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Kyoto University
Asymptotic profile of a radially symmetric solution with transition layers for an unbalanced bistable equation

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1 Introduction and Main Results

In this paper, we consider the following boundary value problem:

\[(P_\varepsilon) \begin{cases} -\varepsilon^2 \Delta u = h(|x|)^2(u - a(|x|))(1 - u^2) & \text{in } B_1(0) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_1(0) \end{cases} \]

where \( \varepsilon > 0 \) is a small parameter, \( B_1(0) \) is a unit ball in \( \mathbb{R}^N \) centered at the origin and the function \( a \) is a \( C^1 \) function on \([0, 1]\) satisfying \(-1 < a(|x|) < 1 \) and \( a'(0) = 0 \). The function \( h \) is a positive \( C^1 \) function on \([0, 1]\) satisfying \( h'(0) = 0 \). We set \( r = |x| \).

Problem \((P_\varepsilon)\) appears in various models such as population genetics, chemical reactor theory and phase transition phenomena. See [1] and the references therein. If the function \( h \) satisfies \( h(r) \equiv 1 \) and the function \( a \) satisfies \( a(r) \neq 0 \), then this problem \((P_\varepsilon)\) has been studied in [1], [4] and [7]. In this case, it is shown that there exist radially symmetric solutions with transition layers near the set \( \{ x \in B_1(0) |a(|x|) = 0 \} \). If the set \( \{ r \in \mathbb{R} |a(r) = 0 \} \) contains an interval \( I \), then the problem to decide the configuration of transition layer on \( I \) is more delicate.

On the other hand, in the case of \( N = 1 \), if the function \( h \) satisfies \( h(r) \neq 1 \) and the function \( a \) satisfies \( a(r) \equiv 0 \), then this problem \((P_\varepsilon)\) has been studied in [8] and [9]. In this case, it is shown that there exist stable solutions with transition layers near prescribed local minimum points of \( h \).

In this paper, we consider the case where the function \( a \) satisfies \( a(r) \neq 0 \) with \( a(r) = 0 \) on some interval \( I \subset (0, 1) \). We show the minimum point of the function \( r^{N-1}h(r) \) on \( I \) has very important role to decide the configuration of transition layer on \( I \) in this case.

We note that in [4], Dancer and Shusen Yan considered a problem similar to ours. They assume that \( N \geq 2 \), \( h \equiv 1 \) and the nonlinear term is \( u(u - a(|x|))(1 - u) \) satisfying \( a(r) = 1/2 \) on \( I = [l_1, l_2] \) and \( a(r) < 1/2 \) for \( l_1 - r > 0 \) small and \( a(r) > 1/2 \) for \( r - l_2 > 0 \) small, then a global minimizer of the corresponding functional has a transition layer near the \( l_1 \), that is, the minimum point of \( r^{N-1} \) on \( I \) (see [4, Theorem 1.3]). In this sense, we can say that our results are natural extention of the results in [4]. We are going to follow throughout the variational
procedure used in [4] with a few modifications prompted by the presence of the function $h$.

Here we state the energy functional corresponding to $(P_{\epsilon})$:

$$J_{\epsilon}(u) = \oint_{B_1(0)} \frac{\epsilon^2}{2} |\nabla u|^2 - F(|x|, u) \, dx,$$

where $F(|x|, u) = \int_{-1}^{u} f(|x|, s) \, ds$ and $f(|x|, u) = h(|x|)^2(u - a(|x|))(1 - u^2)$.

It is easy to see that the following minimization problem has a minimizer

$$\inf\{J_{\epsilon}(u) | u \in H^1(B_1(0))\}. \quad (1.1)$$

Let $A_- = \{ x \in B_1(0) | a(|x|) < 0 \}$ and $A_+ = \{ x \in B_1(0) | a(|x|) > 0 \}$.

In this paper, we will analyze the profile of the minimizer of (1.1). Our main theorem is the following:

**Theorem 1.1.** Let $u_{\epsilon}$ be a global minimizer of (1.1). Then $u_{\epsilon}$ is radially symmetric and

$$u_{\epsilon} \to \begin{cases} 1 & \text{uniformly on any compact subset of } A_- , \\ -1 & \text{uniformly on any compact subset of } A_+ , \end{cases}$$

as $\epsilon \to 0$. In particular $u_{\epsilon}$ converges uniformly near the boundary of $B_1(0)$, that is, if $a(r) < 0$ on $[r_0, 1]$ for some $r_0 > 0$, $u_{\epsilon} \to 1$ uniformly on $B_1(0) \setminus B_{r_0}(0)$ and if $a(r) > 0$ on $[r_0, 1]$ for some $r_0 > 0$, $u_{\epsilon} \to -1$ uniformly on $B_1(0) \setminus B_{r_0}(0)$.

Moreover, for any $0 < r_1 \leq r_2 < 1$ with $a(r_i) = 0$, $i = 1, 2$, $a(r) \neq 0$ for $r_1 - r > 0$ small and for $r - r_2 > 0$ small, $a(r) = 0$ if $r \in [r_1, r_2]$, we have:

(i) If $a(r) < 0$ for $r_1 - r > 0$ small and $a(r) > 0$ for $r - r_2 > 0$, then for any small $\eta > 0$ and for any small $\theta > 0$, there exists a positive number $\epsilon_0$ which has the following properties: For any $\epsilon \in (0, \epsilon_0]$, there exist $t_{\epsilon,1} < t_{\epsilon,2}$ such that

(a) \[ \begin{align*} u_{\epsilon}(r) & > 1 - \eta \quad \text{for } r \in [r_1 - \theta, t_{\epsilon,1}) , \\ u_{\epsilon}(t_{\epsilon,1}) & = 1 - \eta , \\ u_{\epsilon}(t_{\epsilon,2}) & = -1 + \eta , \\ u_{\epsilon}(r) & < -1 + \eta \quad \text{for } r \in (t_{\epsilon,2}, r_2 + \theta] . \end{align*} \]

(b) The function $u_{\epsilon}(r)$ is decreasing in $(t_{\epsilon,1}, t_{\epsilon,2})$

(c) The inequality $0 < R_1 \leq \frac{t_{\epsilon,2} - t_{\epsilon,1}}{\epsilon} \leq R_2$ holds, where $R_1$ and $R_2$ are two constants independent of $\epsilon > 0$.

(d) If $t_{\epsilon,j_1}, t_{\epsilon,j_2} \to \bar{t}$ for some positive sequence $\{\epsilon_j\}$ converging to zero as $j \to \infty$, then $\bar{t}$ satisfies $h(\bar{t})\bar{t}^{N-1} = \min_{s \in [r_1, r_2]} h(s)s^{N-1}$. 

(ii) If $a(r) > 0$ for $r_1 - r > 0$ small and $a(r) < 0$ for $r - r_2 > 0$, then for any small $\eta > 0$ and for any small $\theta > 0$, there exists a positive number $\epsilon_0$ which has the following properties: For any $\epsilon \in (0, \epsilon_0)$, there exist $t_{\epsilon,1} < t_{\epsilon,2}$ such that

\[
\begin{align*}
    u_\epsilon(r) &< -1 + \eta, & \text{for } r \in [r_1 - \theta, t_{\epsilon,1}), \\
    u_\epsilon(t_{\epsilon,1}) &= -1 + \eta, \\
    u_\epsilon(t_{\epsilon,2}) &= 1 - \eta, \\
    u_\epsilon(r) &> 1 - \eta, & \text{for } r \in (t_{\epsilon,2}, r_2 + \theta].
\end{align*}
\]

(b) The function $u_\epsilon(r)$ is increasing in $(t_{\epsilon,1}, t_{\epsilon,2})$.

(c) The inequality $0 < R_1 \leq \frac{t_{\epsilon,2} - t_{\epsilon,1}}{\epsilon} \leq R_2$ holds, where $R_1$ and $R_2$ are two constants independent of $\epsilon > 0$.

(d) If $t_{\epsilon_j,1}, t_{\epsilon_j,2} \to \bar{t}$ for some positive sequence $\{\epsilon_j\}$ converging to zero as $j \to \infty$, then $\bar{t}$ satisfies $h(\bar{t})\bar{t}^{N-1} = \min_{s \in [r_1, r_2]} h(s)s^{N-1}$.

![Graph of $u_\epsilon(r)$ and $r^{N-1}h(r)$]

**Remarks.**  
(i) We note that results from (a) to (c) both in cases (i) and (ii) are not related to the presence of the function $h$. The effect of presence of function $h$ appears in the result (d) in (i) and (ii).

(ii) If $\min_{s \in [r_1, r_2]} s^{N-1}h(s)$ is attained at a unique point $\bar{t}$, we can show $t_{\epsilon,1}, t_{\epsilon,2} \to \bar{t}$ as $\epsilon \to 0$ without taking subsequences.

(iii) If the function $r^{N-1}h(r)$ is constant on $[r_1, r_2]$, it is a very difficult problem to know the location of the point $\bar{t} \in [r_1, r_2]$.

This paper is organized as follows. In section 2, we prepare some preliminary results. We will prove Theorems 1.1 in section 3.
2 Preliminary Results

In this section we prepare some preliminary results.

Let $D$ be a bounded domain in $\mathbb{R}^N$. Let $\overline{f}(x, t)$ be a function defined on $\overline{D} \times \mathbb{R}$ which is bounded on $\overline{D} \times [-1, 1]$. Suppose $\overline{f}$ is continuous on $t \in \mathbb{R}$ for each $x \in \overline{D}$ and is measurable in $D$ for each $t \in \mathbb{R}$. We also assume

$$\overline{f}(x, t) > 0 \text{ for } x \in \overline{D}, \quad \overline{f}(x, t) < 0 \text{ for } x \in \overline{D}.$$  \hspace{1cm} (2.1)

Consider the following minimization problem:

$$\inf \left\{ \overline{J}_{\epsilon}(u, D) := \int_{D} \frac{\epsilon^2}{2} |\nabla u|^2 - \overline{F}(x, u) dx : u - \eta \in H_{0}^{1}(D) \right\},$$  \hspace{1cm} (2.2)

where $\eta \in H^{1}(D)$ with $-1 \leq \eta \leq 1$ on $D$ and

$$\overline{F}(x, t) = \int_{-1}^{t} \overline{f}(x, s) ds.$$  

We can prove next two lemmas by methods similar to [4]. For readers’s convenience we prove these lemmas in this section.

**Lemma 2.1.** Suppose that $\overline{f}(x, t)$ satisfies (2.1). Let $u_{\epsilon}$ be a minimizer of (2.2). Then $-1 \leq u_{\epsilon} \leq 1$ on $D$.

**Proof.** We prove $-1 \leq u_{\epsilon}$ on $D$. Let $M = \{ x : u_{\epsilon}(x) < -1 \}$. Define $\tilde{u}_{\epsilon}$ as follows:

$$\tilde{u}_{\epsilon}(x) = \begin{cases} u_{\epsilon}(x) & \text{if } x \in D \setminus M \\ -1 & \text{if } x \in M. \end{cases}$$

Since $u_{\epsilon}(x) = \eta \geq -1$ on $\partial D$, we see $M$ is compactly contained in $D$. Thus $\tilde{u} - \eta \in H_{0}^{1}(D)$. If the measure $m(M)$ of $M$ is positive, we have $\overline{J}_{\epsilon}(\tilde{u}_{\epsilon}, D) < \overline{J}_{\epsilon}(u_{\epsilon}, D)$. Because $u_{\epsilon}$ is a minimizer, we see $m(M) = 0$, where $m(A)$ denotes the Lebesgue measure of the set $A$. Thus $u_{\epsilon} \geq -1$. Similarly we can prove that $u_{\epsilon} \leq 1$. \hfill $\Box$

**Lemma 2.2.** Suppose that $\overline{f}_{1}(x, t)$ and $\overline{f}_{2}(x, t)$ both satisfy (2.1) and the same regularity assumption on $\overline{f}$. Assume that $\eta_{i} \in H^{1}(D)$ satisfy $-1 \leq \eta_{i} \leq 1$ on $D$ for $i = 1, 2$. Let $u_{\epsilon,i}$ be a corresponding minimizer of (2.2), where $\overline{f} = \overline{f}_{i}$ and $\eta = \eta_{i}$, $i = 1, 2$. Suppose that $\overline{f}_{1}(x, t) \geq \overline{f}_{2}(x, t)$ for all $(x, t) \in \overline{D} \times [-1, 1]$ and $1 \geq \eta_{1} \geq \eta_{2} \geq -1$. Then $u_{\epsilon,1} \geq u_{\epsilon,2}$.

**Proof.** Let $M = \{ x \in D : u_{\epsilon,2} > u_{\epsilon,1} \}$. Define $\varphi_{\epsilon} = (u_{\epsilon,2} - u_{\epsilon,1})^{+}$. Since $\eta_{1} \geq \eta_{2}$, we have $\varphi_{\epsilon} \in H_{0}^{1}(D)$. Set $\overline{F}_{i}(x, u) = \int_{-1}^{u} \overline{f}_{i}(x, s) ds$. Since $u_{\epsilon,i}$ is a minimizer of

$$J_{\epsilon,i}(u) := \int_{D} \frac{\epsilon^2}{2} |\nabla u|^2 - \overline{F}_{i}(x, u) dx$$

we have

$$J_{\epsilon,1}(\varphi_{\epsilon}) = \int_{D} \frac{\epsilon^2}{2} |\nabla \varphi_{\epsilon}|^2 - \overline{F}_{1}(x, \varphi_{\epsilon}) dx = \int_{D} \frac{\epsilon^2}{2} |\nabla \varphi_{\epsilon}|^2 - \overline{F}_{2}(x, \varphi_{\epsilon}) dx = J_{\epsilon,2}(\varphi_{\epsilon}).$$
and $\varphi_{\epsilon} = 0$ for $x \in D \setminus M$, we have

$$0 \leq J_{\epsilon,1}(u_{\epsilon,1} + \varphi_{\epsilon}) - J_{\epsilon,1}(u_{\epsilon,1}) = \int_{M} \frac{\epsilon^{2}}{2} (|\nabla (u_{\epsilon,1} + \varphi_{\epsilon})|^{2} - |\nabla u_{\epsilon,1}|^{2}) dx - \int_{M} \int_{u_{\epsilon,1}}^{u_{\epsilon,1} + \varphi_{\epsilon}} \overline{f}_{1}(x, s) ds$$

$$\leq \int_{M} \frac{\epsilon^{2}}{2} (|\nabla u_{\epsilon,1}|^{2} - |\nabla u_{\epsilon,1}|^{2}) dx - \int_{M} \int_{u_{\epsilon,1}}^{u_{\epsilon,1} + \varphi_{\epsilon}} \overline{f}_{2}(x, s) ds$$

$$= J_{\epsilon,2}(u_{\epsilon,2}) - J_{\epsilon,2}(u_{\epsilon,2} - \varphi_{\epsilon}) \leq 0.$$

This implies that $u_{\epsilon,1} + \varphi_{\epsilon}$ is also a minimizer of $J_{\epsilon,1}(u)$. Let $L > 0$ be large enough such that $\overline{f}_{1}(x, t) + Lt$ is strictly increasing for $x \in \overline{D}$, $t \in [-1, 1]$. From

$$-\epsilon^{2} \Delta (u_{\epsilon,1} + \varphi_{\epsilon}) = \overline{f}_{1}(u_{\epsilon,1} + \varphi_{\epsilon}),$$

we obtain

$$-\epsilon^{2} \Delta \varphi_{\epsilon} = \overline{f}_{1}(u_{\epsilon,1} + \varphi_{\epsilon}) - \overline{f}_{1}(u_{\epsilon,1}).$$

Thus

$$-\epsilon^{2} \Delta \varphi_{\epsilon} + L \varphi_{\epsilon} = \overline{f}_{1}(u_{\epsilon,1} + \varphi_{\epsilon}) + L(u_{\epsilon,1} + \varphi_{\epsilon}) - (\overline{f}_{1}(u_{\epsilon,1}) + Lu_{\epsilon,1}) > 0$$

in $D$. Fix $z_{0} \in M$. Let $x_{0} \in \partial M$ such that $|x_{0} - z_{0}| = \text{dist}(z_{0}, \partial M)$. Using the Strong maximum principle and Hopf's lemma in $B_{\text{dist}(z_{0}, \partial M)}(z_{0})$, we obtain that $\frac{\partial \varphi_{\epsilon}}{\partial \nu}(x_{0}) < 0$, where $\nu = (x_{0} - z_{0})/|x_{0} - z_{0}|$. But $\varphi_{\epsilon}(x) = 0$ for $x \not\in M$. Thus, $\frac{\partial \varphi_{\epsilon}}{\partial \nu}(x_{0}) = 0$. This is a contradiction. Thus we obtain $M = \emptyset$. □

3 Proof of Main Theorem

In this section we prove Theorem 1.1. The following proposition is the first part of Theorem 1.1.

Proposition 3.1. Let $u_{\epsilon}$ be a global minimizer of the problem (1.1). Then $u_{\epsilon}$ satisfies

$$u_{\epsilon} \to \begin{cases} 1 & \text{uniformly on any compact subset of } A_{-} \\ -1 & \text{uniformly on any compact subset of } A_{+} \end{cases}$$

as $\epsilon \to 0$.

Proof. Let $x_{0} \in A_{-}$. Choose $\delta > 0$ small so that $B_{\delta}(x_{0}) \subset A$. Take $b \in (\max_{z \in B_{\delta}(x_{0})} h(z), 1/2)$. Define $f_{x_{0}, \delta, b}(t) = (\min_{z \in B_{\delta}(x_{0})} h(z))^{2}(t - b)(1 - t^{2})$. Then for $x \in B_{\delta}(x_{0})$, $t \in [-1, 1]$, we have $f(|x|, t) \geq f_{x_{0}, \delta, b}(t)$. Let $u_{\epsilon, x_{0}, \delta, b}$ be the minimizer of

$$\inf \left\{ \int_{B_{\delta}(x_{0})} \frac{\epsilon^{2}}{2} |\nabla u|^{2} - F_{x_{0}, \delta, b}(u) dx : u + 1 \in H_{0}^{1}(B_{\delta}(x_{0})) \right\},$$

as $\epsilon \to 0$. □
where $F_{x_{0},\delta,b}(t) = \int_{-1}^{t} f_{x_{0},\delta,b}(s)ds$. It follows from Lemmas 2.1 and 2.2 that

$$u_{\varepsilon,x_{0},\delta,b}(x) \leq u_{\varepsilon}(x) \leq 1,$$

for $x \in B_{\delta}(x_{0})$.

Since $\int_{-1}^{1} f_{x_{0},\delta,b}(s)ds > 0$, it follows from [2, 3] that $u_{\varepsilon,x_{0},\delta,b}(x) \to 1$ as $\varepsilon \to 0$ uniformly in $B_{\delta/2}(x_{0})$, thus $u_{\varepsilon}(x) \to 1$ as $\varepsilon \to 0$ uniformly in $B_{\delta/2}(x_{0})$.

To prove the rest of Theorem 1.1, we need the following proposition and lemma.

**Proposition 3.2.** Let $u$ be a local minimizer of the following problem:

$$\inf \left\{ \int_{B_{1}(0)} \frac{1}{2} |\nabla u|^{2} - G(|x|, u) \ dx : u \in H^{1}(B_{1}(0)) \right\}.$$

Here $G(r, t) = \int_{-1}^{t} g(r, s)ds$, $g(r, t)$ is $C^{1}$ in $t \in \mathbb{R}$ for each $r \geq 0$, $g(r, t)$ and $g_{t}(r, t)$ are measurable on $[0, +\infty)$ for each $t \in \mathbb{R}$, $g(r, t) < 0$ if $t < -1$ or $t > 1$ and $|g(r, t)| + |g_{t}(r, t)|$ is bounded on $[0, k] \times [-2, 2]$ for any $k > 0$. Then $u$ is radial, i.e., $u(x) = u(|x|)$.

**Proof.** See [4, Proposition 2.6].

Before we prove Theorem 1.1, we prepare a lemma.

**Lemma 3.3.** Let $0 < \eta < 1$ be any fixed constant and $w$ satisfies

$$\begin{cases}
- w_{zz} = w(1 - w^{2}) & \text{on } \mathbb{R}, \\
w(0) = -1 + \eta \ (\text{resp. } w(0) = 1 - \eta), \\
w(z) \leq -1 + \eta \ (\text{resp. } w(z) \geq 1 - \eta) & \text{for } z \leq 0, \\
w \text{ is bounded on } \mathbb{R}.
\end{cases}$$

Then $w$ is a unique solution of

$$\begin{cases}
- w_{zz} = w(1 - w^{2}) & \text{on } \mathbb{R}, \\
w(0) = -1 + \eta \ (\text{resp. } w(0) = 1 - \eta), \\
w'(z) > 0 \ (\text{resp. } w'(z) < 0) & \text{for } z \in \mathbb{R}, \\
w(z) \to \pm 1 \ (\text{resp. } w(z) \to \mp 1) & \text{as } z \to \pm \infty.
\end{cases}$$

**Proof.** See for example [6].

Now we prove the rest of Theorem 1.1.

**Proof of Theorem 1.1.** For the sake of simplicity, we prove for the case where $a(r) < 0$ on $(0, r_{1})$, $a(r) = 0$ on $[r_{1}, r_{2}]$ and $a(r) > 0$ on $(r_{2}, 1]$ for some $0 < r_{1} < r_{2} < 1$ (see Figure 1 in Section 1).

**Part 1.** First we show that $u_{\varepsilon}$ converges uniformly near the boundary of $B_{1}(0)$, that is, $u_{\varepsilon} \to -1$ uniformly on $\overline{B_{1}(0)} \setminus B_{r_{2}+\tau}(0)$ for any small $\tau > 0$. We note that
we have \( u_\epsilon \to -1 \) uniformly on \( \overline{B_{1-\tau}(0) \setminus B_{r_2+\tau}(0)} \) as \( \epsilon \to 0 \). Now we claim that \( u_\epsilon(r) \leq u_\epsilon(1 - \tau) := T_\epsilon \) for \( r \in [1 - \tau, 1] \). We define the function \( \tilde{u}_\epsilon \) as follows:

\[
\tilde{u}_\epsilon(r) = \begin{cases} 
  u_\epsilon(r) & \text{if } r \in [0, 1 - \tau] \\
  u_\epsilon(r) & \text{if } u_\epsilon(r) < T_\epsilon \text{ and } r \in [1 - \tau, 1], \\
  T_\epsilon & \text{if } u_\epsilon(r) \geq T_\epsilon \text{ and } r \in [1 - \tau, 1]. 
\end{cases}
\]

We note that \( \tilde{u}_\epsilon \in H^1(B_1(0)) \) and \(-F(r, T_\epsilon) \leq -F(r, t)\) for \( \epsilon > 0 \) and \(|r - 1| \text{ small and } t \geq T_\epsilon \). Hence we obtain \( J_\epsilon(\tilde{u}_\epsilon) < J_\epsilon(u_\epsilon) \) and we have a contradiction if we assume that the measure of the set \( \{ r \in [0, 1]|u_\epsilon(r) > T_\epsilon \text{ and } r \in [1 - \tau, 1]\} \) is positive. Hence \(-1 < u_\epsilon(r) \leq T_\epsilon \) and \( u_\epsilon \to -1 \) uniformly on \( \overline{B_1(0) \setminus B_{r_2+\tau}(0)} \).

**Part 2.** Next we remark that, by Proposition 3.2, \( u_\epsilon \) is radially symmetric and we note that for any \( t_2 > t_1 \), \( u_\epsilon \) is a minimizer of the following problem

\[
\inf\{J_\epsilon(u, B_{t_2}(0) \setminus B_{t_1}(0)) : u - u_\epsilon \in H^1_0(B_{t_2}(0) \setminus B_{t_1}(0))\},
\]

where

\[
J_\epsilon(u, M) = \int_M \frac{\epsilon^2}{2} |\nabla u|^2 - F(|x|, u) dx
\]

for any open set \( M \). Let \( m_{\epsilon, t_1, t_2} \) be the minimum value of this minimization problem.

In this part we show that \( u_\epsilon \) has exactly one layer near the interval \([r_1, r_2]\).

**Step 2.1.** First we estimate the energy of transition layer.

Let \( \eta > 0 \) and \( \theta > 0 \) be small numbers. Since \( u_\epsilon \to 1 \) uniformly on \([0, r_1 - \theta]\) and \( u_\epsilon \to -1 \) uniformly on \([r_2 + \theta, 1 - \theta]\), we can find \( \tilde{r}_\epsilon \in (r_1 - \theta, r_2 + \theta) \) such that \( u_\epsilon(r) \geq 1 - \eta \) if \( r \in [0, \tilde{r}_\epsilon] \), \( u_\epsilon(r) < 1 - \eta \) for \( r - \tilde{r}_\epsilon > 0 \) small. Let \( \tilde{r}_\epsilon > \tilde{r}_\epsilon \) be such that \( u_\epsilon(r) \leq \eta \) if \( r \in [\tilde{r}_\epsilon, 1 - \theta] \), \( u_\epsilon(r) > \eta \) for \( \tilde{r}_\epsilon - r > 0 \) small. We may assume that \( \tilde{r}_\epsilon \to \tilde{r} \in [r_1, r_2] \) and \( \tilde{r}_\epsilon \to \tilde{r} \in [r_1, r_2] \).

We employ the so-called blow-up argument. Let \( v_\epsilon(t) = u_\epsilon(\epsilon t + \tilde{r}_\epsilon) \). Then

\[
-v_\epsilon'' - \epsilon \frac{N - 1}{\epsilon t + \tilde{r}_\epsilon} v_\epsilon' = f(\epsilon t + \tilde{r}_\epsilon, v_\epsilon),
\]

\(-1 \leq v_\epsilon \leq 1 \) and \( v_\epsilon(0) = 1 - \eta \). Since \( \tilde{r}_\epsilon \to \tilde{r} \in [r_1, r_2] \), it is easy to see that \( v_\epsilon \to v \) in \( C^1_{\text{loc}}(\mathbb{R}) \) and

\[
-v'' = h(\tilde{r})^2(v - v^3), \quad t \in \mathbb{R}.
\]

and \( v(t) \geq 1 - \eta \) for \( t \leq 0 \). If we set \( v(t) = V(h(\tilde{r}) t) \), the function \( V(t) \) satisfies

\[
\begin{align*}
-V'' &= V - V^3 & \text{on } \mathbb{R}, \\
V(0) &= 1 - \eta, \\
V'(t) &\geq 1 - \eta & t \leq 0.
\end{align*}
\]
Hence by Lemma 3.3, the function $V$ is a unique solution for

$$
\begin{align*}
-V'' &= V - V^3 \quad \text{on } \mathbb{R}, \\
V(0) &= 1 - \eta, \\
V'(t) &< 0 \quad t \leq 0, \\
V(t) &\to \pm 1 \quad \text{as } t \to \pm\infty.
\end{align*}
$$

Thus, we can find an $R > 0$ large, such that $v(R) = \eta$. Since $v_\varepsilon \to v$ in $C^1_\text{loc}(\mathbb{R})$, we can find an $R_\varepsilon \in (R - 1, R + 1)$, such that $v'_\varepsilon(r) < 0$ if $r \in [0, R_\varepsilon]$ and $v_\varepsilon(R_\varepsilon) = -1 + \eta$. Hence $u'_\varepsilon(r) < 0$ if $r \in [\overline{r}_\varepsilon, \overline{r}_\varepsilon + \varepsilon R_\varepsilon]$ and $u_\varepsilon(\overline{r}_\varepsilon + \varepsilon R_\varepsilon) = -1 + \eta$. Then we have

$$
J_\varepsilon(u_\varepsilon, B_{R_\varepsilon + \varepsilon R_\varepsilon}(0) \setminus B_{\overline{r}_\varepsilon}(0))
= \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) \int_{\overline{r}_\varepsilon}^{\overline{r}_\varepsilon + \varepsilon R_\varepsilon} \left(\frac{\varepsilon^2}{2} |u'_\varepsilon|^2 - F(t, u_\varepsilon)\right) dt
= \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) \int_0^{R_\varepsilon} \left(\frac{1}{2} |v'_\varepsilon|^2 - F(\varepsilon t + \overline{r}_\varepsilon, v_\varepsilon)\right) dt
= \omega_{N-1}(\overline{r}_\varepsilon^{N-1} + o_\varepsilon(1)) (\beta_{h(s)} + O(\eta) + o_\varepsilon(1)) \varepsilon,
$$

where $\omega_{N-1}$ is the area of the unit sphere in $\mathbb{R}^N$, $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, $\beta_{h(s)}$ is the positive value defined by

$$
\beta_{h(s)} = \int_{-\infty}^{+\infty} \left(\frac{1}{2} |w_{h(s)}(t)|^2 + h(s)^2 \frac{(w_{h(s)}^2 - 1)^2}{4}\right) dt
= h(s) \int_{-\infty}^{+\infty} \frac{1}{2} |V'(t)|^2 + \frac{(V(t)^2 - 1)^2}{4} dt
= h(s) \beta_1
$$

and $w_{h(s)}(t) = V(h(s)t)$ for $s \in [0, 1]$. We note that although the function $V$ depends on $\eta$, the value

$$
\beta_1 = \int_{-\infty}^{+\infty} \frac{1}{2} |V'(t)|^2 + \frac{(V(t)^2 - 1)^2}{4} dt
$$

is independent of $\eta$.

**Step 2.2.** We claim $u_\varepsilon$ has exactly one layer near the interval $[r_1, r_2]$. To show $u_\varepsilon$ has exactly one layer near the interval $[r_1, r_2]$, it sufficient to prove the following claim:

**Claim.** $\overline{r}_\varepsilon = \overline{r}_\varepsilon + \varepsilon R_\varepsilon$.

Suppose that the claim is not true. Then we can find a $t_\varepsilon > \overline{r}_\varepsilon + R_\varepsilon \varepsilon$ such that $u_\varepsilon(r) < -1 + \eta$ if $r \in (\overline{r}_\varepsilon + R_\varepsilon \varepsilon, t_\varepsilon)$, $u_\varepsilon(t_\varepsilon) = -1 + \eta$. Thus we can use the blow-up argument again at $t_\varepsilon$ to deduce that there is a $\tilde{t}_\varepsilon = t_\varepsilon + \varepsilon \tilde{R}_\varepsilon$ with $u'_\varepsilon(r) > 0$ if
$r \in (t_{\epsilon}, \tilde{t}_{\epsilon}), u_{\epsilon}(\tilde{t}_{\epsilon}) = 1 - \eta$. We may assume that $t_{\epsilon}, \tilde{t}_{\epsilon} \rightarrow \tilde{t}$ as $\epsilon \rightarrow 0$ for some $\tilde{t} \in [r_{2}, r_{3}]$. Moreover
\[ J_{\epsilon}(u_{\epsilon}, B_{t_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)}) = \omega_{N-1}(t_{\epsilon}^{N-1} + o_{\epsilon}(1) + O(\eta))\epsilon + o_{\epsilon}(1) \quad (3.4) \]

Now we claim $\tilde{t}_{\epsilon} \geq r_{1}$. Suppose $\tilde{t}_{\epsilon} < r_{1}$.

Let $F_{a}(t) = \int_{-1}^{t}(v-a)(1-v^{2})dv$. Then for any $t > 0$ small and $s \in [-1+t, 1-t]$,
\[
F_{a}(1-t) - F_{a}(s) = F_{0}(1-t) - F_{0}(s) + F_{a}(1-t) - F_{0}(1-t) - F_{a}(s) + F_{0}(s) \quad (3.5)
\]

Thus it follows from (3.5) that if $a < 0$ then
\[ F_{a}(1-t) - F_{a}(s) > 0 \quad (3.6) \]

for $s \in [-1+t, 1-t]$. Define
\[
\overline{u}_{\epsilon}(r) := \begin{cases} 
1 - \eta & r \in [\overline{r}_{\epsilon}, \overline{r}_{\epsilon} + R_{\epsilon}\epsilon] \cup [t_{\epsilon}, \tilde{t}_{\epsilon}], \\
-u_{\epsilon}(r) & r \in [\overline{r}_{\epsilon} + R_{\epsilon}\epsilon, t_{\epsilon}]. 
\end{cases}
\]

By the assumption that $\tilde{t}_{\epsilon} < r_{1}$ and using (3.6), we see $F(r, u_{\epsilon}) < F(r, \overline{u}_{\epsilon})$ if $r \in [\overline{r}_{\epsilon}, \tilde{t}_{\epsilon}]$. Hence, we obtain
\[
J_{\epsilon}(u_{\epsilon}, B_{t_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)}) < J_{\epsilon}(u_{\epsilon}, B_{t_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)}).
\]

Thus we obtain a contradiction. Therefore we have that $\tilde{t}_{\epsilon} \geq r_{1}$.

Since $a(r) \geq 0$ for $r \in [r_{1}, 1]$, we see $F(r, t) - F(r, -1) = 0$ if $r \in [r_{1}, 1]$.

Since $u_{\epsilon}(r) \in (-1, -1+\eta)$ for $r \in [\overline{r}_{\epsilon} + R_{\epsilon}\epsilon, t_{\epsilon}]$, we have
\[
m_{\epsilon, \overline{r}_{\epsilon}, r_{\epsilon}} = J_{\epsilon}(\overline{u}_{\epsilon}, B_{\overline{r}_{\epsilon}+R_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)}) + J_{\epsilon}(\overline{u}_{\epsilon}, B_{t_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)})
+ J_{\epsilon}(u_{\epsilon}, B_{t_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)}) + J_{\epsilon}(u_{\epsilon}, B_{t_{\epsilon}}(0) \backslash \overline{B_{\epsilon}(0)})
\geq \omega_{N-1}(t^{N-1}_{\epsilon}(\beta_{h(\overline{r}_{\epsilon})} + t^{N-1}_{\epsilon}(\beta_{h(0)}\epsilon) + O(\eta\epsilon) + o(\epsilon)
+ \inf \left\{ - \int_{B_{t_{\epsilon}}(0) \backslash B_{\overline{r}_{\epsilon}+R_{\epsilon}}(0)} F(r, w) : -1 \leq w \leq 1 + \eta \right\}
+ \inf \left\{ - \int_{B_{t_{\epsilon}}(0) \backslash B_{t_{\epsilon}}(0)} F(r, w) : -1 \leq w \leq 1 \right\}
\geq \omega_{N-1}(t^{N-1}_{\epsilon}(\beta_{h(\overline{r}_{\epsilon})} + t^{N-1}_{\epsilon}(\beta_{h(0)}\epsilon) + O(\eta\epsilon) + o(\epsilon)
\]

\[
(3.7)
\]
Now we give an upper bound for \( m_{\epsilon,s_{\epsilon },r_{\epsilon}} \). Let \( R > 0 \) be such that \( V(h(\overline{r}))R = \eta \), where \( V \) is a unique solution to (3.2). Define \( \overline{u} \) as follows:

\[
\overline{u}(r) := \begin{cases} 
V(h(\overline{r}) \frac{r - \overline{r}_{\epsilon}}{\epsilon}) & r \in [\overline{r}_{\epsilon}, \overline{r}_{\epsilon} + \epsilon R] \\
-1 + \eta - \frac{\eta}{\epsilon} (r - \overline{r}_{\epsilon} - \epsilon) & r \in [\overline{r}_{\epsilon} + \epsilon R, \overline{r}_{\epsilon} + \epsilon R + \epsilon] \\
-1 & r \in [\overline{r}_{\epsilon} + \epsilon R + \epsilon, \overline{r}_{\epsilon} - \epsilon] \\
-1 + \frac{\eta}{\epsilon} (r - \overline{r}_{\epsilon} + \epsilon) & r \in [\overline{r}_{\epsilon} - \epsilon, \overline{r}_{\epsilon}] 
\end{cases}
\] (3.8)

Now we note that \(|F(r, t)| = O(\eta)\) for \( r \in [\overline{r}_{\epsilon}, \overline{r}_{\epsilon}]\) and \(-1 \leq t \leq -1 + \eta\). Then we have

\[
m_{\epsilon,s_{\epsilon },r_{\epsilon}} \leq J_{\epsilon}(\overline{u}, B_{\overline{r}_{\epsilon}}(0) \setminus B_{\overline{r}_{\epsilon}}(0)) \\
\leq J_{\epsilon}(\overline{u}, B_{\overline{r}_{\epsilon} + R_{\epsilon}}(0) \setminus B_{\overline{r}_{\epsilon}}(0)) + J_{\epsilon}(\overline{u}, B_{\overline{r}_{\epsilon}}(0) \setminus B_{\overline{r}_{\epsilon} - \epsilon}(0)) \\
+ J_{\epsilon}(\overline{u}, B_{\overline{r}_{\epsilon} - \epsilon}(0) \setminus B_{\overline{r}_{\epsilon} + \epsilon R}(0)) \\
\leq \omega_{N-1} \overline{r}_{\epsilon}^{N-1} (\beta_{h(\overline{r})} + O(\eta)) \epsilon + o(\epsilon) + O(\epsilon \eta) + o(\epsilon) \\
= \omega_{N-1} \overline{r}_{\epsilon}^{N-1} \beta_{h(\overline{r})} + O(\epsilon \eta) + o(\epsilon)
\] (3.9)

By (3.7) and (3.9), we have

\[
\omega_{N-1} \overline{r}_{\epsilon}^{N-1} \beta_{h(\overline{r})} + t_{\epsilon}^{N-1} \beta_{h(\overline{r})} \epsilon \leq \omega_{N-1} \overline{r}_{\epsilon}^{N-1} \beta_{h(\overline{r})} \epsilon + O(\epsilon \eta) + o(\epsilon)
\]

This is a contradiction. So we can conclude \( \overline{r}_{\epsilon} = \overline{r}_{\epsilon} + \epsilon R_{\epsilon} \).

**Part 3.** It remains to prove that if \( \overline{r}_{\epsilon j} \rightarrow \overline{r} \) for some positive sequence \( \{\epsilon_{j}\} \) converging to zero as \( j \rightarrow \infty \) then \( \overline{r} \) satisfies

\[
\overline{r}^{N-1} h(\overline{r}) = \min_{s \in [r_{1}, r_{2}]} s^{N-1} h(s).
\]

**Step 3.1.** First we note that from Part 1, the function \( u_{\epsilon} \) satisfies \(-1 \leq u_{\epsilon} \leq -1 + \eta \) for \( r \in [\overline{r}_{\epsilon} + \epsilon R_{\epsilon}, 1] \) in this case.

**Step 3.2.** Set \( H(s) = s^{N-1} h(s) \). Assume that the result is not true. Then there exists a subsequence of \( \{\overline{r}_{\epsilon}\} \) (denoted by \( \overline{r}_{\epsilon} \)) such that \( \overline{r}_{\epsilon} \rightarrow r' \in [r_{1}, r_{2}] \) and \( H(r') > \min_{s \in [r_{1}, r_{2}]} H(s) \). Then we can find a point \( \overline{\xi} \in (r_{1}, r_{2}) \) such that \( H(r') = H(\overline{\xi}) \).

Next we give a lower estimate for \( J_{\epsilon}(u_{\epsilon}) \). We have

\[
J_{\epsilon}(u_{\epsilon}) = J_{\epsilon}(u_{\epsilon}, B_{\overline{r}_{\epsilon}}(0)) + J_{\epsilon}(u_{\epsilon}, B_{\overline{r}_{\epsilon} + \epsilon R_{\epsilon}}(0) \setminus B_{\overline{r}_{\epsilon}}(0)) \\
+ J_{\epsilon}(u_{\epsilon}, B_{1}(0) \setminus B_{\overline{r}_{\epsilon} + \epsilon R_{\epsilon}}(0)).
\] (3.10)

First we note that \( 1 - \eta \leq u_{\epsilon}(r) \leq 1 \) for \( r \leq \overline{r}_{\epsilon} \) and for sufficiently small \( \eta > 0 \), \(-F(r, u) \geq -F(r, 1) (u \in [1 - \eta, 1]) \). We also remark that since \( a(r) < 0 \) for \( r < r_{1} \) and \( a(r) = 0 \) for \( r_{1} \leq r \leq r_{2} \) and \( a(r) > 0 \) for \( r > r_{2} \), we have \(-F(r, 1) < 0 \) for
$r < r_1$ and $-F(r, 1) = 0$ for $r_1 \leq r \leq r_2$ and $-F(r, 1) > 0$ for $r > r_2$. Hence we have $-\int_{r_1}^{r_2} r^{N-1} F(r, 1) dr \geq 0$ and we obtain the following estimate

$$J_\epsilon(u_\epsilon, B_{r_\epsilon}(0)) \geq -\int_{0}^{r_\epsilon} r^{N-1} F(r, u_\epsilon) dr \geq -\int_{0}^{r_1} r^{N-1} F(r, 1) dr \geq -\int_{0}^{r_1} r^{N-1} F(r, 1) dr =: A.$$

We also obtain

$$J_\epsilon(u_\epsilon, B_{r_\epsilon+R_\epsilon}(0) \setminus B_{r_\epsilon}(0)) \geq \omega_{N-1} H(r') \beta_1 \epsilon + O(\eta \epsilon) + o(\epsilon). \quad (3.11)$$

by methods similar to proof of (3.3).

Since $-1 \leq u_\epsilon(r) \leq -1 + \eta$ for $r \geq r_\epsilon + \epsilon R_\epsilon$ and for sufficiently small $\eta > 0$, $-F(r, u) \geq -F(r, -1) = 0 \quad (u \in [-1, -1 + \eta])$, we obtain the following estimate:

$$J_\epsilon(u_\epsilon, B_{1}(0) \setminus B_{r_\epsilon}(0)) \geq -\int_{r_\epsilon}^{1} r^{N-1} F(r, u_\epsilon) dr \geq -\int_{r_\epsilon}^{1} r^{N-1} F(r, -1) dr = 0. \quad (3.12)$$

Thus we obtain

$$J(u_\epsilon) \geq A + \omega_{N-1} H(r') \beta_1 \epsilon + O(\eta \epsilon) + o(\epsilon). \quad (3.13)$$

Next we give an upper bound for $J_\epsilon(u_\epsilon)$. Consider the following function $\overline{w}_\epsilon$:

$$\overline{w}_\epsilon(r) := \begin{cases} 
1 & \text{if } r \in [0, \overline{t} - \epsilon], \\
1 - \frac{\eta}{\epsilon} (r - \overline{t} + \epsilon) & \text{if } r \in [\overline{t} - \epsilon, \overline{t}], \\
V \left( h(\overline{t}) \frac{r-\overline{t}}{\epsilon} \right) & \text{if } r \in [\overline{t}, \overline{t} + \epsilon R'] \\
-1 - \frac{\eta}{\epsilon} (r - \overline{t} - \epsilon R' - \epsilon) & \text{if } r \in [\overline{t} + \epsilon R', \overline{t} + \epsilon R' + \epsilon], \\
-1 & \text{if } r \in [\overline{t} + \epsilon R' + \epsilon, 1], 
\end{cases}$$

where $R' > 0$ is the number satisfying $V(h(\overline{t})R') = -1 + \eta$. Then we can see

$$J_\epsilon(u_\epsilon) \leq J_\epsilon(\overline{w}_\epsilon) \leq A + \omega_{N-1} H(\overline{t}) \beta_1 \epsilon + O(\eta \epsilon) + o(\epsilon). \quad (3.14)$$

By (3.13) and (3.14) we have a contradiction. The proof of Theorem 1.1 is completed. In the more complicated case, we can show by similar method (see Remark below).
Remark. We briefly show in more complicated case, that is, when $a$ is the function as in Figure 2. More precisely we set $I_1 := [r_1, r_2]$ and $I_2 := [r_3, r_4]$ and we assume $a > 0$ on $[0, r_1) \cup (r_4, 1]$ and $a < 0$ on $(r_3, r_4)$.

Let $\eta > 0$ and $\theta > 0$ be small numbers. As in Part 1, we can find pairs of numbers $(\overline{r}_{1,\epsilon}, \overline{r}_{2,\epsilon})$ and $(R_{1,\epsilon}, R_{2,\epsilon})$ satisfying $\overline{r}_{1,\epsilon} \in (r_1 - \theta, r_2 + \theta)$, $\overline{r}_{2,\epsilon} \in (r_3 - \theta, r_4 + \theta)$, $\sup_{\epsilon} |R_{1,\epsilon}| < \infty$, $\sup_{\epsilon} |R_{2,\epsilon}| < \infty$ and

\[
\begin{align*}
  u_\epsilon(r) &< -1 + \eta & \text{for } 0 < r < \overline{r}_{1,\epsilon} \\
  u_\epsilon(\overline{r}_{1,\epsilon}) &= -1 + \eta \\
  u_\epsilon(\overline{r}_{1,\epsilon} + \epsilon R_{1,\epsilon}) &= 1 - \eta \\
  u_\epsilon(\overline{r}_{2,\epsilon}) &= 1 - \eta \\
  u_\epsilon(\overline{r}_{2,\epsilon} + \epsilon R_{2,\epsilon}) &= -1 + \eta \\
  u_\epsilon(r) &< -1 + \eta & \text{for } \overline{r}_{2,\epsilon} + \epsilon R_{2,\epsilon} < r < 1
\end{align*}
\]

We assume $\overline{r}_{1,\epsilon_j} \rightarrow \overline{r}_1 \in I_1$ and $\overline{r}_{2,\epsilon_j} \rightarrow \overline{r}_2 \in I_2$ for some sequence $\{\epsilon_j\}$ which converges to 0 as $j \rightarrow \infty$. In this case it is easy to show that the energy of global minimizer $J(u_\epsilon)$ is estimated as follows:

\[
J_{\epsilon_j}(u_{\epsilon_j}) \geq J_{\epsilon_j}(u_{\epsilon_j}, B_{r_2-\epsilon}(0)) + \epsilon_j \omega_{N-1} H(\overline{r}_2) \beta_1 + B + O(\epsilon_j \eta) + o(\epsilon_j), \quad (3.15)
\]

where $B = - \int_{r_2}^{r_3} r^{N-1} F(r, 1) dr$.

Let us assume the result does not hold. Then $H(\overline{r}_1) > \min_{s \in I_1} H(s)$ or $H(\overline{r}_2) > \min_{s \in I_2} H(s)$ hold. We assume $H(\overline{r}_1) = \min_{s \in I_1}$ and $H(\overline{r}_2) > \min_{s \in I_2} H(s)$. We also assume $r_1 = \overline{r}_1$. We note that if $H(\overline{r}_1) > \min_{s \in I_1} H(s)$ or $\overline{r}_1 \in \text{int}I_1$, the proof is more easy.
Let we take $\tilde{r}_2 \in \text{int}I_2$ such that $H(\tilde{r}_2) > H(\tilde{r}_2) > \min_{s \in I_2} H(s)$ and consider the following function:

$$
\tilde{u}_\epsilon(r) := \begin{cases} 
  u_\epsilon(r) & \text{on } [0, r_2 - \epsilon) \\
  1 + \frac{n}{\epsilon} (r - r_2) & \text{on } [r_2 - \epsilon, r_2) \\
  1 & \text{on } [r_2, \tilde{r}_2 - \epsilon] \\
  1 - \frac{n}{\epsilon} (r - \tilde{r}_2 + \epsilon) & \text{on } [\tilde{r}_2 - \epsilon, \tilde{r}_2] \\
  V \left( \frac{h(\tilde{r}_2) - r}{\epsilon} \right) & \text{on } [\tilde{r}_2, \tilde{r}_2 + \epsilon R''] \\
  -1 - \frac{n}{\epsilon} (r - \tilde{r}_2 - \epsilon R'' - \epsilon) & \text{on } [\tilde{r}_2 + \epsilon R'', \tilde{r}_2 + \epsilon R'' + \epsilon] \\
  -1 & \text{on } [\tilde{r}_2 + \epsilon R'' + \epsilon, 1],
\end{cases}
$$

where $V$ is the unique solution of (3.2) and $R''$ is the unique value such that $V(h(r_1) R'') = -1 + \eta$.

Since $u_\epsilon$ is global minimizer, we can estimate the energy of $J_\epsilon(\tilde{u}_\epsilon)$ as follows:

$$J_\epsilon(u_\epsilon) \leq J_\epsilon(\tilde{u}_\epsilon) \leq J_\epsilon(u_\epsilon, B_{r_2-\epsilon}(0)) + \varepsilon \omega_{N-1} H(\tilde{r}_2) \beta_1 + B + O(\varepsilon \eta) + o(\varepsilon). \quad (3.16)$$

Then we have a contradiction from (3.15) and (3.16) by taking $\varepsilon = \varepsilon_j$ and sufficiently large $j$.

References


