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Kyoto University
A KINETIC APPROACH TO A COMPARISON THEOREM FOR DEGENERATE PARABOLIC EQUATIONS

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Let Ω be an open bounded subset of $\mathbb{R}^d$ and $T \in (0, +\infty]$. Let $Q$ denote the set $(0, T) \times \Omega$, $\partial \Omega$ the boundary of $\Omega$, $\mathbf{n}(\overline{x})$ the outward unit normal to $\Omega$ at a point $\overline{x} \in \partial \Omega$ and $\Sigma$ the set $(0, T) \times \partial \Omega$. We consider the following parabolic-hyperbolic problem:

$$\partial_t u + \text{div} A(u) - \Delta \beta(u) = 0 \quad \text{in} \quad Q$$

(1.1)

with the initial condition:

$$u(0, x) = u_0(x), \quad x \in \Omega,$$

(1.2)

and the boundary condition:

$$u(t, x) = u_b(t, x), \quad (t, x) \in \Sigma,$$

(1.3)

where the flux function $A$ belongs to $C^1(\mathbb{R})$ and the function $\beta$ is non-decreasing and Lipschitz continuous. This monotonicity assumption of $\beta$ allows us some degenerate diffusion cases which appear in many interesting models, for example, filtration problems in porous media [2,5,8].

In the nondegenerate case (in which the function $\beta$ is strictly increasing), the problem (1.1) is of parabolic type and hence the existence and uniqueness of solutions are well known. In the case where $\beta' \equiv 0$, the problem (1.1) being a nonlinear hyperbolic problem, the uniqueness
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of weak solutions is not ensured, and one must consider a notion of entropy solution, relying on the notion of boundary entropy-flux pairs to recover uniqueness (see [11,16]). When $\beta$ is merely a nondecreasing function, in the case of homogeneous boundary data, i.e., $u_b \equiv 0$, Carrillo [3] succeeded in proving the uniqueness of entropy solutions by mainly using the dedoubling variable technique developed by Kružkov [11]. The equivalence of entropy solutions and weak solutions is also considered in [10]. In the case of nonhomogeneous boundary data existence and uniqueness of entropy solutions to (1.1)-(1.3) have been proved in [1,14,15]. The method used there is also the dedoubling variable technique.

On the other hand Perthame [12,17] proved the uniqueness of entropy solutions to the Cauchy problem of the conservation law (in which $\beta' \equiv 0$ and $\Omega = \mathbb{R}^d$) by using the kinetic formulation which is introduced by Lions, Perthame and Tadmor [12], without relying on the dedoubling variable technique. Imbert and Vovelle [9] developed analogous techniques for conservation laws with boundary conditions, proved the Comparison Theorem for entropy sub- and supersolutions, and applied their results to the BGK-like model. This technique was also applied in [6] to study the parabolic approximation of a multidimensional conservation law with initial and boundary conditions.

The purpose of this note is to give a comparison result for their sub- and supersolutions by using kinetic techniques. Although the $L^1$ contractivity and, therefore, uniqueness of entropy weak solutions has been obtained, it would seem that any comparison theorem for those solutions is not proven.

According to [14] we introduce the definition of entropy sub- and supersolution.

Define

$$\text{sgn}^+(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases} \quad \text{and} \quad \text{sgn}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \geq 0, \end{cases}$$

and $r^\pm = \text{sgn}^\pm(r)r$.

**Definition 1.1.** A function $u$ of $L^1(Q)$ is said to be a weak solution of the problem (1.1) - (1.3) if it satisfies:

$$\beta(u) - \beta(u_b) \in L^2(0,T; H^1_0(\Omega)),$$

$$A(u) \in L^1(Q)^d$$

and

$$\int_Q u\varphi_t + (A(u) - \nabla\beta(u)) \cdot \nabla\varphi dxdt + \int_\Omega u_0\varphi(0,x)dx = 0$$

(1.4)

for any $\varphi \in C^\infty_c([0,T) \times \Omega)$. 


**Definition 1.2.** Let $u \in L^\infty(Q)$. $u$ is said to be an entropy subsolution of (1.1) - (1.3) if it is a weak solution and satisfies:

$$
\int_Q (u - \kappa)^+ \partial_t \varphi + (\mathcal{F}^+(u, \kappa) - \nabla(\beta(u) - \beta(\kappa))^+) \cdot \nabla \varphi \, dx \, dt
+ \int_\Omega (u_0 - \kappa)^+ \varphi(0, x) \, dx + M \int_\Sigma (u_b - \kappa)^+ \varphi \, d\sigma \, dt \geq 0
$$

(1.5)

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)^+$ such that $\text{sgn}^+(\beta(u_b) - \beta(\kappa)) \varphi = 0$ a.e. on $\Sigma$.

$u$ is said to be an entropy supersolution if (1.7) is replaced by

$$
\int_Q (u - \kappa)^- \partial_t \varphi + (\mathcal{F}^-(u, \kappa) - \nabla(\beta(u) - \beta(\kappa))^-) \cdot \nabla \varphi \, dx \, dt
+ \int_\Omega (u_0 - \kappa)^- \varphi(0, x) \, dx + M \int_\Sigma (u_b - \kappa)^- \varphi \, d\sigma \, dt \geq 0
$$

(1.6)

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^d)^+$ such that $\text{sgn}^-(\beta(u_b) - \beta(\kappa)) \varphi = 0$ a.e. on $\Sigma$. Here $C_c^\infty([0, T) \times \mathbb{R}^d)^+$ is the set of nonnegative functions in $C_c^\infty([0, T) \times \mathbb{R}^d)$.

We also set

$$
M = \sup\{|A'(r)|; |r| \leq \max\{\|u_0\|_{L^\infty(\Omega)}, \|u_b\|_{L^\infty(\Sigma)}\}
$$

(1.7)

and

$$
L = \max_{1 \leq i \leq N} \|\Delta A h_i(T_i x)\|_{L^\infty(\Sigma_{\lambda_i})}.
$$

(1.8)

We are now in a position to state the main theorem which obviously extends the $L^1$ contractive property for entropy solutions.

**Theorem**  Assume that the following conditions hold:

(A1) $\Omega$ is a bounded open subset of $\mathbb{R}^d$ whose boundary $\partial \Omega$ is $C^2$, $A \in C^1(\mathbb{R}, \mathbb{R})$ and $\beta : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function.

(A2) $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Sigma)$.

Let $u \in L^\infty(Q)$ be an entropy subsolution of (1.1) - (1.3) with data $(u_0, u_b)$ and let $\tilde{u}$ be an entropy supersolution of (1.1) - (1.3) with data $(\tilde{u}_0, \tilde{u}_b)$. Then we have

$$
\frac{1}{T} \int_0^T \int_\Omega (u(t, x) - \tilde{u}(t, x))^+ \, dx \, dt
\leq \int_\Omega (u_0(x) - \tilde{u}_0(x))^+ \, dx + M \int_0^T \int_{\partial \Omega} (u_b(t, x) - \tilde{u}_b(t, x))^+ \, d\sigma \, dt
+ \frac{L}{2} \int_0^T \int_{\partial \Omega} (\beta(u_b(t, x)) - \beta(\tilde{u}_b(t, x)))^+ \, d\sigma \, dt.
$$

(1.9)
2. SKETCH OF PROOF.

The semi-Kružkov entropies are the convex functions defined by
\[ \eta_{k}^{\pm}(r) = (r-k)^{\pm}, \quad k \in \mathbb{R}, \]
while the corresponding entropy flux are the function defined by
\[ \mathcal{F}^{\pm}(r, k) = \text{sgn}^{\pm}(r-k)(A(r) - A(k)). \]
For a function \( u \in L^\infty(Q) \) and \( \xi \in \mathbb{R} \) we set
\[ f_{\pm}(t, x, \xi) = \text{sgn}^{\pm}(u(t, x) - \xi). \]
We assume that \( \Omega \) is a \( C^2 \) bounded open subset in \( \mathbb{R}^d \). Thus, we can find a finite open cover \( \{B_i\}_{i=0}^{N} \) of \( \overline{\Omega} \) and a partition of unity \( \{\lambda_i\}_{i=0}^{N} \) on \( \overline{\Omega} \) subordinate to \( \{B_i\}_{i=0}^{N} \) such that, for \( i \geq 1 \), up to a change of coordinates represented by an orthogonal matrix \( T_i \), the set \( \Omega \cap B_i \) is the epigraph of a \( C^2 \) function \( h_i : \mathbb{R}^{d-1} \to \mathbb{R} \), that is to say:
\[ \Omega_{\lambda_i} \cap B_i = \{x \in B_i; (T_i x)_d > h_i(T_i \overline{x})\} \]
and
\[ \partial \Omega_{\lambda_i} = \partial \Omega \cap B_i = \{x \in B_i; (T_i x)_d = h_i(T_i \overline{x})\}, \]
where \( x = (\overline{x}, x_d) \in \mathbb{R}^d \) and \( \overline{x} = (x_1, \cdots, x_{d-1}) \). For simplicity we will drop the index \( i \) and we suppose that the change of coordinates is trivial: \( Y_i = Id \). We also write \( Q_{\lambda} = (0, T) \times \Omega_{\lambda} \), \( \Sigma_{\lambda} = (0, T) \times \partial \Omega_{\lambda} \), \( \Pi_{\lambda} = \{\overline{x}; x \in \text{supp}(\lambda) \cap \Omega\} \) and \( \Theta_{\lambda} = (0, T) \times \Pi_{\lambda} \). We denote by \( \mathbf{n}(\overline{x}) \) the outward unit normal to \( \Omega_{\lambda} \) at a point \((\overline{x}, h(\overline{x}))\) of \( \partial \Omega_{\lambda} \) and by \( d\sigma(\overline{x}) \) the \((d-1)\)-dimensional area element in \( \partial \Omega_{\lambda} \):
\[ \mathbf{n}(\overline{x}) = (1 + |\nabla_{\overline{x}} h(\overline{x})|^2)^{-1/2}(\nabla_{\overline{x}} h(\overline{x}), -1), \]\[ d\sigma(\overline{x}) = (1 + |\nabla_{\overline{x}} h(\overline{x})|^2)^{1/2}d\overline{x}. \]

To regularize the functions, for small \( \rho, s > 0 \) let us consider a smooth function \( \theta_{\rho,s} : \mathbb{R} \to \mathbb{R}^+ \) such that \( \text{supp} \theta_{\rho,s} \subset [\rho s/2, (1+\rho)s] \), \( \theta_{\rho,s}(r) = s^{-1} \) for \( \rho s \leq r \leq s \) and \( \int_{\mathbb{R}} \theta_{\rho,s}(r)dr = 1 \). Then, for \( \nu > 0 \) and \( \epsilon = (\epsilon_1, \cdots, \epsilon_d) \in (\mathbb{R}^+)^d \), we set \( \gamma_{\rho,\epsilon}(x) = \Pi_{i=1}^{d} \theta_{\rho,\epsilon_i}(x_i) \) and \( \gamma_{\rho,\nu,\epsilon}(t, x) = \theta_{\rho,\nu}(t) \gamma_{\rho,\epsilon}(x) \).
For simplicity, we will also use the following notations:
\[ \mathbf{n}_1 = \sqrt{1 + |\nabla_{\overline{x}} h(\overline{x})|^2} \mathbf{n}, \]
\[ \overline{x}_r = (\overline{x}, h(\overline{x}) + r) \quad \text{for } \overline{x} = (x_1, \cdots, x_{d-1}), \]
\( \psi^\lambda \) stands for \( \psi \lambda \) and \( \overline{\psi} \) denotes the restriction of \( \psi \) to \( \Sigma \times \mathbb{R}_\xi \), i.e.,
\[
\overline{\psi}(t, \overline{x}, \xi) = \psi(t, \overline{x}, h(\overline{x}, \xi)),
\]
where \( \psi \) is a function on \([0, T) \times \mathbb{R}^{d+1} \) and
\( \lambda \) is an element of the partition of unity \( \{\lambda_i\}_{i=0}^N \). Moreover we set
\[
s \vee t = \max\{s, t\} \quad \text{and} \quad s \wedge t = \min\{s, t\}.
\]

The proof of the theorem will follow from the following three lemmas whose proofs will be given in the forthcoming paper.

**Lemma 2.1.** Let \( u \) be an entropy subsolution with data \( (u_0, u_b) \) and let \( \lambda \) be an element of the partition of unity \( \{\lambda_i\}_{i=0}^N \). Then we have:

(a) There exists \( f_+^{\tau_0} \in L^\infty(\Omega \times \mathbb{R}) \) such that
\[
\lim_{s \to +0} \int_{\Omega \times \mathbb{R}} \left[ \frac{1}{s} \int_0^s f_+(t, x, \xi) dt \right] \phi \, dx d\xi = \int_{\Omega \times \mathbb{R}} f_+^{\tau_0} \phi \, dx d\xi
\]
for any \( \phi \in C_c^\infty(\Omega \times \mathbb{R}) \).

(b) For any \( \psi \in C_c^\infty([0, T) \times \mathbb{R}^{d+1}) \) and any weak* cluster point \( f_+^+ \) of \( \frac{1}{s} \int_0^s f_+(t, \overline{x}_f, \xi) dr \) as \( s \to +0 \) in \( L^\infty(\Theta_\lambda \times \mathbb{R}) \), we have
\[
\int_{Q_\lambda \times \mathbb{R}} (f_+(\partial_t + a \cdot \nabla)\psi^\lambda - \beta' \nabla f_+ \cdot \nabla \psi^\lambda) \, dtdx d\xi
\]
\[
+ \int_{\Theta_\lambda \times \mathbb{R}} \beta'(\nabla h(\overline{x}) \cdot \nabla f_+ b) \overline{\psi^\lambda} \, dtd\overline{x} d\xi
\]
\[
+ \int_{\Theta_\lambda \times \mathbb{R}} (-n_1 \cdot a) f_+ \overline{\psi^\lambda} \, dtd\overline{x} d\xi
\]
\[
\geq \int_{Q_\lambda \times \mathbb{R}} \partial_\xi \psi^\lambda \, d(m_+ + n_+).
\]

**Lemma 2.2.** There exist families of probability measures \( \{\nu_x^{\tau_0}\}_{x \in \Omega} \) and \( \{\tilde{\nu}_x^{\tau_0}\}_{x \in \Omega} \) on \( \mathbb{R}_\xi \), called Young measures, supported in \( (-\infty, ||u||_{L^\infty}) \) and \( (||\tilde{u}||_{L^\infty}, \infty)) \), respectively, and nonnegative functions \( m_+^0(x, \xi) \) and \( \tilde{m}_-^0(x, \xi) \) defined on \( \Omega \times \mathbb{R}_\xi \) such that
\[
m_+^0, \tilde{m}_-^0 \in C(\mathbb{R}_\xi; \mathcal{M}^+(\Omega)),
\]
\[
\lim_{\xi \to \infty} m_+^0(x, \xi) = \lim_{\xi \to -\infty} \tilde{m}_-^0(x, \xi) = 0 \quad \text{for a.e.} \ x \in \Omega,
\]
\[
f_+^{\tau_0}(x, \xi) = \nu_x^{\tau_0}(\xi, \infty) = \partial_\xi m_+^0(x, \xi) + \text{sgn}^+(u_0(x) - \xi)
\]
and
\[
\tilde{f}_-^{\tau_0}(x, \xi) = -\tilde{\nu}_x^{\tau_0}(-\infty, \xi] = \partial_\xi \tilde{m}_-^0(x, \xi) + \text{sgn}^-(\tilde{u}_0(x) - \xi).
\]
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Lemma 2.3. Let $\lambda$ be an element of the partition of unity $\{\lambda_i\}_{i=0}^{N}$ and let $f_+^\tau$ and $\tilde{f}_-^\tau$ be weak* cluster points of $\frac{1}{s} \int_0^s f_+(t, \bar{x}_r, \xi)dr$ and $\frac{1}{s} \int_0^s \tilde{f}_-(t, \bar{x}_r, \xi)dr$, respectively, as $s \to +0$, in $L^\infty(\Theta_\lambda \times \mathbb{R})$. There exist Young measures $\{\nu_{t,y}^\tau\}_{(t,y)\in\Sigma}$ and $\{\tilde{\nu}_{t,y}^\tau\}_{(t,y)\in\Sigma}$ on $\mathbb{R}_{\xi}$, supported in $(-\infty, ||u||_{L^\infty}]$ and $[-||\tilde{u}||_{L^\infty}, \infty))$, respectively, and nonnegative functions $m_+^b(t, y, \xi)$ and $\tilde{m}_-^b(t, y, \xi)$ defined on $\Sigma \times \mathbb{R}_{\xi}$ such that

$$
\lim_{\xi \to -\infty} m_+^b(t, y, \xi) = \lim_{\xi \to \infty} \tilde{m}_-^b(t, y, \xi) = 0 \quad \text{for a.e.} \quad (t, y) \in \Sigma,
$$

$$
f_+^\tau(t, y, \xi) = \nu_{t,y}^\tau([\xi, \infty)), \quad \tilde{f}_-^\tau = -\tilde{\nu}_{t,y}^\tau((-\infty, \xi]),
$$

$$
(-a \cdot n_1)f_+^\tau = \partial_\xi m_+^b + M\text{sgn}^+(u_b - \xi)
$$

(2.4)

$$
(-a \cdot n_1)\tilde{f}_-^\tau = \partial_\xi \tilde{m}_-^b + M\text{sgn}^-(\tilde{u}_b - \xi),
$$

$$
\int_{\Theta_{\lambda}} m_+^b(t, \bar{x}_0, \xi)\bar{\varphi}^\lambda(t, \bar{x}_0)dtd\bar{x} \geq 0
$$

(2.5)

for any $\varphi \in C(\Sigma)^+$ satisfying $\text{sgn}^+(\beta(u_b) - \beta(\xi))\varphi = 0$ a.e. on $\Sigma$ and

$$
\int_{\Theta_{\lambda}} \tilde{m}_-^b(t, \bar{x}_0, \xi)\bar{\varphi}^\lambda(t, \bar{x}_0)dtd\bar{x} \geq 0
$$

for any $\varphi \in C(\Sigma)^+$ satisfying $\text{sgn}^-(\beta(\tilde{u}_b) - \beta(\xi))\varphi = 0$ a.e. on $\Sigma$.

We continue the proof of Theorem. Let $f_+, n_+$ and $m_+$ be the functions defined for $u$ as above. $f_+^{\tau_0}$ denotes the time kinetic traces and $f_+^\tau$ a cluster point of space kinetic traces associated with $u$. The corresponding ones associated with $\tilde{u}$ will be denoted by $\tilde{f}_-, n_-, \tilde{m}_-$, $f_-^{\tau_0}$ and $\tilde{f}_-^\tau$, respectively. We set for $(t, \bar{x}, \xi) \in \Theta_\lambda \times \mathbb{R},$

$$
F_+(t, \bar{x}, \xi) = -n_1(\bar{x}_0) \cdot a(\xi)f_+^\tau(t, \bar{x}_0, \xi) + \beta'(\xi)\nabla_\bar{x}h(\bar{x}) \cdot \nabla_\bar{x}f_+^b(t, \bar{x}_0, \xi)
$$

and

$$
\tilde{F}_-(t, \bar{x}, \xi) = -n_1(\bar{x}_0) \cdot a(\xi)\tilde{f}_-^\tau(t, \bar{x}_0, \xi) + \beta'(\xi)\nabla_\bar{x}h(\bar{x}) \cdot \nabla_\bar{x}\tilde{f}_-^b(t, \bar{x}_0, \xi)
$$

where $\tilde{f}_b = \text{sgn}^-(\tilde{u}_b - \xi)$. For $\rho, \nu \in \mathbb{R}_+$ and $\epsilon = (\epsilon, \epsilon_d) \in \mathbb{R}_+^d$, set

$$
\begin{align*}
& f_+^{\rho,\nu,\epsilon} = (f_+ \times 1_{Q_{\lambda}}) * \gamma_{\rho,\nu,\epsilon}, \quad f_+^{\tau_0,\rho,\epsilon} = (f_+^{\tau_0} \times 1_{\Omega_{\lambda}}) * \gamma_{\rho,\epsilon}, \\
& F_+^{\rho,\nu,\epsilon} = (F_+ \times 1_{\Sigma_{\lambda}}) * \gamma_{\rho,\nu,\epsilon}, \quad m_+^{\rho,\nu,\epsilon} = (m_+ \times 1_{Q_{\lambda}}) * \gamma_{\rho,\nu,\epsilon}
\end{align*}
$$

and $n_+^{\rho,\nu,\epsilon} = (n_+ \times 1_{Q_{\lambda}}) * \gamma_{\rho,\nu,\epsilon}$. 

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As for \( \tilde{f}_{-}, \tilde{f}_{-}^{\tau_{0}}, \tilde{F}_{-}, \tilde{F}_{-}^{\tau_{0}}, \) etc., their regularizations \( \tilde{f}_{-}^{\eta,\mu,\delta}, \tilde{f}_{-}^{\tau_{0}\eta,\mu,\delta}, \tilde{F}_{-}^{\eta,\mu,\delta}, \) etc., are similarly defined in the same manner as above, but with different parameters \( \eta, \mu, \delta. \) Let \( \psi \in C_{c}^{\infty}([0, T) \times \mathbb{R}^{d+1})^{+} \) and apply (2.2) in Lemma 2.1 to the test function \( \psi^{\lambda} \ast \tilde{\gamma}_{\rho,\nu,\epsilon}, \) where \( \tilde{\gamma}_{\rho,\nu,\epsilon} \) is defined by \( \tilde{\gamma}_{\rho,\nu,\epsilon}(t, x, \xi) = \gamma_{\rho,\nu,\epsilon}(-t, -x, -\xi): \)

\[
\int_{\mathbb{R}^{d+2}} \left( f_{+}^{\rho,\nu,\epsilon} (\partial_{t} + a \cdot \nabla) \psi^{\lambda} - \beta' \nabla f_{+}^{\rho,\nu,\epsilon} \cdot \nabla \psi^{\lambda} \right) d\xi dt dx 
+ (f_{+}^{\tau_{0}\rho,\epsilon} \theta_{\rho,\nu} + F_{+}^{\rho,\nu,\epsilon}) \psi^{\lambda} \right) d\xi dt dx 
\geq \int_{\mathbb{R}^{d+2}} \partial_{\xi} \psi^{\lambda} d(m_{+}^{\rho,\nu,\epsilon} + n_{+}^{\rho,\nu,\epsilon}).
\]

On the other hand, we can regularize the equation satisfied by \( \tilde{f}_{-} \) by the same method and obtain for same \( \psi^{\lambda} \)'

\[
- \int_{\mathbb{R}^{d+2}} \left( \tilde{f}_{-}^{\eta,\mu,\delta} (\partial_{t} + a \cdot \nabla) \psi^{\lambda} + \beta' \nabla \tilde{f}_{-}^{\eta,\mu,\delta} \cdot \nabla \psi^{\lambda} \right) d\xi dt dx 
+ (\tilde{f}_{-}^{\tau_{0}\eta,\delta} \theta_{\eta,\mu} + \tilde{F}_{-}^{\eta,\mu,\delta}) \psi^{\lambda} \right) d\xi dt dx 
\geq - \int_{\mathbb{R}^{d+2}} \partial_{\xi} \psi^{\lambda} d(\tilde{m}_{-}^{\eta,\mu,\delta} + \tilde{n}_{-}^{\eta,\mu,\delta}).
\]

Now let us fix a test function \( \varphi(t, x) \in C_{c}^{\infty}([0, T) \times \mathbb{R}^{d})^{+} \). Apply (2.6) to \( \psi = -\tilde{f}_{-}^{\eta,\mu,\delta}(t, x, \xi) \varphi(t, x) \) and (2.7) to \( \psi = f_{+}^{\rho,\nu,\epsilon}(t, x, \xi) \varphi(t, x) \), and add the two equations together. After integrating by parts the left hand side of the resultant inequality, we obtain

\[
\int_{\mathbb{R}^{d+2}} \left( -f_{+}^{\rho,\nu,\epsilon} \tilde{f}_{-}^{\eta,\mu,\delta} (\partial_{t} + a \cdot \nabla + \beta' \Delta + 2\beta' \nabla f_{+}^{\rho,\nu,\epsilon} \cdot \nabla \tilde{f}_{-}^{\eta,\mu,\delta}) \varphi^{\lambda} d\xi dt dx 
- \int_{\mathbb{R}^{d+2}} \left( f_{+}^{\tau_{0}\rho,\epsilon} \theta_{\rho,\nu} f_{-}^{\eta,\mu,\delta} + f_{-}^{\tau_{0}\eta,\delta} \theta_{\eta,\mu} f_{+}^{\rho,\nu,\epsilon} 
+ F_{+}^{\rho,\nu,\epsilon} \tilde{f}_{-}^{\eta,\mu,\delta} + \tilde{f}_{-}^{\tau_{0}\eta,\delta} \tilde{f}_{+}^{\rho,\nu,\epsilon} \varphi^{\lambda} d\xi dt dx 
\geq - \int_{\mathbb{R}^{d+2}} \partial_{\xi} \tilde{f}_{-}^{\eta,\mu,\delta} \varphi^{\lambda} d(\tilde{m}_{+}^{\rho,\nu,\epsilon} + \tilde{n}_{+}^{\rho,\nu,\epsilon}).
\]

Notice that if \( \xi \in F \), then \( f_{+}(t, x, \xi) = \text{sgn}^{+}(\beta(u(t, x)) - \beta(\xi)) \) and hence \( \nabla f_{+}^{\rho,\nu,\epsilon} = [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\epsilon} \equiv \delta(\xi - u) \times 1_{Q} \ast \gamma_{\rho,\nu,\epsilon}. \) Similarly, we have \( \nabla \tilde{f}_{-}^{\eta,\mu,\delta} = [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]^{\eta,\mu,\delta}. \) On the other hand, it is easy to see that \( \partial_{\xi} f_{+}^{\rho,\nu,\epsilon} = -\delta(\xi - u)^{\rho,\nu,\epsilon} \equiv -[\delta(\xi - u) \times 1_{Q}] \ast \gamma_{\rho,\nu,\epsilon} \) and \( \partial_{\xi} \tilde{f}_{-}^{\eta,\mu,\delta} = -\delta(\xi - \tilde{u}) \eta_{\mu,\delta}. \) Noting also that \( m_{+} \) and \( m_{-} \) are nonnegative
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measures, we have

\[
\begin{align*}
\int_{\mathbb{R}^{d+2}} (-f_{+}^{\rho,\nu,\epsilon} \tilde{f}_{-}^{\eta,\mu,\delta} (\partial_{t} + a \cdot \nabla + \beta' \Delta \\
+ 2\beta' [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\epsilon} [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]^{\eta,\mu,\delta}) \varphi^\lambda d\xi dtdx \\
- \int_{\mathbb{R}^{d+2}} (f_{+}^{\tau_0 \rho,\epsilon} \theta_{\rho,\nu} \tilde{f}_{-}^{\eta,\mu,\delta}
+ \tilde{f}_{-}^{\tau_0 \eta,\delta} \theta_{\eta,\mu} f_{+}^{\rho,\nu,\epsilon}
+ F_{+}^{\rho,\nu,\epsilon} \tilde{f}_{-}^{\eta,\mu,\delta}
+ \tilde{F}_{-}^{\eta,\mu,\delta} f_{+}^{\rho,\nu,\epsilon}) \varphi^\lambda d\xi dtdx \\
\geq \int_{\mathbb{R}^{d+2}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda d\nu_{+}^{\rho,\nu,\epsilon} + \int_{\mathbb{R}^{d+2}} \delta(\xi - u)^{\rho,\nu,\epsilon} \varphi^\lambda d\tilde{\nu}_{-}^{\eta,\mu,\delta}
\end{align*}
\]

Let successively $\eta, \mu, \tilde{\delta}$ and $\delta_d$ go to $+0$:

\[
\int_{Q_{\lambda} \times \mathbb{R}} (-f_{+}^{\rho,\nu,\epsilon} \tilde{\tilde{f}}_{-}(\partial_{t} + a \cdot \nabla + \beta' \Delta \\
+ 2\beta' [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\epsilon} [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]) \varphi^\lambda d\xi dtdx \\
- \int_{Q_{\lambda} \times \mathbb{R}} (f_{+}^{\tau_0 \rho,\epsilon} \theta_{\rho,\nu} \tilde{\tilde{f}}_{-}
+ \tilde{\tilde{f}}_{-}^{\tau_0 \eta,\delta} \theta_{\eta,\mu} f_{+}^{\rho,\nu,\epsilon}
+ F_{+}^{\rho,\nu,\epsilon} \tilde{\tilde{f}}_{-}^{\eta,\mu,\delta}
+ \tilde{\tilde{F}}_{-}^{\eta,\mu,\delta} f_{+}^{\rho,\nu,\epsilon}) \varphi^\lambda d\xi dtdx \\
\geq \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda d\nu_{+}^{\rho,\nu,\epsilon} + \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - u)^{\rho,\nu,\epsilon} \varphi^\lambda d\tilde{\nu}_{-}^{\eta,\mu,\delta}.
\]

Here we used the fact that regularized functions equal zero at $t = 0$ and at the boundary. Then, let successively $\rho, \nu, \tilde{\epsilon}$ and $\epsilon_d$ go to $+0$ and use (2.2) in Lemma 2.1 to obtain

\[
\int_{Q_{\lambda} \times \mathbb{R}} (-f_{+}^{\rho,\nu,\epsilon} \tilde{f}_{-}(\partial_{t} + a \cdot \nabla + \beta' \Delta \\
+ 2\beta' [\delta(\xi - u) \nabla \beta(u)]^{\rho,\nu,\epsilon} [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]) \varphi^\lambda d\xi dtdx \\
- \int_{\Omega_{\lambda} \times \mathbb{R}} f_{+}^{\tau_0 \rho,\epsilon} \varphi^\lambda(0, \cdot) d\xi dx + \int_{\Sigma_{\lambda} \times \mathbb{R}} ((n_1 \cdot a)f_{+}^{b,\nu} \tilde{f}_{-}^{b}) \overline{\varphi}^\lambda d\xi d\tilde{x} \\
\geq \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda d\nu_{+} + \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - u)^{\rho,\nu,\epsilon} \varphi^\lambda d\tilde{\nu}_{-}.
\]

Next, let successively $\rho, \nu, \tilde{\epsilon}$ and $\epsilon_d$ go to $+0$ and then let successively $\eta, \mu, \tilde{\delta}$ and $\delta_d$ go to $+0$: For any weak* cluster point $\tilde{f}^\tau_-$ and for some
weak* cluster point $f^+_\tau$, we have

\[
\int_{Q_\lambda \times \mathbb{R}} \left(-f_+\tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta + 2\beta' \delta(\xi - u)\delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u}))\varphi^\lambda d\xi dt dx \right.
\]

\[
- \int_{\Omega_\lambda \times \mathbb{R}} f^+_{\tau_0} \tilde{f}^-_{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx
\]

\[
+ \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a)f^+_\tau \tilde{f}^-_\tau - \beta' (\nabla_x h \cdot \nabla_x (f^+_b \tilde{f}^-_b))\right) \overline{\varphi}^\lambda d\xi d\overline{x}
\]

\[
\geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u})\varphi^\lambda d\tilde{n}_- + \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - u)\varphi^\lambda d\tilde{n}_-
\]

Adding (2.8) and (2.9) yields

\[
\int_{Q_\lambda \times \mathbb{R}} \left(-f_+\tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta - 2\beta'(\xi)\delta(\xi - u)\delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u}))\varphi^\lambda d\xi dt dx \right.
\]

\[
- \int_{\Omega_\lambda \times \mathbb{R}} f^+_{\tau_0} \tilde{f}^-_{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx
\]

\[
+ \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a)f^+_\tau \tilde{f}^-_\tau - \frac{1}{2}\beta' \nabla_x h \cdot \nabla_x (f^+_b \tilde{f}^-_b)\right) \overline{\varphi}^\lambda d\xi d\overline{x}
\]

\[
\geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u})n_+ + \delta(\xi - u)\tilde{n}_-\varphi^\lambda d\xi dt dx.
\]

for some weak* cluster points $f^+_\tau$ and $\tilde{f}^-_\tau$. Since

\[2\beta'(\xi)\delta(\xi - u)\delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u}) \leq 1_f(\xi)\delta(\xi - u) \delta(\xi - \tilde{u})(|\nabla \beta(u)|^2 + |\nabla \beta(\tilde{u})|^2)
\]

\[= \delta(\xi - \tilde{u})n_+(t, x, \xi) + \delta(\xi - u)\tilde{n}_-(t, x, \xi),
\]

we arrive at

\[
- \int_{Q_\lambda \times \mathbb{R}} f_+\tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta)\varphi^\lambda d\xi dt dx \quad (2.10)
\]

\[
\geq \int_{Q_\lambda \times \mathbb{R}} f^+_{\tau_0} \tilde{f}^-_{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx - \int_{\Sigma_\lambda \times \mathbb{R}} \left((\mathbf{n}_1 \cdot a)f^+_\tau \tilde{f}^-_\tau - \frac{1}{2}\beta' \nabla_x h \cdot \nabla_x (f^+_b \tilde{f}^-_b)\right) \overline{\varphi}^\lambda d\xi d\overline{x}.
\]
We compute each term of (2.26). Firstly,

\[- \int_{Q_{\lambda} \times \mathbb{R}} f^+ \tilde{f}^- (\partial_t + a \cdot \nabla + \beta \Delta) \varphi^\lambda d\xi dtdx\]

\[= \int_{Q_{\lambda}} \left( (u - \tilde{u})^+ + F^+(u, \tilde{u}) \nabla \varphi^\lambda + (\beta(u) - \beta(\tilde{u}))^+ \Delta \varphi^\lambda \right) dtdx. \tag{2.11} \]

Secondly, by virtue of Lemma 2.2 and by using integration by parts one can calculate:

\[
\int_{\mathbb{R}} f^+_o \tilde{f}^-_o d\xi
\]

\[= \int_{-\infty}^{\tilde{u}_o} \nu_x^0 ([\xi, \infty)) \partial_\xi \tilde{m}_-^0 d\xi - \int_{-\infty}^{u_o \vee \tilde{u}_o} \nu_x^0 ([\xi, \infty)) \tilde{\nu}_x^0 ((-\infty, \xi]) d\xi
\]

\[= \nu_x^0 ([\tilde{u}_o, \infty)) \tilde{m}_-^0 (\cdot, \tilde{u}_o) + \int_{-\infty}^{\tilde{u}_o} \tilde{m}_-^0 d\nu_x^0 - \int_{-\infty}^{\tilde{u}_o} \tilde{m}_-^0 d\tilde{\nu}_x^0 + \int_{u_o \vee \tilde{u}_o}^{\infty} \partial_\xi \tilde{m}_+^0 d\tilde{\nu}_x^0
\]

\[\geq - \int_{\tilde{u}_o}^{u_o \vee \tilde{u}_o} d\xi = -(u - \tilde{u}_o)^+. \]

Here we used the fact that \(d \nu_x^0 ([\xi, \infty))/d\xi = -d \nu_x^0 (\xi)\) and \(d \tilde{\nu}_x^0 ((-\infty, \xi])/d\xi = d \tilde{\nu}_x^0 (\xi)\). Thus we have

\[
\int_{\Omega_{\lambda} \times \mathbb{R}} f^+_o \tilde{f}^-_o \varphi^\lambda (0, \cdot) d\xi dx \geq - \int_{\Omega_{\lambda}} (u_o - \tilde{u}_o)^+ \varphi^\lambda (0, \cdot) dx. \tag{2.12} \]

Finally, we calculate analogously the boundary term by using Lemma 2.4:

\[
\int_{\mathbb{R}} (n_1 \cdot a) f^+_\tau \tilde{f}^-_\tau d\xi
\]

\[= - \int_{-\infty}^{\tilde{u}_b} \partial_\xi \tilde{m}_+^b \nu_{t,y}^\tau ([\xi, \infty)) d\xi - \int_{u_b \vee \tilde{u}_b}^{\infty} (n_1 \cdot a) \nu_{t,y}^\tau ([\xi, \infty)) \tilde{\nu}_{t,y}^\tau ((-\infty, \xi]) d\xi
\]

\[+ \int_{u_b \vee \tilde{u}_b}^{\infty} \partial_\xi m_+^b \nu_{t,y}^\tau ((-\infty, \xi]) d\xi
\]

\[\leq M \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} d\xi = M (u_b - \tilde{u}_b)^+, \]

where \(y\) stands for \(\bar{x}_0\) and we used the fact that \(d \nu_{t,y}^\tau ([\xi, \infty))/d\xi = -d \nu_{t,y}^\tau (\xi)\) and \(d \tilde{\nu}_{t,y}^\tau ((-\infty, \xi])/d\xi = d \tilde{\nu}_{t,y}^\tau (\xi)\) as well as the fact that \(m_+^b \geq\)
0 for $\xi \geq u_b$ and $\tilde{m}_-^b \geq 0$ for $\xi \leq \tilde{u}_b$ by virtue of (2.21) and the corresponding inequality associated with $\tilde{u}$, respectively. This implies
\[
\int_{\Sigma_{\lambda} \times \mathbb{R}} (n_1 \cdot a) f_+^b \tilde{f}_-^b \varphi^\lambda d\xi dtd\overline{x} \leq M \int_{\Sigma_{\lambda}} (u_b - \tilde{u}_b)^+ \varphi^\lambda dtd\overline{x}.
\] (2.13)

Moreover
\[
\int_{\Sigma_{\lambda} \times \mathbb{R}} \beta'((\xi) \nabla_{\overline{x}} h(\overline{x}) \cdot \nabla_{\overline{x}} (f_+^b \tilde{f}_-^b) \varphi^\lambda d\xi dtd\overline{x} = - \int_{\Sigma_{\lambda}} (\beta(u_b) - \beta(\tilde{u}_b))^+ (\nabla_{\overline{x}} h(\overline{x}) \cdot \nabla_{\overline{x}} \varphi^\lambda) dtd\overline{x}
\] (2.14)
\[
\geq \int_{\Sigma_{\lambda}} (\beta(u_b) - \beta(\tilde{u}_b))^+ (-L \varphi^\lambda + \nabla_{\overline{x}} h(\overline{x}) \cdot \nabla_{\overline{x}} \varphi^\lambda) dtd\overline{x}.
\]

Combining (2.10) with (2.11) through (2.14) and choosing appropriate test functions $\varphi'$s, we arrive at the estimate
\[
\frac{1}{T} \int_{Q_{\lambda}} (u - \tilde{u})^+ dtdx \\
\leq \int_{\Omega_{\lambda}} (u_0 - \tilde{u}_0)^+ dx + M \int_{\Sigma_{\lambda}} (u_b - \tilde{u}_b)^+ dtd\overline{x} + \frac{L}{2} \int_{\Sigma_{\lambda}} (\beta(u_b) - \beta(\tilde{u}_b))^+ dtd\overline{x}.
\]

By summing over $i = 0, 1, \cdots, N$, we obtain the desired estimate (1.9) and the proof of Theorem is complete.

REFERENCES

Comparison theorem for degenerate PDEs


