)}(\mathcal{D}) = \inf \{ \gamma(\delta) : \delta \in R^{(\nu+1) N}, \sigma \left( A_i + \sum_{j=1}^{N} \delta_{ij} B_{ij} \right) \not\subset \mathcal{D} \}.$$

In particular case of $\mathcal{D} = D(0, 1)$, the problems of computing of the stability radii of the linear discrete-time systems (1) under single perturbations, affine perturbations and multi-perturbations have been done recently by ourselves (see [5], [9], [10]). We summarize here some existing results of these problems. Recall that the system (1) is positive if and only if system matrices $A_0, A_1, \ldots, A_\nu$ are nonnegative.

Theorem 3.2. [9] Suppose the linear discrete time-delay system (1) is $D(0, 1)$-stable, positive and the system matrices $A_i, i \in \nu$ are subjected to affine perturbations of the form (6) where $B_{ij} \in R_{l_{i} x n}, i \in \nu, j \in \underline{N}$. Then, $r_{C}^{(\alpha)}(D(0, 1)) = r_{R}^{(\alpha)}(D(0, 1)) = \frac{1}{\rho(P(1)^{-1}B)}$, where $B := \sum_{j=1}^{N} B_{0j} + \sum_{j=1}^{N} B_{1j} + \ldots + \sum_{j=1}^{N} B_{\nu j}$.

Remark 3.3. In the proof of Theorem 3.2, we showed that the real perturbation $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu1}, \ldots, \delta_{\nuN})) \in R^{(\nu+1) N}$, $\delta_{ij} = \frac{1}{\rho(P(1)^{-1}B)}, (i \in \nu, j \in \underline{N})$ is a minimal size destabilizing perturbation. This fact will be used in the sequel.

Theorem 3.4. [10] Let the linear discrete time-delay system (1) be positive, $D(0, 1)$-stable. Assume that the system matrices $A_i, i \in m$ are subjected to the multi-perturbations of the form (5) where $D_{ij} := D \in R_{l_{i} x l}$, $E_{ij} \in R_{q_{ij} x n}$ for all $i \in \nu, j \in \underline{N}$ or $E_{ij} := E \in R_{l_{i} x n}, D_{ij} \in R_{l_{i} x l_{ij}}$ for all $i \in \nu, j \in \underline{N}$. Then, $r_{C}^{(m)}(D(0, 1)) = r_{R}^{(m)}(D(0, 1)) = \frac{1}{\rho([P(1)^{-1}D_{ij}|| \not\subset \nu, j \in \underline{N}]}.$
Theorem 3.5. Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the coefficient matrices $A_i$, $i \in \overline{\nu}$, are subjected to the multi-perturbations (5), where $D_{ij} := D \in \mathbb{R}_{+}^{n \times l}$, $E_{ij} \in \mathbb{R}_{+}^{q_{ij} \times n}$ ($i \in \overline{\nu}, j \in N$) or $D_{ij} \in \mathbb{R}_{+}^{n \times l_{ij}}, E_{ij} := E \in \mathbb{R}_{+}^{q_{ij} \times n}$ ($i \in \overline{\nu}, j \in N$). If $\alpha \leq 0$ and $A_0, A_1, \ldots, A_{\nu-1}, (A_\nu - \alpha I_n) \in \mathbb{R}_{+}^{n \times n}$, then

$$r_{c}^{\nu}(D(\alpha, r)) = r_{c}^{\nu}(D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in N} \|\langle \alpha + r \rangle E_{ij} P(\alpha + r)^{-1} D_{ij} \|}.$$

Proof. Assume $D_{ij} := D \in \mathbb{R}_{+}^{n \times l}$, $E_{ij} \in \mathbb{R}_{+}^{q_{ij} \times n}$ ($i \in \overline{\nu}, j \in N$). Consider the companion matrix of the polynomial matrix $P(z) = (z^{\nu+1} I_{n} - A_{\nu} z^{\nu} - A_{\nu-1} z^{\nu-1} - \ldots - A_{0})$:

$$A_c := \begin{bmatrix} 0 & I_{n} & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{n} \\ A_{0} & A_{1} & \cdots & \cdots & \cdots & A_{\nu} \end{bmatrix} \in \mathbb{R}^{(\nu+1)n \times (\nu+1)n},$$

and similarly $A_c(\Delta)$ for the perturbed polynomial matrix $P_{\Delta}(z) := z^{\nu+1} I_{n} - \sum_{i=0}^{\nu} (A_{i} + \sum_{j=1}^{N} D_{ij} \Delta_{E_{ij}}) z^{i}$, where $\Delta := (\Delta_{0}, \Delta_{1}, \ldots, \Delta_{\nu})$, $\Delta_{i} := (\Delta_{i1}, \Delta_{i2}, \ldots, \Delta_{iN}) \in \mathbb{R}^{l \times q_{i1} \times \ldots \times l_{ij} \times q_{ij}, i \in \overline{\nu}}$. Then the matrix $A_c(\Delta)$ can be represented by the following form

$$A_c(\Delta) = A_c + \sum_{j=1}^{N} \tilde{D} \Delta_{0j} \tilde{E}_{0j} + \sum_{j=1}^{N} \tilde{D} \Delta_{ij} \tilde{E}_{ij} + \cdots + \sum_{j=1}^{N} \tilde{D} \Delta_{\nu j} \tilde{E}_{\nu j},$$

where $\tilde{D}_{ij} := \tilde{D} \in \mathbb{R}_{+}^{(\nu+1)n \times l}$, $\tilde{E}_{ij} := [E_{ij}, 0, \ldots, 0] \in \mathbb{R}^{q_{ij} \times (\nu+1)n}$, $\tilde{E}_{0j} := [E_{0j}, 0, \ldots, 0] \in \mathbb{R}^{q_{0j} \times (\nu+1)n}$, $\tilde{E}_{ij} := [0, \ldots, 0, E_{ij}] \in \mathbb{R}^{l_{ij} \times (\nu+1)n}$, for every $i \in \overline{\nu}, j \in N$. It follows from the equality $\det P_{\Delta}(z) = \det (z I_{(\nu+1)n} - A_c(\Delta))$ that $\sigma((A_{i} + \sum_{j=1}^{N} D_{ij} \Delta_{ij} E_{ij})_{i \in \overline{\nu}}) = \sigma(A_c(\Delta))$. So, we get $r_{c}^{\nu}(D(\alpha, r)) = \sigma_{c}(A_{c}, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i \in \overline{\nu}, j \in N}; D(\alpha, r))$. By the assumption $\alpha \leq 0$, $A_0, A_1, \ldots, A_{\nu-1}, (A_\nu - \alpha I_n) \in \mathbb{R}_{+}^{n \times n}$, $D_{ij} \in \mathbb{R}_{+}^{n \times l}$, $E_{ij} \in \mathbb{R}_{+}^{q_{ij} \times n}$ ($i \in \overline{\nu}, j \in N$), we have $A_c \geq \alpha I_{(\nu+1)n}$ and $\tilde{D} \in \mathbb{R}_{+}^{(\nu+1)n \times l}$, $\tilde{E}_{ij} \in \mathbb{R}_{+}^{q_{ij} \times (\nu+1)n}$ ($i \in \overline{\nu}, j \in N$). Hence, from Theorem 2.4, we get $r_{c}(A_{c}, (\tilde{D}_{ij}, \tilde{E}_{ij})_{i \in \overline{\nu}, j \in N}; D(\alpha, r)) = \frac{1}{\max_{i \in \overline{\nu}, j \in N} \|\langle \alpha + r \rangle I_{(\alpha+r)_{(\nu+1)n} - A_c}^{-1} D_{ij} \|}$. On the other hand, it is easy to check that

$$(z I_{(\nu+1)n} - A_c)^{-1} \tilde{D} = \begin{pmatrix} P(z)^{-1} \\ z P(z)^{-1} \\ \vdots \\ z^\nu P(z)^{-1} \end{pmatrix}.$$
Therefore, $\max_{(\nu, i) \in \mathbb{N}^2} \left\| (\alpha + r)^{\nu} E_{ij} P^{(\alpha + r)^{-1} B_{ij}} \right\|$. The proof of the case of $D_{ij} \in \mathbb{R}^{n \times n}_{+}$, $E_{ij} := E \in \mathbb{R}^{q \times n}_{+}$ ($i \in \overline{\nu}, j \in N$), can be done by a similar way. This completes our proof.

We now turn to the problem of computing of the complex, real $D$-stability radius under affine perturbations (6). For every $i \in \overline{\nu}$, let us define

$$A_{i}^{*} := \frac{1}{r_{\nu+1-i}} \left( C_{\nu-i}^{\nu-i} A_{\nu} + C_{\nu-1-i}^{\nu-1-i} A_{\nu-1} + \ldots + A_{i} - C_{\nu+1-i}^{\nu+1-i} I_{n} \right),$$

(7)

where $C_{\nu}^{u} := \frac{u!}{v!(u-v)!}$, $u, v \in \mathbb{N}, u \geq v$. The following theorem is an extension of Theorem 3.2 to the general case of $D = D(\alpha, r)$.

**Theorem 3.6.** Let the linear discrete time-delay system (1) be $D(\alpha, r)$-stable. Suppose the system matrices $A_{i}, i \in \overline{\nu}$ are subjected to affine perturbations (6), where $B_{ij} \in \mathbb{R}^{n \times n}_{+}$ ($i \in \overline{\nu}, j \in N$). If either $\alpha \leq 0$ and $A_{0}, A_{1}, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}^{n \times n}$, or $\alpha > 0$ and $A_{i}^{*} \in \mathbb{R}^{n \times n}_{+}, i \in \overline{\nu}$, then $r_{C}^{\nu}(D(\alpha, r)) = r_{R}^{\nu}(D(\alpha, r)) = \frac{1}{\max_{(\nu, i, j) \in \mathbb{N}^2} \left\| (\alpha+r)^{\nu} E_{ij} P^{(\alpha+r)^{-1} B_{ij}} \right\|}$.

**Proof.** In the case of $\alpha \leq 0$ and $A_{0}, A_{1}, \ldots, A_{\nu-1}, (A_{\nu} - \alpha I_{n}) \in \mathbb{R}^{n \times n}$, the proof is similar to that of Theorem 3.5, based on the result of Theorem 2.4(i). Then, we have $r_{C}^{\nu}(D(\alpha, r)) = r_{R}^{\nu}(D(\alpha, r)) = \frac{1}{\max_{(\nu, i, j) \in \mathbb{N}^2} \left\| (\alpha+r)^{\nu} E_{ij} P^{(\alpha+r)^{-1} B_{ij}} \right\|}$. We now assume that $\alpha > 0$ and $A_{i}^{*} \in \mathbb{R}^{n \times n}_{+}, i \in \overline{\nu}$. Denote by $P^{*}(z) := z^{\nu+1} I_{n} - A_{\nu} z^{\nu} - \ldots - A_{0}$. Let $s \in \mathbb{C}, |s - \alpha| \geq r$ satisfy $\text{det} P(s) = 0$. Setting $z = \frac{s-\alpha}{r}$, $|z| \geq 1$, by a direct computation, we have $\text{det} P(s) = 0$ if and only if $\text{det} P^{*}(z) = 0$. So the discrete time-delay system (1) is $D(\alpha, r)$-stable if and only if the following discrete time-delay system

$$x(k+1) = A_{\nu}^{*} x(k) + A_{\nu-1}^{*} x(k-1) + \ldots + A_{0}^{*} x(k-\nu), \quad k \in \mathbb{N}, \ k \geq \nu,$$

(8)

is $D(0, 1)$-stable. Similarly, the perturbed system

$$x(k+1) = (A_{\nu} + \sum_{j=1}^{N} \delta_{ij} B_{ij}) x(k) + \ldots + (A_{0} + \sum_{j=1}^{N} \delta_{ij} B_{ij}) x(k-\nu), \quad k \in \mathbb{N}, \ k \geq \nu,$$

(9)

is $D(\alpha, r)$-stable if and only if the following discrete time-delay system is $D(0, 1)$-stable

$$x(k+1) = (A_{\nu}^{*} + B_{ij}) x(k) + \ldots + (A_{0}^{*} + B_{ij}) x(k-\nu), \quad k \in \mathbb{N}, \ k \geq \nu.$$

Here, $B_{ij} := \sum_{j=1}^{N} \delta_{ij} (1 - \frac{1}{r^{\nu+1-i}} C_{\nu}^{\nu-i} B_{ij}) + \ldots + \sum_{j=1}^{N} \delta_{ij} (1 - \frac{1}{r^{\nu+1-i}} B_{ij})$, $i \in \overline{\nu}$. Since $B_{ij} \in \mathbb{R}^{n \times n}_{+}, (i \in \overline{\nu}, j \in N)$, we have

$$\frac{1}{r^{\nu+1-i}} C_{\nu}^{\nu-i} B_{ij}, \quad \frac{1}{r^{\nu+1-i}} C_{\nu+1-i}^{\nu-1-i} B_{(\nu+1)-j}, \ldots, \frac{1}{r^{\nu+1-i}} B_{ij} \in \mathbb{R}^{n \times n}_{+}, \quad i \in \overline{\nu}, j \in N.
It follows from Theorem 3.2 that the system (10) is $D(0, 1)$-stable for every $\delta$ satisfying $\max_{i \in \overline{\nu}, j \in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(P^*(1)^{-1}G)}$, where

$$G := \sum_{i=0}^{\nu} \left( \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} C_{\nu}^{\nu-i} \alpha^{\nu-i} B_{\nu j} + \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} C_{\nu-1}^{\nu-1-i} \alpha^{\nu-1-i} B_{(\nu-1)j} + \ldots + \sum_{j=1}^{N} \frac{1}{r^{\nu+1-i}} B_{ij} \right).$$

Hence, the perturbed system (9) is $D(\alpha, r)$-stable for every complex perturbation $\delta$ such that $\max_{i \in \overline{\nu}, j \in \underline{N}} |\delta_{ij}| < \frac{1}{\rho(P^*(1)^{-1}G)}$. By the definition of the complex $D(\alpha, r)$-stability radius of the system (1) under affine perturbations of the form (6), we get $r^a_c(D(\alpha, r)) \geq \frac{1}{\rho(P^*(1)^{-1}G)}$. On the other hand, taking Remark 3.3 into account, the system (10) is not $D(0, 1)$-stable if $\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu 1}, \ldots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}$; $\delta_{ij} = \frac{1}{\rho(P^*(1)^{-1}G)}$ ($i \in \overline{\nu}, j \in \underline{N}$). Then the perturbed system (9) is not $D(\alpha, r)$-stable if

$$\delta := ((\delta_{01}, \ldots, \delta_{0N}); \ldots; (\delta_{\nu 1}, \ldots, \delta_{\nu N})) \in \mathbb{R}^{(\nu+1)N}; \delta_{ij} = \frac{1}{\rho(P^*(1)^{-1}G)} (i \in \overline{\nu}, j \in \underline{N}).$$

We derive that $r^a_R(D(\alpha, r)) \leq \frac{1}{\rho(P^*(1)^{-1}G)}$. So we get the following inequalities

$$\frac{1}{\rho(P^*(1)^{-1}G)} \leq r^a_c(D(\alpha, r)) \leq r^a_R(D(\alpha, r)) \leq \frac{1}{\rho(P^*(1)^{-1}G)}.$$

Therefore $r^a_c(D(\alpha, r)) = r^a_R(D(\alpha, r)) = \frac{1}{\rho(P^*(1)^{-1}G)}$. Finally, by a direct computation, we get $P^*(1)^{-1}G = P(\alpha + r)^{-1}B$. This completes our proof. \hfill \square

References


