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Kyoto University
A KINETIC APPROACH TO A COMPARISON THEOREM FOR DEGENERATE PARABOLIC EQUATIONS

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1. STATEMENT OF THE RESULT.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^d$ and $T \in (0, +\infty]$. Let $Q$ denote the set $(0, T) \times \Omega$, $\partial \Omega$ the boundary of $\Omega$, $\mathbf{n}(\overline{x})$ the outward unit normal to $\Omega$ at a point $\overline{x} \in \partial \Omega$ and $\Sigma$ the set $(0, T) \times \partial \Omega$. We consider the following parabolic-hyperbolic problem:

$$\partial_t u + \text{div} A(u) - \Delta \beta(u) = 0 \quad \text{in} \quad Q$$

(1.1)

with the initial condition:

$$u(0, x) = u_0(x), \quad x \in \Omega,$$

(1.2)

and the boundary condition:

$$u(t, x) = u_b(t, x), \quad (t, x) \in \Sigma,$$

(1.3)

where the flux function $A$ belongs to $C^1(\mathbb{R})$ and the function $\beta$ is non-decreasing and Lipschitz continuous. This monotonicity assumption of $\beta$ allows us some degenerate diffusion cases which appear in many interesting models, for example, filtration problems in porous media [2,5,8].

In the nondegenerate case (in which the function $\beta$ is strictly increasing), the problem (1.1) is of parabolic type and hence the existence and uniqueness of solutions are well known. In the case where $\beta' \equiv 0$, the problem (1.1) being a nonlinear hyperbolic problem, the uniqueness
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of weak solutions is not ensured, and one must consider a notion of entropy solution, relying on the notion of boundary entropy-flux pairs to recover uniqueness (see [11,16]). When \( \beta \) is merely a nondecreasing function, in the case of homogeneous boundary data, i.e., \( u_b \equiv 0 \), Carrillo [3] succeeded in proving the uniqueness of entropy solutions by mainly using the dedoubling variable technique developed by Kružkov [11]. The equivalence of entropy solutions and weak solutions is also considered in [10]. In the case of nonhomogeneous boundary data, i.e., \( u_b \equiv 0 \), Carrillo [3] succeeded in proving the uniqueness of entropy solutions by mainly using the dedoubling variable technique developed by Kružkov [11].

On the other hand Perthame [12,17] proved the uniqueness of entropy solutions to the Cauchy problem of the conservation law (in which \( \beta' \equiv 0 \) and \( \Omega = \mathbb{R}^d \)) by using the kinetic formulation which is introduced by Lions, Perthame and Tadmor [12], without relying on the dedoubling variable technique. Imbert and Vovelle [9] developed analogous techniques for conservation laws with boundary conditions, proved the Comparison Theorem for entropy sub- and supersolutions, and applied their results to the BGK-like model. This technique was also applied in [6] to study the parabolic approximation of a multidimensional conservation law with initial and boundary conditions.

The purpose of this note is to give a comparison result for their sub- and supersolutions by using kinetic techniques. Although the \( L^1 \) contractivity and, therefore, uniqueness of entropy weak solutions has been obtained, it would seem that any comparison theorem for those solutions is not proven.

According to [14] we introduce the definition of entropy sub- and supersolution.

Define

\[
\text{sgn}^+(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \leq 0, \end{cases} \quad \text{and} \quad \text{sgn}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \geq 0, \end{cases}
\]

and \( r^\pm = \text{sgn}^\pm(r)r \).

**Definition 1.1.** A function \( u \) of \( L^1(Q) \) is said to be a weak solution of the problem (1.1) - (1.3) if it satisfies:

\[
\beta(u) - \beta(u_b) \in L^2(0,T; H_0^1(\Omega)), \quad A(u) \in L^1(Q)^d \quad \text{and} \quad \oint_{Q} u \varphi_t + (A(u) - \nabla \beta(u)) \cdot \nabla \varphi dx dt + \oint_{\Omega} u_0 \varphi(0,x) dx = 0
\]

for any \( \varphi \in C_c^\infty([0,T) \times \Omega) \).
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**Definition 1.2.** Let $u \in L^\infty(Q)$. $u$ is said to be an entropy subsolution of (1.1) - (1.3) if it is a weak solution and satisfies:

$$
\int_Q (u - \kappa)^+ \partial_t \varphi + (\mathcal{F}(u, \kappa) - \nabla (\beta(u) - \beta(\kappa))^+) \cdot \nabla \varphi dxdt
+ \int_\Omega (u_0 - \kappa)^+ \varphi(0, x) dx + M \int_\Sigma (u_b - \kappa)^+ \varphi d\sigma dt \geq 0
$$

(1.5)

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C^\infty_c([0, T) \times \mathbb{R}^d)^+$ such that $\text{sgn}^+(\beta(u_b) - \beta(\kappa)) \varphi = 0$ a.e. on $\Sigma$.

$u$ is said to be an entropy supersolution if (1.7) is replaced by

$$
\int_Q (u - \kappa)^- \partial_t \varphi + (\mathcal{F}(u, \kappa) - \nabla (\beta(u) - \beta(\kappa))^-) \cdot \nabla \varphi dxdt
+ \int_\Omega (u_0 - \kappa)^- \varphi(0, x) dx + M \int_\Sigma (u_b - \kappa)^- \varphi d\sigma dt \geq 0
$$

(1.6)

for any $\kappa \in \mathbb{R}$ and any $\varphi \in C^\infty_c([0, T) \times \mathbb{R}^d)^+$ such that $\text{sgn}^-(\beta(u_b) - \beta(\kappa)) \varphi = 0$ a.e. on $\Sigma$. Here $C^\infty_c([0, T) \times \mathbb{R}^d)^+$ is the set of nonnegative functions in $C^\infty_c([0, T) \times \mathbb{R}^d)$.

We also set

$$
M = \sup\{|A'(r)|; |r| \leq \max\{|u_0||_{L^\infty(\Omega)}, |u_b||_{L^\infty(\Sigma)}\}
$$

(1.7)

and

$$
L = \max_{1 \leq i \leq N} ||\Delta_x h_i(T_i \overline{x})||_{L^\infty(\Sigma_{\lambda_i})}.
$$

(1.8)

We are now in a position to state the main theorem which obviously extends the $L^1$ contractive property for entropy solutions

**Theorem** Assume that the following conditions hold:

(A1) $\Omega$ is a bounded open subset of $\mathbb{R}^d$ whose boundary $\partial \Omega$ is $C^2$, $A \in C^1(\mathbb{R}, \mathbb{R})$ and $\beta: \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function.

(A2) $u_0 \in L^\infty(\Omega)$ and $u_b \in L^\infty(\Sigma)$.

Let $u \in L^\infty(Q)$ be an entropy subsolution of (1.1) - (1.3) with data $(u_0, u_b)$ and let $\tilde{u}$ be an entropy supersolution of (1.1) - (1.3) with data $(\tilde{u}_0, \tilde{u}_b)$. Then we have

$$
\frac{1}{T} \int_0^T \int_\Omega (u(t, x) - \tilde{u}(t, x))^+ dxdt
\leq \int_\Omega (u_0(x) - \tilde{u}_0(x))^+ dx + M \int_0^T \int_{\partial \Omega} (u_b(t, x) - \tilde{u}_b(t, x))^+ d\sigma dt
+ \frac{L}{2} \int_0^T \int_{\partial \Omega} (\beta(u_b(t, x)) - \beta(\tilde{u}_b(t, x)))^+ d\sigma dt.
$$

(1.9)
2. **Sketch of Proof.**

The semi-Kružkov entropies are the convex functions defined by

$$\eta_{k}^{\pm}(r) = (r - k)^{\pm}, \quad k \in \mathbb{R},$$

while the corresponding entropy flux are the function defined by

$$\mathcal{F}^{\pm}(r, k) = \text{sgn}^{\pm}(r - k)(A(r) - A(k)).$$

For a function \(u \in L^{\infty}(Q)\) and \(\xi \in \mathbb{R}\) we set

$$f_{\pm}(t, x, \xi) = \text{sgn}^{\pm}(u(t, x) - \xi).$$

We assume that \(\Omega\) is a \(C^{2}\) bounded open subset in \(\mathbb{R}^{d}\). Thus, we can find a finite open cover \(\{B_{i}\}_{i=0}^{N}\) of \(\overline{\Omega}\) and a partition of unity \(\{\lambda\}_{i=0}^{N}\) on \(\overline{\Omega}\) subordinate to \(\{B_{i}\}_{i=0}^{N}\) such that, for \(i \geq 1\), up to a change of coordinates represented by an orthogonal matrix \(T_{i}\), the set \(\Omega \cap B_{i}\) is the epigraph of a \(C^{2}\) function \(h_{i}: \mathbb{R}^{d-1} \rightarrow \mathbb{R}\), that is to say:

$$\Omega_{\lambda_{i}} \cap B_{i} = \{x \in B_{i}; (T_{i}x)_{d} > h_{i}(\overline{T_{i}x})\}$$

and

$$\partial\Omega_{\lambda_{i}} = \partial\Omega \cap B_{i} = \{x \in B_{i}; (T_{i}x)_{d} = h_{i}(\overline{T_{i}x})\},$$

where \(x = (\overline{x}, x_{d}) \in \mathbb{R}^{d}\) and \(\overline{x} = (x_{1}, \cdots, x_{d-1})\). For simplicity we will drop the index \(i\) and we suppose that the change of coordinates is trivial: \(Y_{i} = Id\). We also write \(Q_{\lambda} = (0, T) \times \Omega_{\lambda}\), \(\Sigma_{\lambda} = (0, T) \times \partial\Omega_{\lambda}\), \(\Pi_{\lambda} = \{\overline{x}; x \in \text{supp}(\lambda) \cap \Omega\}\) and \(\Theta_{\lambda} = (0, T) \times \Pi_{\lambda}\). We denote by \(n(\overline{x})\) the outward unit normal to \(\Omega_{\lambda}\) at a point \((\overline{x}, h(\overline{x}))\) of \(\partial\Omega_{\lambda}\) and by \(d\sigma(\overline{x})\) the \((d-1)\)-dimensional area element in \(\partial\Omega_{\lambda}\):

$$n(\overline{x}) = (1 + |\nabla_{\overline{x}}h(\overline{x})|^{2})^{-1/2}(\nabla_{\overline{x}}h(\overline{x}), -1),$$

$$d\sigma(\overline{x}) = (1 + |\nabla_{\overline{x}}h(\overline{x})|^{2})^{1/2}d\overline{x}.$$

To regularize the functions, for small \(\rho, s > 0\) let us consider a smooth function \(\theta_{\rho, s}: \mathbb{R} \rightarrow \mathbb{R}^{+}\) such that \(\text{supp} \theta_{\rho, s} \subset [\rho s/2, (1 + \rho)s]\), \(\theta_{\rho, s}(r) = s^{-1}\) for \(\rho s \leq r \leq s\) and \(\int_{\mathbb{R}} \theta_{\rho, s}(r)dr = 1\). Then, for \(\nu > 0\) and \(\epsilon = (\epsilon_{1}, \cdots, \epsilon_{d}) \in (\mathbb{R}^{+})^{d}\), we set \(\gamma_{\rho, \epsilon}(x) = \prod_{i=1}^{d} \theta_{\rho, \epsilon_{i}}(x_{i})\) and \(\gamma_{\rho, \nu, \epsilon}(t, x) = \theta_{\rho, \nu}(t)\gamma_{\rho, \epsilon}(x)\).

For simplicity, we will also use the following notations:

$$n_{1} = \sqrt{1 + |\nabla_{\overline{x}}h(\overline{x})|^{2}} \quad n,$$

$$\overline{x}_{r} = (\overline{x}, h(\overline{x}) + r) \quad \text{for} \quad \overline{x} = (x_{1}, \cdots, x_{d-1}),$$
ψ^λ stands for ψλ and \( \overline{\psi} \) denotes the restriction of ψ to \( \Sigma \times \mathbb{R}_\xi \), i.e.,
\[ \overline{\psi}(t, \overline{x}, \xi) = \psi(t, \overline{x}, h(\overline{x}, \xi)) \]
where ψ is a function on \([0, T) \times \mathbb{R}^{d+1}\) and λ is an element of the partition of unity \( \{\lambda_i\}_{i=0}^{N}\). Moreover we set
\[ s \vee t = \max\{s, t\} \quad \text{and} \quad s \wedge t = \min\{s, t\}. \]

The proof of the theorem will follow from the following three lemmas whose proofs will be given in the forthcoming paper.

**Lemma 2.1.** Let \( u \) be an entropy subsolution with data \((u_0, u_b)\) and let \( \lambda \) be an element of the partition of unity \( \{\lambda_i\}_{i=0}^{N}\). Then we have:

(a) There exists \( f^\tau_+ \in L^\infty(\Omega \times \mathbb{R}) \) such that
\[
\lim_{s \to 0^+} \int_{\Omega \times \mathbb{R}} \left[ \frac{1}{s} \int_0^s f_+(t, x, \xi) dt \right] \phi \, dx d\xi = \int_{\Omega \times \mathbb{R}} f^\tau_+ \phi \, dx d\xi
\]
for any \( \phi \in C_c^\infty(\Omega \times \mathbb{R}) \).

(b) For any \( \psi \in C_c^\infty([0, T) \times \mathbb{R}^{d+1})^+ \) and any weak* cluster point \( f^\tau_+ \) of \( \frac{1}{s} \int_0^s f_+(t, \overline{x}_f, \xi) dr \) as \( s \to 0^+ \) in \( L^\infty(\Theta_\lambda \times \mathbb{R}) \), we have
\[
\int_{Q_\lambda \times \mathbb{R}} (f_+ (\partial_t + a \cdot \nabla) \psi^\lambda - \beta' \nabla f_+ \cdot \nabla \psi^\lambda) dtdx d\xi
\]
\[
+ \int_{\Omega \times \mathbb{R}} f^\tau_+ \psi^\lambda(0, x) dx d\xi
\]
\[
+ \int_{\Theta_\lambda \times \mathbb{R}} \beta' (\nabla h(\overline{x}) \cdot \nabla f_+^b) \overline{\psi} \, dtd\overline{x} d\xi
\]
\[
+ \int_{\Theta_\lambda \times \mathbb{R}} (-n_1 \cdot a) f^\tau_+ \overline{\psi} \, dtd\overline{x} d\xi
\]
\[
\geq \int_{Q_\lambda \times \mathbb{R}} \partial_\xi \psi^\lambda d(m_+ + n_+).
\]

**Lemma 2.2.** There exist families of probability measures \( \{\nu^\tau_0 \}_{x \in \Omega} \) and \( \{\tilde{\nu}^\tau_0 \}_{x \in \Omega} \) on \( \mathbb{R}_\xi \), called Young measuers, supported in \((-\infty, ||u||_{L^\infty}) \) and \((-||\tilde{u}||_{L^\infty}, \infty))\), respectively, and nonnegative functions \( m^0_+(x, \xi) \) and \( \tilde{m}^0_-(x, \xi) \) defined on \( \Omega \times \mathbb{R}_\xi \) such that
\[
m^0_+, \tilde{m}^0_- \in C(\mathbb{R}_\xi; w-\mathcal{M}^+(\Omega)),
\]
\[
\lim_{\xi \to \infty} m^0_+(x, \xi) = \lim_{\xi \to -\infty} \tilde{m}^0_-(x, \xi) = 0 \quad \text{for a.e. } x \in \Omega,
\]
\[
f^\tau_+(x, \xi) = \nu^\tau_0([\xi, \infty)) = \partial_\xi m^0_+(x, \xi) + \text{sgn}^+(u_0(x) - \xi)
\]
\[
\text{and}
\]
\[
\tilde{f}^\tau_-(x, \xi) = -\tilde{\nu}^\tau_0((-\infty, \xi]) = \partial_\xi \tilde{m}^0_-(x, \xi) + \text{sgn}^-(\tilde{u}_0(x) - \xi).
\]
Lemma 2.3. Let $\lambda$ be an element of the partition of unity $\{\lambda_i\}_{i=0}^{N}$ and let $f_+^t$ and $\tilde{f}_-^t$ be weak* cluster point of $\frac{1}{s} \int_{0}^{s} f_+(t, \overline{x}_r, \xi)dr$ and $\frac{1}{s} \int_{0}^{s} \tilde{f}_-(t, \overline{x}_r, \xi)dr$, respectively, as $s \to +0$, in $L^\infty(\Theta_\lambda \times \mathbb{R})$. There exist Young measures $\{\nu^\ell_{t,y}\}_{(t,y) \in \Sigma}$ and $\{\tilde{\nu}^\ell_{t,y}\}_{(t,y) \in \Sigma}$ on $\mathbb{R}_{\xi}$, supported in $(-\infty, ||u||_{L^\infty}]$ and $[-||\tilde{u}||_{L^\infty}, \infty)$, respectively, and nonnegative functions $m_+^b(t, y, \xi)$ and $\tilde{m}_-^b(t, y, \xi)$ defined on $\Sigma \times \mathbb{R}_{\xi}$ such that

$$\lim_{\xi \to -\infty} m_+^b(t, y, \xi) = \lim_{\xi \to +\infty} \tilde{m}_-^b(t, y, \xi) = 0 \quad \text{for a.e.} \quad (t, y) \in \Sigma,$$

$$f_+^t(t, y, \xi) = \nu^r_{t,y}([\xi, \infty)), \quad \tilde{f}_-^t = -\tilde{\nu}^r_{t,y}((\infty, \xi]),$$

$$(-a \cdot n_1)f_+^t = \partial_\xi m_+^b + Ms\text{gn}^+(u_b - \xi)$$

$$(-a \cdot n_1)\tilde{f}_-^t = \partial_\xi \tilde{m}_-^b + Ms\text{gn}^-(\tilde{u}_b - \xi),$$

$$\int_{\Theta_\lambda} m_+^b(t, \overline{x}_0, \xi)\overline{\varphi}_+^\lambda(t, \overline{x}_0)dtd\overline{x} \geq 0 \quad (2.5)$$

for any $\overline{\varphi} \in C(\Sigma)^+$ satisfying $\text{sgn}^+(\beta(u_b) - \beta(\xi))\overline{\varphi} = 0$ a.e. on $\Sigma$ and

$$\int_{\Theta_\lambda} \tilde{m}_-^b(t, \overline{x}_0, \xi)\overline{\varphi}_-^\lambda(t, \overline{x}_0)dtd\overline{x} \geq 0$$

for any $\overline{\varphi} \in C(\Sigma)^+$ satisfying $\text{sgn}^-(\beta(\tilde{u}_b) - \beta(\xi))\overline{\varphi} = 0$ a.e. on $\Sigma$.

We continue the proof of Theorem. Let $f_+, n_+$ and $m_+$ be the functions defined for $u$ as above. $f_+^{t_0}$ denotes the time kinetic traces and $f_+^t$ a cluster point of space kinetic traces associated with $u$. The corresponding ones associated with $\tilde{u}$ will be denoted by $\tilde{f}_-, \tilde{n}_-, \tilde{m}_-$, $\tilde{f}_{-t_0}$ and $\tilde{f}_-^t$, respectively. We set for $(t, \bar{x}, \xi) \in \Theta_\lambda \times \mathbb{R},$

$$F_+(t, \bar{x}, \xi) = -n_1(\bar{x}_0) \cdot a(\xi)f_+^t(t, \bar{x}_0, \xi) + \beta'(\xi)\nabla_\overline{x} h(\bar{x}) \cdot \nabla_\overline{x} f_+^b(t, \bar{x}_0, \xi)$$

and

$$\tilde{F}_-(t, \bar{x}, \xi) = -n_1(\bar{x}_0) \cdot a(\xi)\tilde{f}_-^t(t, \bar{x}_0, \xi) + \beta'(\xi)\nabla_\overline{x} h(\bar{x}) \cdot \nabla_\overline{x} \tilde{f}_-^b(t, \bar{x}_0, \xi)$$

where $\tilde{f}_b^b = \text{sgn}^-(\tilde{u}_b - \xi)$. For $\rho, \nu \in \mathbb{R}_+$ and $\epsilon = (\overline{\epsilon}, \epsilon_d) \in \mathbb{R}_+^d$, set

$$f_+^{\rho,\nu,\epsilon} = (f_+ \times 1_{Q_\lambda}) \ast \gamma_{\rho,\nu,\epsilon}, \quad f_+^{t_0,\rho,\epsilon} = (f_+^{t_0} \times 1_{Q_\lambda}) \ast \gamma_{\rho,\epsilon},$$

$$F_+^{\rho,\nu,\epsilon} = (F_+ \times 1_{Q_\lambda}) \ast \gamma_{\rho,\nu,\epsilon}, \quad m_+^{\rho,\nu,\epsilon} = (m_+ \times 1_{Q_\lambda}) \ast \gamma_{\rho,\nu,\epsilon}$$

and

$$n_+^{\rho,\nu,\epsilon} = (n_+ \times 1_{Q_\lambda}) \ast \gamma_{\rho,\nu,\epsilon}.$$
As for \( \tilde{f}_-, \tilde{f}_-^{\tau_0}, \tilde{F}_- \), etc., their regularizations \( \tilde{f}_-^{\eta,\mu,\delta}, \tilde{f}_-^{\tau_0\eta,\delta}, \tilde{F}_-^{\eta,\mu,\delta} \), etc. are similarly defined in the same manner as above, but with different parameters \( \eta, \mu, \delta \). Let \( \psi \in C_c^\infty([0, T) \times \mathbb{R}^{d+1})^+ \) and apply (2.2) in Lemma 2.1 to the test function \( \psi^\lambda \ast \tilde{\gamma}_{\rho,\nu,\epsilon} \), where \( \tilde{\gamma}_{\rho,\nu,\epsilon} \) is defined by \( \tilde{\gamma}_{\rho,\nu,\epsilon}(t, x, \xi) = \gamma_{\rho,\nu,\epsilon}(-t, -x, -\xi) \):

\[
\int_{\mathbb{R}^{d+2}} \left( \tilde{f}_+^{\rho,\nu,\epsilon}(\partial_t + a \cdot \nabla)\psi^\lambda - \beta' \nabla \tilde{f}_+^{\rho,\nu,\epsilon} \cdot \nabla \psi^\lambda \right) + \beta' \nabla \tilde{f}_+^{\rho,\nu,\epsilon} \cdot \nabla \psi^\lambda \right) d\xi dt dx 
\geq \int_{\mathbb{R}^{d+2}} \partial_\xi \psi^\lambda d(m_+^{\rho,\nu,\epsilon} + n_+^{\rho,\nu,\epsilon}).
\]

On the other hand we can regularize the equation satisfied by \( \tilde{f}_- \) by the same method and obtain for same \( \psi^\lambda \)’s,

\[
- \int_{\mathbb{R}^{d+2}} \left( \tilde{f}_-^{\eta,\mu,\delta}(\partial_t + a \cdot \nabla)\psi^\lambda + \beta' \nabla \tilde{f}_-^{\eta,\mu,\delta} \cdot \nabla \psi^\lambda \right) + \beta' \nabla \tilde{f}_-^{\eta,\mu,\delta} \cdot \nabla \psi^\lambda \right) d\xi dt dx 
\geq - \int_{\mathbb{R}^{d+2}} \partial_\xi \psi^\lambda d(\tilde{m}_-^{\eta,\mu,\delta} + \tilde{n}_-^{\eta,\mu,\delta}).
\]

Now let us fix a test function \( \varphi(t, x) \in C_c^\infty([0, T) \times \mathbb{R}^d)^+ \). Apply (2.6) to \( \psi = -\tilde{f}_-^{\eta,\mu,\delta}(t, x, \xi)\varphi(t, x) \) and (2.7) to \( \psi = f_+^{\rho,\nu,\epsilon}(t, x, \xi)\varphi(t, x) \), and add the two equations together. After integrating by parts the left hand side of the resultant inequality, we obtain

\[
\int_{\mathbb{R}^{d+2}} \left( f_+^{\rho,\nu,\epsilon}(\partial_t + a \cdot \nabla)\varphi^\lambda + f_+^{\rho,\nu,\epsilon} d\xi dtx 
- \int_{\mathbb{R}^{d+2}} \left( f_+^{\rho,\nu,\epsilon}(\partial_t + a \cdot \nabla)\varphi^\lambda + f_+^{\rho,\nu,\epsilon} d\xi dtx 
\right) + \beta' \nabla \tilde{f}_-^{\eta,\mu,\delta} \cdot \nabla \psi^\lambda \right) d\xi dt dx 
\geq - \int_{\mathbb{R}^{d+2}} \partial_\xi \tilde{f}_-^{\eta,\mu,\delta} d(\tilde{m}_-^{\eta,\mu,\delta} + \tilde{n}_-^{\eta,\mu,\delta}).
\]

Notice that if \( \xi \in F \), then \( f_+^\lambda(t, x, \xi) = \text{sgn}^+(\beta(u(t, x)) - \beta(\xi)) \) and hence \( \nabla f_+^{\rho,\nu,\epsilon} = [\delta(\xi - u(t, x, \xi)) \nabla \beta(u)]^{\rho,\nu,\epsilon} = \delta(\xi - u) \times 1_Q \ast \gamma_{\rho,\nu,\epsilon} \). Similarly, we have \( \nabla \tilde{f}_-^{\eta,\mu,\delta} = [\delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})]^{\eta,\mu,\delta} \). On the other hand, it is easy to see that \( \partial_\xi f_+^{\rho,\nu,\epsilon} = -\delta(\xi - u)^{\rho,\nu,\epsilon} \equiv -[\delta(\xi - u) \times 1_Q] \ast \gamma_{\rho,\nu,\epsilon} \) and \( \partial_\xi \tilde{f}_-^{\eta,\mu,\delta} = -\delta(\xi - \tilde{u})^{\eta,\mu,\delta} \). Noting also that \( m_+ \) and \( \tilde{m}_- \) are nonnegative
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measures, we have

$$\int_{\mathbb{R}^{d+2}} (-f_+^{\rho,\nu,\epsilon} \tilde{f}_-^{\eta,\mu,\delta} (\partial_t + a \cdot \nabla + \beta' \Delta \\
+ 2\beta'[\delta(\xi - u)\nabla \beta(u)]^{\rho,\nu,\epsilon} \delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})) \varphi^\lambda d\xi dtdx \\
- \int_{\mathbb{R}^{d+2}} (f_+^{\tau_0,\rho,\epsilon} \theta_{\rho,\nu} f_-^{\eta,\mu,\delta} + f_+^{\tau_0,\rho,\epsilon} \theta_{\eta,\mu} f_-^{\rho,\nu,\epsilon} \\
+ \beta' \delta(\xi-u) \nabla \beta(u) f_+^{\rho,\nu,\epsilon} + \beta' \delta(\xi-\tilde{u}) \nabla \beta(\tilde{u}) f_+^{\eta,\mu,\delta}) \varphi^\lambda d\xi dtdx \\
\geq \int_{\mathbb{R}^{d+2}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda dn_+^{\rho,\nu,\epsilon} + \int_{\mathbb{R}^{d+2}} \delta(\xi - u)^{\rho,\nu,\epsilon} \varphi^\lambda d\tilde{n}_-^{\eta,\mu,\delta}$$

Let successively $\eta, \mu, \tilde{\delta}$ and $\delta_d$ go to +0:

$$\int_{Q_{\lambda} \times \mathbb{R}} (-f_+^{\rho,\nu,\epsilon} \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \\
+ 2\beta'[\delta(\xi - u)\nabla \beta(u)]^{\rho,\nu,\epsilon} \delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})) \varphi^\lambda d\xi dtdx \\
- \int_{Q_{\lambda} \times \mathbb{R}} (f_+^{\tau_0,\rho,\epsilon} \theta_{\rho,\nu} f_-^{\eta,\mu,\delta} + f_+^{\tau_0,\rho,\epsilon} \theta_{\eta,\mu} f_-^{\rho,\nu,\epsilon} \\
+ \beta' \delta(\xi-u) \nabla \beta(u) f_+^{\rho,\nu,\epsilon} + \beta' \delta(\xi-\tilde{u}) \nabla \beta(\tilde{u}) f_+^{\eta,\mu,\delta}) \varphi^\lambda d\xi dtdx \\
\geq \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda dn_+^{\rho,\nu,\epsilon} + \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - u)^{\rho,\nu,\epsilon} \varphi^\lambda d\tilde{n}_-^{\eta,\mu,\delta}$$

Here we used the fact that regularized functions equal zero at $t = 0$ and at the boundary. Then, let successively $\rho, \nu, \tilde{\epsilon}$ and $\epsilon_d$ go to +0 and use (2.2) in Lemma 2.1 to obtain

$$\int_{Q_{\lambda} \times \mathbb{R}} (-f_+^{\rho,\nu,\epsilon} \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \\
+ 2\beta'[\delta(\xi - u)\nabla \beta(u)]^{\rho,\nu,\epsilon} \delta(\xi - \tilde{u}) \nabla \beta(\tilde{u})) \varphi^\lambda d\xi dtdx \\
- \int_{Q_{\lambda} \times \mathbb{R}} (f_+^{\tau_0,\rho,\epsilon} \theta_{\rho,\nu} f_-^{\eta,\mu,\delta} + f_+^{\tau_0,\rho,\epsilon} \theta_{\eta,\mu} f_-^{\rho,\nu,\epsilon} \\
+ \beta' \delta(\xi-u) \nabla \beta(u) f_+^{\rho,\nu,\epsilon} + \beta' \delta(\xi-\tilde{u}) \nabla \beta(\tilde{u}) f_+^{\eta,\mu,\delta}) \varphi^\lambda d\xi dtdx \\
\geq \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - \tilde{u})^{\eta,\mu,\delta} \varphi^\lambda dn_+^{\rho,\nu,\epsilon} + \int_{Q_{\lambda} \times \mathbb{R}} \delta(\xi - u)^{\rho,\nu,\epsilon} \varphi^\lambda d\tilde{n}_-^{\eta,\mu,\delta}$$

Next, let successively $\rho, \nu, \tilde{\epsilon}$ and $\epsilon_d$ go to +0 and then let successively $\eta, \mu, \tilde{\delta}$ and $\delta_d$ go to +0: For any weak* cluster point $\tilde{f}_-^\tau$ and for some
weak* cluster point $f_+^\tau$, we have

\begin{align*}
\int_{Q_\lambda \times \mathbb{R}} & \left( -f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \
+ 2\beta' \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u})) \right) \varphi^\lambda d\xi dt dx \\
& - \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx \\
& + \int_{\Sigma_\lambda \times \mathbb{R}} \left( (n_1 \cdot a) f_+^{\tau} \tilde{f}_-^{\tau} - \beta' (\nabla_x h \cdot \nabla_x \tilde{f}_-^{\tau}) f_+^{\tau} \right) \overline{\varphi}^\lambda d\xi d\overline{t} \\
& \geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u}) \varphi^\lambda dn_+ + \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - u) \varphi^\lambda d\tilde{n}_-.
\end{align*}

Adding (2.8) and (2.9) yields

\begin{align*}
\int_{Q_\lambda \times \mathbb{R}} & \left( -f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta \
+ 2\beta' \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u})) \right) \varphi^\lambda d\xi dt dx \\
& - \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx \\
& + \int_{\Sigma_\lambda \times \mathbb{R}} \left( (n_1 \cdot a) f_+^{\tau} \tilde{f}_-^{\tau} - \frac{1}{2} \beta' \nabla_x h \cdot \nabla_x (f_+^{\tau} \tilde{f}_-^{\tau}) \right) \overline{\varphi}^\lambda d\xi d\overline{t} \\
& \geq \int_{Q_\lambda \times \mathbb{R}} \delta(\xi - \tilde{u}) n_+ + \delta(\xi - u) \tilde{n}_- \varphi^\lambda d\xi dt dx.
\end{align*}

for some weak* cluster points $f_+^\tau$ and $\tilde{f}_-^\tau$. Since

\begin{align*}
2\beta'(\xi) \delta(\xi - u) \delta(\xi - \tilde{u}) \nabla \beta(u) \cdot \nabla \beta(\tilde{u}) \\
& \leq 1_{F}(\xi) \delta(\xi - u) \delta(\xi - \tilde{u}) (|\nabla \beta(u)|^2 + |\nabla \beta(\tilde{u})|^2) \\
& = \delta(\xi - \tilde{u}) n_+(t, x, \xi) + \delta(\xi - u) \tilde{n}_-(t, x, \xi),
\end{align*}

we arrive at

\begin{align}
& - \int_{Q_\lambda \times \mathbb{R}} f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta) \varphi^\lambda d\xi dt dx \\
& \geq \int_{\Omega_\lambda \times \mathbb{R}} f_+^{\tau_0} \tilde{f}_-^{\tau_0} \varphi^\lambda(0, \cdot) d\xi dx - \int_{\Sigma_\lambda \times \mathbb{R}} \left( (n_1 \cdot a) f_+^{\tau} \tilde{f}_-^{\tau} - \frac{1}{2} \beta' \nabla_x h \cdot \nabla_x (f_+^{\tau} \tilde{f}_-^{\tau}) \right) \overline{\varphi}^\lambda d\xi d\overline{t}.
\end{align}
Comparison theorem for degenerate PDEs

We compute each term of (2.26). Firstly,

\[
- \int_{Q_{λ} \times \mathbb{R}} f_+ \tilde{f}_- (\partial_t + a \cdot \nabla + \beta' \Delta ) \varphi^λ dξ dt dx 
= \int_{Q_λ} ((u - \tilde{u})^+ + F^+(u, \tilde{u}) \nabla \varphi^λ + (\beta(u) - \beta(\tilde{u}))^+ \Delta \varphi^λ ) dt dx.
\]

Secondly, by virtue of Lemma 2.2 and by using integration by parts one can calculate:

\[
\int_{\mathbb{R}} \nu^0_x (\xi, \infty)) \partial_\xi m_+^0 dξ = - \int_{\mathbb{R}} \partial_\xi \tilde{m}_-^0 \tilde{\nu}_x^0 (\tilde{u}_0, \xi) dξ 
+ \left( \int_{u_b \vee \tilde{u}_b}^\infty \partial_\xi m_+^b \tilde{\nu}_x^0 (\tilde{u}_0, \tilde{u}_0) dξ 
+ \int_{u_b \vee \tilde{u}_b}^\infty m_+^b (\cdot, u_b \vee \tilde{u}_b) \tilde{\nu}_x^0 (\cdot, \tilde{u}_0) dξ \right)
\]

\[
\geq - \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} dξ = -(u - \tilde{u}_b)^+.
\]

Here we used the fact that \( d\nu^0_x (\xi, \infty)) / dξ = -d\tilde{\nu}_x^0 (\xi) \) and \( d\tilde{\nu}_x^0 ((-\infty, \xi)] / dξ = d\tilde{\nu}_x^0 (\xi) \). Thus we have

\[
\int_{Ω_λ \times \mathbb{R}} f_+ \tilde{f}_- \varphi^λ(0, \cdot) dξ dx \geq - \int_{Ω_λ} (u_0 - \tilde{u}_0)^+ \varphi^λ(0, \cdot) dx.
\]

Finally, we calculate analogously the boundary term by using Lemma 2.4:

\[
\int_{\mathbb{R}} (n_1 \cdot a) f_+ \tilde{f}_- dξ 
= - \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} \partial_ξ \tilde{m}_-^b \nu_{t,y}^0 (\tilde{u}_0, \xi) dξ 
- \int_{u_b \vee \tilde{u}_b}^{\tilde{u}_b} (n_1 \cdot a) \nu_{t,y}^0 (\tilde{u}_0, \xi) \tilde{\nu}_{t,y}^0 ((-\infty, \xi)] dξ 
+ \int_{u_b \vee \tilde{u}_b}^{\tilde{u}_b} \partial_ξ m_+^b \nu_{t,y}^0 ((-\infty, \xi)] dξ 
\leq M \int_{\tilde{u}_b}^{u_b \vee \tilde{u}_b} dξ = M(u_b - \tilde{u}_b)^+,
\]

where \( y \) stands for \( \tilde{x}_0 \) and we used the fact that \( d\nu_{t,y}^0 (\xi, \infty)) / dξ = -d\nu_{t,y}^0 (\xi) \) and \( d\tilde{\nu}_{t,y}^0 ((-\infty, \xi)] / dξ = d\tilde{\nu}_{t,y}^0 (\xi) \) as well as the fact that \( m_+^b \geq \)

\[
\]
0 for $\xi \geq u_b$ and $\tilde{m}_-^b \geq 0$ for $\xi \leq \tilde{u}_b$ by virtue of (2.21) and the corresponding inequality associated with $\tilde{u}$, respectively. This implies

$$\int_{\Sigma_\lambda \times \mathbb{R}} (\mathbf{n} \cdot \mathbf{a}) f^\tau_+ \tilde{f}^\tau_- \overline{\varphi}^\lambda \, d\xi dtd\overline{x} \leq M \int_{\Sigma_\lambda} (u_b - \tilde{u}_b)^+ \overline{\varphi}^\lambda \, dtd\overline{x}. \quad (2.13)$$

Moreover

$$\int_{\Sigma_\lambda \times \mathbb{R}} \beta'(\xi) \nabla_{\overline{x}} h(\overline{x}) \cdot \nabla_{\overline{x}} (f^b_+ \tilde{f}^b_-) \overline{\varphi}^\lambda \, d\xi dtd\overline{x} \quad (2.14)$$

$$= - \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ (\Delta_\overline{x} h(\overline{x}) \overline{\varphi}^\lambda + \nabla_\overline{x} h(\overline{x}) \cdot \nabla_\overline{x} \overline{\varphi}^\lambda) \, dtd\overline{\xi}$$

$$\geq \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ (-L \overline{\varphi}^\lambda + \nabla_\overline{x} h(\overline{x}) \cdot \nabla_\overline{x} \overline{\varphi}^\lambda) \, dtd\overline{\xi}. \quad$$

Combining (2.10) with (2.11) through (2.14) and choosing appropriate test functions $\varphi$'s, we arrive at the estimate

$$\frac{1}{T} \int_{Q_\lambda} (u - \tilde{u})^+ \, dtdx$$

$$\leq \int_{\Omega_\lambda} (u_0 - \tilde{u}_0)^+ \, dx + M \int_{\Sigma_\lambda} (u_b - \tilde{u}_b)^+ \, dtd\overline{x} + \frac{L}{2} \int_{\Sigma_\lambda} (\beta(u_b) - \beta(\tilde{u}_b))^+ \, dtd\overline{x}. \quad$$

By summing over $i = 0, 1, \cdots, N$, we obtain the desired estimate (1.9) and the proof of Theorem is complete.

REFERENCES

Comparison theorem for degenerate PDEs


