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Kyoto University
Integer-valued and almost integer-valued functions

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Abstract

In this article, we discuss conditions so that complex entire functions are integer-valued, by means of methods based on Diophantine problems. We also describe how are deduced conditions to be “almost” integer-valued.

Keywords: Integer-valued functions, Almost integer-valued functions, Lattice, Geometry of numbers, Transcendence method.

1 Introduction

We first consider a naive question as follows. Denote by $\overline{\mathbb{Q}}$ the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Let $F(z)$ be a complex function in one variable which satisfies $F(u) \in \overline{\mathbb{Q}}$ for any $u \in \mathbb{Q}$.

Could we specify any properties for the function $F(z)$?

We know that such $F(z)$ is NOT necessarily algebraic function; we had examples in 1894 due to P. Stäckel of a transcendental function taking algebraic values at all algebraic points. In fact, Stäckel's showed a more general statement: for any countable set $S$, and for any set $T$ being dense in the complex plane, there exists an entire function $F(z) = \sum_{n=0}^{\infty} f_n z^n$ with rational $f_n$, satisfying $F(u) \in T$ for any $u \in S$. Thus we have $F(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$ for $S = T = \overline{\mathbb{Q}}$.

Moreover he constructed in 1902 a transcendental function $F(z) = \sum_{n=0}^{\infty} f_n z^n$ with rational $f_n$, analytic in a neighbourhood of the origin, with the property that the both $F(z)$ and its inverse function take algebraic values at all algebraic points $\in \overline{\mathbb{Q}}$ in the neighbourhood.
We also recall that the Hermite-Lindemann theorem shows $\exp(\alpha) \not\in \overline{\mathbb{Q}}$ for any $\alpha \in \mathbb{Q}, \alpha \neq 0$, which says, such transcendental function always takes transcendental values at any non-trivial algebraic point. Stäckel's result notices that this is not true for all transcendental functions. F. Beukers and J. Wolfart gave in 1988 a condition for the algebraicity of the values of Gauss' hypergeometric function $F(z)$ and Wolfart made a criterion to distinguish whether a Gauss' hypergeometric function takes algebraic values at algebraic points or not [Be-Wo]: if Gauss' hypergeometric function $F(z)$ is algebraic over $\mathbb{C}(z)$, then $F(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$, otherwise there are hypergeometric functions, either $F(z)$ with $F(\xi) \in \overline{\mathbb{Q}}$ for only finitely many $\xi \in \overline{\mathbb{Q}}$. or, $F(z)$ such that there is a subset $E \subset \overline{\mathbb{Q}}$ which is dense in $\mathbb{C}$ satisfying $F(\xi) \in \overline{\mathbb{Q}}$ whenever $\xi \in E$.

On the other hand, there are many examples of transcendental functions such that $F(N) \subset \mathbb{Z}$, or $F(\mathbb{Q}) \subset \overline{\mathbb{Q}}$, e. g. $F(z) = 2^z$.

Now we ask, what it is, the function $F(z)$ with $F(N) \subset \mathbb{Z}$ or $F(\mathbb{Q}) \subset \overline{\mathbb{Q}}$. We have the following fundamental result due to G. Pólya [Po] in 1915 (see also [Ha]).

**Definition 1** Let $F(z)$ be an entire function in $\mathbb{C}$. Write $|F|_r = \sup_{|z| \leq r} |F(z)|$ and define $\tau(F)$ the order of exponential type of $F(z)$; $\tau(F) = \lim_{r \to +\infty} \sup \frac{\log |F|_r}{r}$.

**Theorem A** (Pólya) Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(N) \subset \mathbb{Z}$. Suppose $\tau(F) < \log 2$. Then $F(z)$ is a polynomial.

Pólya considered also the case $F(\mathbb{Z}) \subset \mathbb{Z}$ (see also [Ca]).

**Theorem B** (Pólya) Let $f'(z)$ be an entire function in $\mathbb{C}$ with $f'(\mathbb{Z}) \subset \mathbb{Z}$. Suppose $\tau(F) < \log \left(\frac{3 + \sqrt{5}}{2}\right)$. Then $F(z)$ is a polynomial.

In Theorem A and B, we see that $f'(z)$ is necessarily a polynomial in $\mathbb{Q}[z]$, but not in $\mathbb{Z}[z]$ (consider for example, $\frac{1}{2}z(z+1)$).

The bounds $\log 2$ and $\log \left(\frac{3 + \sqrt{5}}{2}\right)$ are optimal because of $2^z$ and $\left(\frac{3 + \sqrt{5}}{2}\right)^z + \left(\frac{3 - \sqrt{5}}{2}\right)^z$.

2 **Results**

We see, $F(z)$ is a polynomial with coefficients in $\mathbb{Q}$, which is equivalent to say the functions $z^hF(z)^k (h,k \in \mathbb{N} \cup \{0\})$ are linearly dependent over $\mathbb{Q}$. Then a natural generalization of works of Pólya is to seek a sufficient condition such that several functions $f_1, \cdots, f_L$ are linearly dependent over $\mathbb{Q}$.

For $\zeta_1, \zeta_2, \cdots \in \mathbb{C}$ denote by $r(N) := \max_{1 \leq n \leq N} |\zeta_n|$.

We then show
Theorem 1 Let $L$ and $N_0$ be rational integers with $1 < N_0 < L$. There are constants $C_1 > 0$ and $C_2 > 0$ depending only on $L, N_0$ satisfying the following. Let $\zeta_1, \zeta_2, \ldots \in \mathbb{C}$ be infinite complex points pairwise distinct. Let $f_1, \ldots, f_L$ be entire functions in $\mathbb{C}$. Suppose $f_j(\zeta_n) \in \mathbb{Z}$ for any $j, n$ with $1 \leq j \leq L$ and $n \geq 1$. If we have $\max_{1 \leq j \leq L} \log |f_j|_{C_1^{|\zeta|}} \leq C_2 N$ for any $N \geq N_0$, then the functions $f_1, \ldots, f_L$ are linearly dependent over $\mathbb{Q}$.

We are able to calculate $C_1 > 0$ and $C_2 > 0$ in an explicit manner. Several consequences of Theorem 1 are, for instance, as follows:

Corollary 2 Let $F(z)$ be an entire function in $\mathbb{C}$ with $F(\mathbb{N}) \subset \mathbb{Z}$. Suppose $\tau(F) \leq \frac{1}{40}$. Then $F(z)$ is a polynomial over $\mathbb{Q}$.

Proof We consider $f_1, \ldots, f_L$ as $\frac{z(z-1)\cdots(z-h+1)}{h!} F(z)^k (h, k \in \mathbb{N} \cup \{0\})$. $\square$

Corollary 3 Let $F$ be an entire function with $F(\mathbb{N}) \subset \mathbb{Z}$ with $\tau(F) \leq \frac{1}{14}$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that $F$ satisfies the functional equation

$$\sum_{h=0}^{N_1} \sum_{k=0}^{N_2} a_{hk} z^h F(z+k) = 0$$

with $a_{hk} \in \mathbb{Q}$ not all zero.

3 Almost integer-valued functions

Let us try to relax our condition in Theorem 1. We still want as consequence the functions to be polynomials over $\mathbb{Q}$ or linearly dependent over $\mathbb{Q}$, however we DOUBT if we really need the functions are integer-valued in the sufficient condition. Indeed we succeeded in proving the following.

Put $\delta(N) := \min_{1 \leq h < k \leq N} |\zeta_h - \zeta_k|$ for $N \geq 2$ and $\zeta_1, \zeta_2, \ldots \zeta_N \in \mathbb{C}$ distinct.

Set also $||z|| := \min_{m \in \mathbb{Z}} |z-m|$.

Theorem 4 Let $L$ and $N_0$ be rational integers with $1 < N_0 < L$. There are constants $C_1 > 19$, $C_2 > 0$ and $C_3 > 0$ depending only on $L, N_0$ satisfying the following. Let $\zeta_1, \zeta_2, \ldots \in \mathbb{C}$ be infinite complex points pairwise distinct. Let $f_1, \ldots, f_L$ be entire functions in $\mathbb{C}$. Assume

$$\max_{1 \leq j \leq L} \log |f_j|_{C_1^{|\zeta|}} \leq C_2 N$$

for any $N \geq N_0, 1 \leq j \leq L$. For $1 \leq \forall j \leq L$, $\forall n \geq 1$ assume also

$$||f_j(\zeta_n)|| \leq C_3 e^{-3n} n^{\frac{1}{20}} \left( \frac{\delta(2n+1)}{r(2n+1)} \right)^{n+1}$$

Then the functions $f_1, \ldots, f_L$ are linearly dependent over $\mathbb{Q}$. 
We may explicitly calculate $C_1$, $C_2$ and $C_3$. We present consequences of Theorem 4:

**Corollary 5** Let $F(z)$ be an entire function in $C$. Suppose $\tau(F) \leq \frac{1}{451}$ and $\|F(n)\| \leq e^{-5n}$ for any $n \in \mathbb{N}$. Then $F(z)$ is a polynomial over $\mathbb{Q}$.

4 Outline of the proof of theorems

We prove the theorems by Schneider's method which is one of main transcendence methods to deal with Diophantine problems.

**Proof of Theorem 1**

[First step] : construction of an auxiliary function

We may suppose that each $f_j(z)$, $(1 \leq j \leq L)$ is not identically zero. Then we know that the zeroes of the functions are isolated. We start to construct the function $F(z) := \sum_{j=1}^{L} p_j f_j(z)$ such that the coefficients $p_j \in \mathbb{Z} \ (1 \leq j \leq L)$ are not all zero and that, for any $n, 1 \leq n \leq N_0$, $F(\zeta_n) = 0$. This requires to solve a system in unknowns $\in \mathbb{Z}$ by Siegel's Lemma (the algebraic one, see [Da-Hi] and [Hi]). We use the Lemma to conclude that there exist $p_j \in \mathbb{Z} \ (1 \leq j \leq L)$ not all zero such that

$$\max_{1 \leq j \leq L} \log |p_j| \leq \frac{1}{L - N_0} \log L + \frac{N_0(N_0 - 1)}{L} \log \frac{C_1}{2}.$$

[Second step] : extrapolation

We consider for each $N \geq N_0$ the following properties:

$A(N) : F(\zeta_n) = 0$ for $1 \leq \forall n \leq N$

$B(N) : |F|_{r<N+1)} < 1.$

We shall show $A(N) \implies B(N)$ and $B(N) \implies A(N+1)$.

[Proof of $A(N) \implies B(N)$] : By the hypothesis $A(N)$, we see that the function $F$ has at least $N$ zeroes in the set $\{z \in C : |z| \leq r(N+1)\}$. The Lemma of Schwarz (in fact the residue formula; see [Gr] and [Gr-Mi-Wa]) shows

$$\log |F|_{r(N+1)} \leq \log |F|_{C_1 r(N+1)} - N \log \frac{C}{2}$$

then we have

$$\log |F|_{r(N+1)} \leq \frac{1}{L - N_0} \log L - \frac{N_0}{L} \log \frac{C_1}{2}.$$

The suitable choice of $C_1$ allows us conclude.

[Proof of $B(N) \implies A(N+1)$] : The property $B(N)$ implies $|F(\zeta_n)| < 1$ for any $1 \leq n \leq N + 1$. Since $f_j(\zeta_n) \in \mathbb{Z}$ then we obtain $F(\zeta_n) \in \mathbb{Z}$ namely $F(\zeta_n) = 0$. 


[Third step]: conclusion
The property \( A(N_0) \) holds by the construction of \( F \) in the First step. Therefore by the Second and the Third steps we see that \( A(N) \) and \( B(N) \) are true for any \( N \geq N_0 \). If \( F \) is not identically zero, the zeroes are isolated, which implies \( r(N) \rightarrow \infty \), thus by Liouville's theorem \( |F|_{r(N)} \rightarrow \infty \) for \( N \rightarrow \infty \), that contradicts with \( B(N) \). Then \( F \) is identically zero. \( \square \)

Proof of Theorem 4

[First step]: construction of an auxiliary function
We choose the constants \( C_1, C_2 \) so as to satisfy the both:

\[
\frac{1}{4} \log \frac{C_1 - 1}{2} - \frac{1}{2} \log 3 - \frac{L}{L - N_0} \log L - \frac{N_0}{2(L - N_0)} - \log \frac{C_1}{C_1 - 1} > 0,
\]

\[
C_2 N \leq \frac{(L - N_0)(N - 1)}{L} \left( \frac{1}{2} \log \frac{C_1 - 1}{2} - \log 3 \right) - \log L - \frac{N_0}{2L} - \frac{L - N_0}{L} \log \frac{C_1}{C_1 - 1}.
\]

By hypothesis, there are integers \( a_{jn} \) such that for each \( 1 \leq j \leq L, 1 \leq n \leq N_0 \):

\[ ||f_j(\zeta_n)|| = |f_j(\zeta_n) - a_{jn}|. \]

The obvious inequality

\[ \log |a_{jn}| \leq \log \left( \frac{1}{2} + |f_j(\zeta_n)| \right) \leq \max \left( \frac{1}{2} + \log |f_j(\zeta_n)|, \ \log \frac{3}{2} \right) \]

and the assumption of Theorem 4 give us for any \( 1 \leq j \leq L, 1 \leq n \leq N_0 \):

\[ \log |a_{jn}| \leq \frac{1}{2} + C_2 N_0. \]

We construct the function \( F(z) := \sum_{j=1}^{L} p_j f_j(z) \) such that the coefficients \( p_j \in \mathbb{Z} \) \( (1 \leq j \leq L) \) are not all zero and for any \( n, \frac{N_0}{2} \leq n \leq N_0 \) that

\[ \sum_{j=1}^{L} p_j a_{jn} = 0. \]

Since \( L > [N_0/2 + 1] \) we may solve the system again by Siegel's Lemma to get \( p_j \in \mathbb{Z} \) \( (1 \leq j \leq L) \) not all zero with

\[ \max_{1 \leq j \leq L} \log |p_j| \leq \frac{N_0}{L - N_0} \left( \log L + \frac{1}{2} + C_2 N_0 \right). \]

We then have

\[ \log |F|_{C_1 r(N)} \leq \frac{L}{L - N_0} \left( \log L + C_2 N + \frac{N_0}{2L} \right). \]
[Second step]: extrapolation
We consider for each $N \geq N_0$ the following properties:

$A(N)$: $\sum_{j=1}^{L} p_j a_{jn} = 0$ for $\frac{N}{2} \leq n \leq N$

$B(N)$: $|F|_{r(N+1)} < 3^{-N} + 3^{-1}$.

We shall show that $A(N) \implies B(N)$ and $B(N) \implies A(N+1)$.

[Proof of $A(N) \implies B(N)$]: Put

$$C_3 = e^{-3} \left( L \cdot \exp \left( \frac{N_0}{L - N_0} \left( \frac{1}{2} + C_2 N_0 \right) \right) \right)^{-1}.$$ 

By the hypothesis $A(N)$, we see

$$|F(\zeta_n)| = \left| \sum_{j=1}^{L} p_j (f_j(\zeta_n) - a_{jn}) \right| \leq C_3^{-1} e^{-3} \max_{1 \leq j \leq L} \|f_j(\zeta_n)\|$$

which shows under the assumption of $\|f_j(\zeta_n)\|$:

$$|F(\zeta_n)| \leq n^{\frac{N_0}{2}} \left( \frac{\delta(2n+1)}{e^3 r(2n+1)} \right)^{n+1}.$$ 

Because of $\frac{\delta(2n+1)}{e^3 r(2n+1)} < 1$,

$$\frac{\delta(n)}{r(n)} \geq \frac{\delta(n+1)}{r(n+1)}$$

we have

$$\max_{\frac{N}{2} \leq n \leq N} |k'(\zeta_n)| \leq e^{-3} \frac{N_0 N^{1/2}}{r(N+1)}.$$ 

On the other hand, use the inequality from the residue formula:

Let $f$ be a function analytic in $|z| \leq R$ in $\mathbb{C}$ and be $\zeta_0, \zeta_1, \cdots \zeta_{l} \in \mathbb{C}$ in $|z| \leq R$. Then we have

$$|f(\zeta_0)| \leq E_1 + E_2$$

where

$$E_1 = |f|_R \cdot \frac{R}{R - |\zeta_0|} \prod_{n=1}^{l} \frac{|\zeta_0 - \zeta_n|}{R - |\zeta_n|}$$

$$E_2 = \prod_{k=1}^{l} \frac{|\zeta_0 - \zeta_k|}{R - |\zeta_k|} \sum_{n=1}^{l} \frac{|f(\zeta_n)|}{|\zeta_0 - \zeta_n|} \cdot \frac{1}{\prod_{i=1, i \neq n}^{l} |\zeta_i - \zeta_n|}.$$ 

Take $\zeta_0 \in \mathbb{C}$ with $|\zeta_0| = r = r(N+1)$ and $\zeta_0 \neq \zeta_i (\frac{N}{2} \leq i \leq N)$. We now get thanks to the above inequality:
\[ |F|_r \leq |F|_R \cdot \frac{R}{R-r} \prod_{\frac{N}{2} \leq n \leq N} \frac{|\zeta_0 - \zeta_n|}{R - |\zeta_n|} + \prod_{\frac{N}{2} \leq k \leq N} \sum_{\frac{N}{2} \leq n \leq N} \left( \frac{|F(\zeta_n)|}{|\zeta_0 - \zeta_n|} \cdot \frac{1}{\prod_{i \neq n} |\zeta_i - \zeta_n|} \right). \]

For \( R = C_1 r \), this implies \( |F'|_r \leq T_1 + T_2 \) where

\[ T_1 = |F|_R \cdot \frac{C_1}{C_1 - 1} \left( \frac{C_1 - 1}{2} \right)^{-M}, \]

\[ T_2 = M \max_{\frac{N}{2} \leq n \leq N} |F(\zeta_n)| 6^{M-1} \left( \frac{3}{M} \right)^{\frac{M}{2}} \left( \frac{r(N + 1)}{\delta(N)} \right)^{M-1} \]

with \( M = \lceil N/2 + 1 \rceil \).

The choice of \( C_2 \) and the construction of the auxiliary function let us obtain

\[ \log |T_1| \leq -N \log 3. \]

Next, we see that the disc of radius \( r(N) + \delta(N)/2 \) contains \( N \) disjoint discs of radius \( \delta(N)/2 \) thus we have \( N \left( \frac{\delta(N)}{2} \right)^2 \leq \left( r(N) + \frac{\delta(N)}{2} \right)^2 \), therefore putting \( \sigma(N) := \frac{\delta(N)}{r(N)} \) we obtain \( \sigma(N) \sqrt{N} \leq 2 + \sigma(N) \) then

\[ \sigma(N + 1) \leq \frac{1}{2(\sqrt{2} - 1)\sqrt{N}} \]

for \( N \geq 1 \) and

\[ \sigma(N + 1) \leq \frac{2\sqrt{2}}{(\sqrt{3} - 1)\sqrt{N}} \]

for \( N \geq 2 \). The upper bound for \( \max_{\frac{N}{2} \leq n \leq N} |F(\zeta_n)| \) gives us

\[ T_2 \leq M e^{-3 - \frac{3M}{2}} N^{\frac{M}{2}} \sigma(N + 1)^{\frac{M}{2} - 1 - |N/2|}. \]

Consequently we have \( T_2 < \frac{1}{3} \) for \( N \geq 2 \) and then

\[ |F|_{r(N+1)} \leq T_1 + T_2 \leq 3^{-N} + 3^{-1}. \]

[Proof of \( B(N) \Rightarrow A(N + 1) \) : ]

By the assumption on \( \| f_j(\zeta_n) \| \) of the theorem and the property on \( \sigma(N) \), we see

\[ \| f_j(\zeta_{N+1}) \| \leq C_3 e^{-7} \]

for any \( N \geq 2 \).
Using the bound for $\log |p_j|$: the coefficient of the auxiliary function, and the choice of $C_3$, we get

$$|\sum_{j=1}^L p_ja_{j,N+1}| \leq L \max_{1 \leq j \leq L} (|p_j| \| f_j(\zeta_{N+1}) \|) + |F(\zeta_{N+1})| \leq C_3^{-1} e^{-3} C_3 e^{-7} + 3^{-2} + 3^{-1} < 1$$

which implies that the integer vanishes, namely $\sum_{j=1}^L p_ja_{j,N+1} = 0$.

[Third step]: conclusion
Since $B(N)$ is true for any $N \geq N_0$, we have

$$|F|_{r(N+1)} < 3^{-N} + 3^{-1}$$

then Liouville's theorem assures us that $F$ is identically zero. $\square$

5 Higher dimensional case

Now we ask to ourselves what happens on $\mathbb{C}^m$ in several variables case, or algebraic integer-valued case. Let $K$ be an algebraic number field of degree $[K : \mathbb{Q}] = d$ and $\mathcal{O}_K$ be the integer ring of $K$. In such cases, we have also a sequence of proven results. The works due to Seigo Fukasama (Seigo Morimoto) around 1920's together with the related results of A. O. Gel'fond concerning the entire function in $\mathbb{C}$ with $F(\mathbb{Z}[i]) \subset \mathbb{Z}[i]$ consist of some fundamental concept in this area. Daihachiro Sato also investigated integer-valued functions in 1960's. F. Gramain obtained the best possible bound for the order of the entire function in $\mathbb{C}$ with $F(\mathcal{O}_K) \subset \mathcal{O}_K$ to be a polynomial when $K$ is imaginary quadratic [Gr]. For the historical survey we refer his article. C. Pisot dealt with not only integer-valued functions but also almost integer-valued functions by interpolation method. A version in characteristic $p > 0$ is studied by M. Car and D. Adam.

**Definition 2** Let $\delta = d$ if $K \subset \mathbb{R}$ and $\delta = \frac{d}{2}$ otherwise.

We claim that the highly important fact

$$a \in \mathbb{Z}, |a| < 1 \implies a = 0$$

is not true in the case of algebraic integers, so we use instead, the following Size Inequality:

$$a \in \mathcal{O}_K, a \neq 0 \implies \log |\overline{a}| \geq -(\delta - 1) \log |\overline{a}|$$

where $|\overline{a}|$ denotes the maximum of absolute values of all the conjugates of $a$ over $\mathbb{Q}$.

**Definition 3** Let $F(z)$ be an entire function in $\mathbb{C}^m$ and $|z|$ denotes Euclidean norm in $\mathbb{C}^m$.
Write $|F|_r = \sup_{z \in \mathbb{C}^m, |z| \leq r} |F(z)|$ and define $\tau(F)$ the order of exponential type

of $F(z)$ by $\tau(F) = \lim_{r \to +\infty} \sup \frac{\log |F|_r}{r}$. 


We then quote a theorem due to Gramain:

**Theorem C** (Gramain) Let \( K \) be an algebraic number field of degree \([ K : \mathbb{Q} ] = d\) and \( \mathcal{O}_K \) be the integer ring of \( K \). Let \( F(z) \) be an entire function in \( \mathbb{C}^m \) of \( \tau(F) \leq \alpha \). Suppose \( F(N^m) \subset \mathcal{O}_K \). Assume further there exists a constant \( c > 0 \) such that for \( n \in \mathbb{N}^m \) we have

\[
\limsup_{|n| \to \infty} \frac{\log |F(n)|}{|n|} \leq c.
\]

Then under the condition \( \log(e^\alpha - 1) < -(\delta - 1) \log(1 + e^\gamma) \) the function \( F \) is a polynomial with coefficients in \( K \).

We collect the results as follows by Pólya, G. H. Hardy, Pisot, Gramain, V. Avanissian & R. Gay, A. Baker and A. Martineau. Below assume the function \( F(z) \) is entire.

(1) \( F(N) \subset \mathbb{Z}, \tau(F') < \log 2 \implies F(z) \) is a polynomial.

(2) \( F(Z) \subset \mathbb{Z}, \tau(F) < \log \frac{2 + \sqrt{2}}{2} \implies F(z) \) is a polynomial.

(3) \( F(N) \subset \mathcal{O}_K, \log(e^\alpha - 1) < -(\delta - 1) \log(1 + e^\gamma) \implies F(z) \) is a polynomial.

(4) \( F(Z) \subset \mathcal{O}_K, \log(2 \sinh(\frac{\alpha}{2})) < -\frac{\delta - 1}{2} \log(2 + e^\gamma + e^{-\gamma}) \implies F(z) \) is a polynomial.

(5) \( F(N^m) \subset \mathbb{Z}, \tau(F) < \log 2 \implies F(z) \) is a polynomial.

(6) \( F(Z^m) \subset \mathbb{Z}, \tau(F) < \log \frac{2 + \sqrt{2}}{2} \implies F(z) \) is a polynomial.

(7) \( F(N^m) \subset \mathcal{O}_K, \log(e^\alpha - 1) < -(\delta - 1) \log(1 + e^\gamma) \implies F(z) \) is a polynomial.

(8) \( F'(Z^m) \subset \mathcal{O}_K, \log(2 \sinh(\frac{\alpha}{2})) < -\frac{\delta - 1}{2} \log(2 + e^\gamma + e^{-\gamma}) \implies F'(z) \) is a polynomial.

The upper bounds \( \log 2 \) and \( \log \frac{2 + \sqrt{2}}{2} \) are only optimal.

We get the following in the case where we consider several variables and algebraic integers:

**Theorem 6** Let \( K \) be an algebraic number field of degree \([ K : \mathbb{Q} ] = d\) and \( \mathcal{O}_K \) be the integer ring of \( K \). Let \( L \) and \( N_0 \) be rational integers with \( 1 < N_0 < L \). Let \( m \in \mathbb{N} \). Then there are constants \( C_1 > 0 \) and \( C_2 > 0 \) depending on \( L, N_0, m \) satisfying the following. Let \( \zeta_1, \zeta_2, \ldots \) be infinite points in \( \mathbb{C}^m \) pairwise distinct. Let \( f_1, \ldots, f_L \) be entire functions in \( \mathbb{C}^m \) with \( f_j(\zeta_n) \in \mathcal{O}_K \) \( (1 \leq j \leq L, \forall n \geq 1) \). Suppose

\[
\max_{1 \leq j \leq L} \frac{\log |f_j|_{C^1 r(N)}}{N} \leq C_2 \quad \forall N \geq N_0.
\]

Assume further that there exist \( C_3 > 0 \) and \( C_4 > 0 \) such that

\[
\max_{1 \leq j \leq L} \frac{\log |f_j(\zeta_N)|}{N} \leq C_3, \quad |\zeta_N| < C_4 N \quad \forall N \geq N_0.
\]

Then under the condition \( \log(e^{C_2} - 1 + m) < -(\delta - 1) \log(1 + e^{C_4}) \) we have that the functions \( f_1, \ldots, f_L \) are linearly dependent over \( K \).
References


