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A Classification of Semiregular RDS's with $k = 12$

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1. Introduction

Definition 1.1. An incidence structure $(\mathcal{P}, \mathcal{B})$ is called a square $(m, u, k, \lambda)$-divisible design if the following conditions (i)-(iii) are satisfied.

(i) $|\mathcal{P}| = |\mathcal{B}| = mu$.

(ii) There exists a partition $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_m$ of $\mathcal{P}$ satisfying $|\mathcal{P}_1| = \cdots = |\mathcal{P}_m| = u$ and

$$[p, q] = \begin{cases} 0 & \text{if } p, q \in \mathcal{P}_i, \exists i, \\ \lambda & \text{otherwise.} \end{cases} \quad (p \neq q \in \mathcal{P}).$$

(iii) $|\mathcal{B}| = k$ \quad (\forall B \in \mathcal{B}).

The following hold.

$$k(k - 1) = (m - 1)u\lambda, \quad [p] = k \quad (\forall p \in \mathcal{P}) \quad (1)$$

$$k \geq u\lambda \quad \text{(Bose–Connor[1])} \quad (2)$$

Let $p \in \mathcal{P}_1$ and $B \in \mathcal{B}$ and assume that an automorphism group $G$ of $(\mathcal{P}, \mathcal{B})$ acts regularly on both $\mathcal{P}$ and $\mathcal{B}$. Set

$$D = \{x \in G \mid px \in B\} \quad \text{and} \quad U = \{x \in G \mid px \in \mathcal{P}_1\}.$$ 

Then $|D| = k$ and $U$ is a subgroup of $G$ of order $u$ satisfying

$$DD^{(-1)} = k + \lambda(G - U). \quad (3)$$

The equation (3) is equivalent to the following.

$$|aD \cap bD| = \begin{cases} 0 & \text{if } aU = bU, \\ \lambda & \text{otherwise.} \end{cases} \quad (a \neq b \in G) \quad (4)$$

Definition 1.2. Let $G$ be a group of order $mu$ and $U$ a subgroup of $G$ of order $u$. A $k$-subset $D$ is called a $(m, u, k, \lambda)$-difference set relative to $U$ if $D$ satisfies (3). $D$ is also called a relative difference set (RDS) relative to $U$. 
Conversely, given a \((m, u, k, \lambda)\)-difference set \(D\) in \(G\) relative to \(U\). Then we can show that \(\text{dev}(D)\) is a square \((m, u, k, \lambda)\)-divisible design, where

\[ \text{dev}(D) := (G, \{DX \mid x \in G\}). \]

**Definition 1.3.** A square \((m, u, k, \lambda)\)-divisible design is said to be symmetric if its dual is also a square \((m, u, k, \lambda)\)-divisible design. In other words, there is a partition \(\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m\) of \(\mathcal{B}\) satisfying

\[
|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathcal{B}_i, \exists i, (B \neq C \in \mathcal{B}) \\ \lambda & \text{otherwise.} \end{cases}
\]

**Result 1.4.** (W. S. Connor [3]) Let \((\mathcal{P}, \mathcal{B})\) be a square \((m, u, k, \lambda)\)-divisible design such that \(k > u\lambda\). If \((k, \lambda) = 1\), then \((\mathcal{P}, \mathcal{B})\) is symmetric.

**Remark 1.5.** Let \(D\) be a \((m, u, k, \lambda)\)-difference set in \(G\) relative to a subgroup \(U\). If \(DD^{(-1)} = D^{(-1)}D\), then \(\text{dev}(D)\) is symmetric.

**Result 1.6.** (D. Jungnickel [9]) If \(G \geq U\), then \(DD^{(-1)} = D^{(-1)}D\).

Concerning this, we have the following results.

**Proposition 1.7.** \(\text{dev}(D)\) is symmetric if and only if \(D^{(-1)}D = u\lambda + \lambda(G - V)\) for a subgroup \(V\) of \(G\).

**Proof.** Set \((\mathcal{P}, \mathcal{B}) = \text{dev}(D)\) and assume \((\mathcal{P}, \mathcal{B})\) is symmetric. Then, there exists a partition \(\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m\) of \(\mathcal{B}\) such that

\[
|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathcal{B}_i, \exists i, (B \neq C \in \mathcal{B}) \\ \lambda & \text{otherwise.} \end{cases}
\]

Set \(\mathcal{B}_1 = \{Dg_1, Dg_2, \cdots, Dg_u\}\), where \(g_1 = 1\). As \(Dd_i \cap Dg_j = \emptyset\) for any distinct \(i, j \in \{1, 2, \cdots, u\}\), for each \(\mathcal{B}_k\) there is an element \(g \in G\) so that \(\mathcal{B}_k = \{Dg_1g, Dg_2g, \cdots, Dg_ug\}\).

We note that

\[
Dg_i \cap Dg_j = \emptyset \quad (\iff \{(d_1, d_2) \mid d_1, d_2 \in D, d_1g_i = d_2g_j = \emptyset\} = \emptyset)
\]

\[
\iff \{(d_1, d_2) \mid d_1, d_2 \in D, \quad d_1^{-1}d_2 = g_kg_j^{-1}\} = \emptyset \quad (*)
\]

Set \(V = \{g_1(=1), g_2, \cdots, g_u\}\). Let \(g_i, g_j \in V\). Then, by \((*)\), \(Dg_ig_j^{-1} \cap D = \emptyset\).

Hence \(Dg_ig_j^{-1} = Dg_k\) for some \(g_k \in V\). Thus \(g_kg_j^{-1} = g_k \in V\) and so \(V\) is a subgroup of \(G\) of order \(u\). By \((*)\), we have the lemma. 

**Corollary 1.8.** Let \(D\) be an RDS. Then \(D^{(-1)}\) is also an RDS if and only if \(\text{dev}(D)\) is symmetric.

**Definition 1.9.** An RDS \(D\) is called symmetric if \(\text{dev}(D)\) is symmetric, otherwise non-symmetric.

If the equality in (2) holds, then \(k = m = u\lambda\) and so \((m, u, k, \lambda) = (u\lambda, u, u\lambda, \lambda)\).
Definition 1.10. A square $(m, u, k, \lambda)$-divisible design $(\mathcal{P}, \mathcal{B})$ is called a *transversal design* and denoted by $\text{TD}_\lambda(k;u)$ if $|B \cap P_i| = 1$ for $\forall B \in \mathcal{B}$ and $\forall i \in \{1,2,\cdots,m\}$.

Therefore, a square $(m, u, k, \lambda)$-divisible design is a transversal design iff $k = m(= u\lambda)$.

$\iff k = m(= u\lambda)$.

Definition 1.11. If $k = m = u\lambda$, then a $(m, u, k, \lambda)$-difference set $D$ in a group $G$ is said to be *semiregular*. Clearly $|G| = u^2\lambda$.

Remark 1.12. Under the above assumption, $DD^{(-1)} \neq D^{(-1)}D$ in general. However, every known transversal design obtained from semiregular RDS is symmetric.

In this talk we give examples of semiregular RDS's $D$ which do not satisfy the condition of Proposition 1.7. Then it gives us examples so that $\text{dev}(D)$'s are non-symmetric, and consequently non-symmetric transversal designs.

2. Known non-normal semiregular RDS's

Let $D$ be a $(u\lambda, u, u\lambda, \lambda)$-difference set (i.e. semiregular RDS) in $G$ relative to $U$. Then $D$ is called normal and non-normal according as $U \triangleleft G$ and $U \not\triangleleft G$, respectively. We note that $\text{dev}(D)$ is symmetric for every normal RDS applying Jungnickel's result.

Example 2.1. ([6], [7]) The following are all known examples of non-normal semiregular RDS's.

(i) $(u, \lambda) = (2, 2), G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1}\rangle (\simeq D_8)$,
$U = \langle y \rangle : (4, 4, 4, 2)-DS$

(ii) $(u, \lambda) = (4, 4), G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1}\rangle (\simeq Q_{16})$
$U = \langle y \rangle : (4, 4, 4, 1)-DS$

(iii) $(u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^7\rangle (\simeq SD_{32})$
$U = \langle y \rangle : (16, 2, 16, 8)-DS$

(iv) $(u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^3\rangle (\simeq M_8(2))$
$U = \langle y \rangle : (16, 2, 16, 8)-DS$

(v) Let $A$ be a $(4m^2, 2m^2 - m, m^2 - m)$-difference set in a group $N$. Assume $t$ is an automorphism of $N$ of order 2. Then $D = A \cup (N \setminus A^t)t$ is a non-normal $(4m^2, 2, 4m^2, 2m^2)-DS$ in a group $N(t)$ relative to $(t)$. 
3. Semiregular RDS's with $|D| = 12$

From now on we assume that $D$ is a semiregular RDS in a group $G$ relative to a subgroup $U$ with $|D| = 12$. Set $u = |U|$. Then $G = 12u$ and $D$ is a $(12, u, 12, \lambda)$-DS, where $u\lambda = 12$. Thus $|D| = 12$ and one of the following holds.

(i) $(u, \lambda) = (2, 6)$, $|G| = 24$, $|U| = 2$.
(ii) $(u, \lambda) = (3, 4)$, $|G| = 36$, $|U| = 3$.
(iii) $(u, \lambda) = (4, 3)$, $|G| = 48$, $|U| = 4$.
(iv) $(u, \lambda) = (6, 2)$, $|G| = 72$, $|U| = 6$.
(v) $(u, \lambda) = (12, 1)$, $|G| = 144$, $|U| = 12$.

**Remark 3.1.** Let $D$ be a semiregular RDS in a group $G$ relative to $U$ and let $s$ be an automorphism of $G$. Then $D^s$ is also a semiregular RDS (with the same parameters as $D$) in $G$ relative to $U^s$.

**CASE** $(u, \lambda) = (2, 6)$, $|G| = 24$, $|U| = 2$

The following lemma holds.

**Lemma 3.2.** ([7]) If there exists a $(2n, 2, 2n, n)$-difference set in $G$ relative to $U$ such that $G = NU$ for a subgroup $N$ of $G$ of index 2, then $n^* = 2$.

By Lemma 3.2 we have the following.

**Lemma 3.3.** If $(u, \lambda) = (2, 6)$, then $[G, G] \geq U$.

**Lemma 3.4.** N. Ito ([8]) If a group $G$ of order $4n (> 4)$ contains a normal $(2n, 2, 2n, n)$-DS relative to $U$, then a Sylow 2-subgroup of $G$ is neither cyclic nor dihedral.

By Remark 3.1, Lemmas 3.3, 3.4, there are five possibilities.

1. $G = Q_8 \times Z_3$, $U = Z(Q_8) \times 1$.
2. $G = Q_{24}$, $U = Z(Q_{24})$.
3. $G = Z_2 \times A_4$ and there are three possibilities for $U(\simeq Z_2)$.
4. $G = SL(2, 3)$, $U = Z(SL(2, 3))$.
5. $G = S_4$ and there are two possibilities for $U(\simeq Z_2)$.

By a computer search, we have the following.

**Lemma 3.5.** Assume $(u, \lambda) = (2, 6)$. Then, a group $G$ of order 24 contains a $(12, 2, 12, 6)$-DS if and only if $G \simeq Q_8 \times Z_3$, $Q_{24}$ or $SL(2, 3)$. 
CASE \((u, \lambda) = (3, 4)\), \(|G| = 36, |U| = 3\).

In this case \((m, u, k, \lambda) = (12, 3, 12, 4)\).

Lemma 3.6. Let \(G\) be a nonabelian group of order 36. Then there are eleven possibilities.

(i) \(G \simeq D_{36}\),
(ii) \(G \simeq Q_{36}\),
(iii) \(G \simeq D_{18} \times \mathbb{Z}_{2}\),
(iv) \(G \simeq (\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)) \times \mathbb{Z}_2\), \(|Z(G)| = 2\),
(v) \(G \simeq (\mathbb{Z}_2 \times (\mathbb{Z}_3 \times \mathbb{Z}_3)) \times \mathbb{Z}_2\), where an element of order 4 inverts \(O_3(G)\),
(vi) \(G \simeq (\mathbb{Z}_4 \ltimes \mathbb{Z}_3) \times \mathbb{Z}_3\), \(|Z(G)| = 6\),
(vii) \(G \simeq (\mathbb{Z}_4 \times \mathbb{Z}_3) \times \mathbb{Z}_2\), \(|Z(G)| = 6\),
(viii) \(G \simeq (\mathbb{Z}_4 \times \mathbb{Z}_3) \times \mathbb{Z}_2\),
(ix) \(G = \langle d \rangle \langle a, b \rangle \simeq \mathbb{Z}_4 \beta \langle (\mathbb{Z}_3 \times \mathbb{Z}_3) \rangle\), \(a^3 = b^3 = d^4 = \langle a, b \rangle = 1\), \(a^d = b^{-1}\), \(b^d = a\),
(x) \(G \simeq A_4 \times \mathbb{Z}_3\),
(xi) \(G \simeq \mathbb{Z}_9 K (\mathbb{Z}_2 \times \mathbb{Z}_2)\).

By a computer search we have the following.

Lemma 3.7. Assume \((u, \lambda) = (3, 4)\). Then, a nonabelian group \(G\) of order 36 contains a \((12, 3, 12, 4) - DS\) if and only if

\(G \simeq (\mathbb{Z}_4 \ltimes \mathbb{Z}_3) \times \mathbb{Z}_3\), \(|Z(G)| = 6\), \(U = O_3(Z(G))\),
\(A_4 \times \mathbb{Z}_3\) \((U = 1 \times \mathbb{Z}_3)\), or \(S_3 \times \mathbb{Z}_6\).

The first and the second cases have been previously known. But the third is a new one and has unusual properties.

Example 3.8. Let \(G = \langle a, b, c \mid a^3 = b^2 = c^6 = 1, b^{-1}ab = a^{-1}, ac = ca, bc = cb \rangle\) and set \(D = \{1, c, c^2, a, ac, b, a^2bc^5, abc^4, a^2bc, bc^4, abc\}\). Then \(D\) is a non-symmetric \((12, 3, 12, 4) - DS\) relative to \(U = \langle ac^2 \rangle \simeq \mathbb{Z}_3\).

Let \((\mathcal{P}, \mathcal{B})(= \text{dev}(D))\) be the corresponding transversal design and let \(A\) be an incidence matrix of \((\mathcal{P}, \mathcal{B})\). Then

\[AA^T = \begin{bmatrix}
\end{bmatrix}\]

where, \(I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\) and \(J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\).
However,

\[
A^T A = \begin{bmatrix}
12 & 4 & 4 & 4 & \cdots & 5 & 4 & 4 & 2 \\
4 & 12 & 4 & 4 & \cdots & 5 & 4 & 5 & 4 \\
4 & 4 & 12 & 4 & \cdots & 2 & 4 & 5 & 4 \\
4 & 4 & 4 & 12 & \cdots & 4 & 2 & 4 & 5 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
4 & 5 & 4 & 2 & \cdots & 4 & 12 & 4 & 4 \\
5 & 4 & 5 & 4 & \cdots & 4 & 12 & 4 & 4 \\
4 & 5 & 4 & 5 & \cdots & 4 & 4 & 12 & 4 \\
2 & 4 & 5 & 4 & \cdots & 4 & 4 & 4 & 12
\end{bmatrix}
\]

An RDS $D$ is called symmetric if $D^{(-1)}$ is also an RDS. Since $AA^T$ has entries $2, 5 \not\in \{0, 4, 12\}$, the dual of $\text{dev}(D)$ is not a transversal design. Applying Proposition 1.7, $D^{(-1)}$ is a non-symmetric RDS. As far as I know, this is the only known non-symmetric RDS.

**CASE $(u, \lambda) = (4, 3)$, $|G| = 48$, $|U| = 4$**

In this case $(m, u, k, \lambda) = (12, 4, 12, 3)$.

We use the following two lemmas to settle the present case.

**Lemma 3.9.** If $2 \mid u$ and $2 \nmid \lambda$, then $U$ contains every involution of $G$.

**Lemma 3.10.** (Elliott-Butson) Let $D$ be a $(u \lambda, u, u \lambda, \lambda)$-DS in $G$ relative to $U$. If $U$ contains a normal subgroup $N$ of $G$ of order $v$, then $DN/N$ is a $(u \lambda, u/v, u \lambda, v \lambda)$-DS in $G/N$ relative to $U/N$.

Applying Lemmas 3.5 and 3.10, it suffices to check the following case.

\[
G = \langle a, b, c \mid a^4 = b^6 = c^3 = 1, c^{-1}ac = ab, c^{-1}bc = ab^2 \rangle, \quad U = \langle a^2, b^2 \rangle
\]

We have the following by a computer search.

**Lemma 3.11.** There is no $(12, 4, 12, 3)$-DS.

**CASE $(u, \lambda) = (6, 2)$, $|G| = 72$, $|U| = 6$.**

In this case $(m, u, k, \lambda) = (12, 6, 12, 2)$.

**Observation.** For every known semiregular RDS, $|U|$ is a power of a prime.

The smallest undecided case is $u = 6$. 
Lemma 3.12. Let $G$ be a nonabelian group of order 72. Then there are five possibilities.

(i) $G \simeq SL(2,3) \times \mathbb{Z}_3$,
(ii) $G \simeq A_4 \times \mathbb{Z}_6$,
(iii) $G \simeq A_4 \times S_3$,
(iv) $G \simeq \langle t \rangle \langle M \times T \rangle$, $t^2 = 1$, $M \simeq A_4$, $T \simeq \mathbb{Z}_3$,
$\langle t \rangle M \simeq S_4$, $\langle t \rangle T \simeq S_3$,
(v) $G \triangleright Q$, $|Q| = 9$.

Applying Lemmas 3.5, 3.7 and 3.10, we have the following by a computer search.

Lemma 3.13. There is no $(12,6,12,2)$-DS.

CASE $(u, \lambda) = (12,1)$, $|G| = 144$, $|U| = 12$.

In this case $(m,u,k,\lambda) = (12,12,12,1)$.

By Lemma 1 of [2], the following holds.

Theorem 3.14. Every transversal design with $\lambda = 1$ is symmetric.

The above theorem implies that if there is a $(u,u,1)$-DS in a group $G$, then the corresponding transversal design can be extended to a projective plane of order $u$ which admits $G$ as a collineation group of order $u^2$. Thus $u \neq 12$ by Baumert-Hall [4] and the following holds.

Lemma 3.15. There is no $(12,12,12,1)$-DS.

By Lemmas 3.5, 3.7, 3.11, 3.13 and 3.15, we have the following.

Theorem 3.16. A group $G$ contains a $(u\lambda,u,\lambda)-DS$ $D$ with $|D| = 12$ if and only if $G$ is isomorphic to one of the following.

(i) $(u, \lambda) = (2,6)$, $G = Q_8 \times \mathbb{Z}_3$, $U = Z(Q_8) \times 1 \simeq \mathbb{Z}_2$.
(ii) $(u, \lambda) = (2,6)$, $G = Q_{24}$, $U = Z(Q_{24}) \simeq \mathbb{Z}_2$.
(iii) $(u, \lambda) = (2,6)$, $G = SL(2,3)$, $U = Z(SL(2,3)) \simeq \mathbb{Z}_2$.
(iv) $(u, \lambda) = (3,4)$, $G = S_3 \times \mathbb{Z}_6$, $U \simeq \mathbb{Z}_3$ (a non-symmetric RDS).
(v) $(u, \lambda) = (3,4)$, $G = (Z_4 \ltimes Z_3) \times \mathbb{Z}_3$, $|Z(G)| = 6$, $U = O_3(Z(G))$.
(vi) $(u, \lambda) = (3,4)$, $G = A_4 \times \mathbb{Z}_3$, $U = 1 \times \mathbb{Z}_3$.
(vii) $(u, \lambda) = (3,4)$, $G = \mathbb{Z}_6 \times \mathbb{Z}_6$, $U \simeq \mathbb{Z}_3$. 
4. Construction of non-symmetric RDS's

In this section we show the following.

**Theorem 4.1.** There exists a non-symmetric \((2^{2m}3^{m}, 3, 2^{2m}3^{m}, 2^{2m}3^{m-1})\) \((m-1)\) times difference set in \((S_3 \times \mathbb{Z}_6) \times (\mathbb{Z}_6 \times S_2) \times \cdots \times (\mathbb{Z}_6 \times S_2)\) relative to \(U \times 1 \times \cdots \times 1\), where \(D\) is a non-symmetric \((12, 3, 12, 4)\)-difference set in \(S_3 \times \mathbb{Z}_6\) relative to its non-normal subgroup \(U\) of order 3 (see Example 3.8).

**Corollary 4.2.** There exists a non-symmetric \(TD_{2m-1}2^{2m-3}3\) for every \(m \in \mathbb{N}\).

**Example 4.3.**

In order to prove Theorem 4.1, we need the following lemma.

**Lemma 4.4.** Let \(L = G \times H\), where \(G\) be a group of order \(u^2\lambda\) and \(H\) is a group of order \(u\mu\). Let \(D\) be a \((u\lambda, u, u\lambda, \lambda)DS\) in \(G\) relative to a subgroup \(U\) of \(G\) of order \(u\) and let \(C\) be a \((u\mu, u, u\mu, \mu)DS\) in \(U \times H\) relative to \(U\). Then

(i) \(CD\) is a \((u^2\lambda u, u^2\lambda u, u\lambda u, u\lambda)DS\) in \(L\) relative to \(U\).

(ii) \(CD\) is symmetric if and only if \(D\) is symmetric.

**Proof.** Let \(c_1, c_2 \in C\) and \(d_1, d_2 \in D\) and assume \(c_1d_1 = c_2d_2\). Then \(c_1^{-1}c_2 = d_1^{-1}d_2 \in UH \cap G = U\). Thus \(d_1 = d_2\) and so \(c_1 = c_2\). Therefore \(CD\) is a subset of \(L\).

By assumption, the following hold.

\[
DD^{(-1)} = u\lambda + \lambda(G - U) \quad (5)
\]

\[
CC^{(-1)} = u\mu + \mu(UH - U) \quad (6)
\]

\[
G = UD, \quad UC = UH \quad (7)
\]

Hence \((CD)(CD)^{(-1)} = C(DD^{(-1)})C^{(-1)} = C(u\lambda + \lambda(G - U))C^{(-1)} = u\lambda CC^{(-1)} + \lambda CGC^{(-1)} - \lambda CUC^{(-1)}\). As \(C, U \subset UH > U\), \(CU = UC\). Similarly \(GC = CG\). It follows that \((CD)(CD)^{(-1)} = u\lambda(\mu U + \mu(UH - U)) + \lambda GCC^{(-1)} - \lambda UCC^{(-1)} = u^2\mu\lambda + u\lambda UH - u\mu U^{(-1)} + \lambda G(\mu U + \mu UH - \mu U) - \lambda U(\mu U + \mu UH - \mu U) = u^2\mu\lambda + u\mu(L - U)\). Thus we have (i).

Since \(UH > U\), \(C^{(-1)}C = CC^{(-1)}\). Hence \((CD)^{(-1)}CD = D^{(-1)}(CC^{(-1)})D = D^{(-1)}(\mu U + \mu(UH - U))D\). By (7), the following holds.

\[
(CD)^{(-1)}CD = u\mu D^{(-1)}D + u\mu\lambda L - u\mu\lambda G \quad (8)
\]

Assume \(CD\) is symmetric. Then \((CD)^{(-1)}CD = u^2\mu\lambda + u\mu\lambda(L - V)\) for a subgroup \(V\) of \(L\) of order \(u\). By (8), \(u\mu D^{(-1)}D - u\mu\lambda G = u^2\mu\lambda - u\mu\lambda V\). Thus \(D^{(-1)}D = u\lambda + \lambda(G - V)\). In particular, \(V\) is a subgroup of \(G\) of order \(u\) and so \(D\) is symmetric. Conversely, assume \(D\) is symmetric. Then \(D^{(-1)}D = u\lambda + \lambda(G - V)\) for a subgroup \(V\) of \(G\) of order \(u\). Then, by (8), \((CD)^{(-1)}CD = u\mu(\mu U + \lambda(G - V)) + u\mu\lambda L - u\mu\lambda G = u^2\mu\lambda + u\lambda(L - V)\). Therefore \(CD\) is symmetric. Thus we have (ii).
We note that Lemma 4.4(i) is a modification of Result 2.4 of [11], where $N$ is assumed to be normal in $G$.

**Proof of Theorem**

Let $D$ be a non-symmetric $(12, 3, 12, 4)\text{DS}$ in $M = S_3 \times \mathbb{Z}_6$ relative to a non-normal subgroup $U$ of $M$ (see Example 3.8). Let $H = \langle a \rangle \times \langle b \rangle (\simeq \mathbb{Z}_6 \times \mathbb{Z}_3)$. We note that an abelian group $H \times \langle c \rangle (\simeq \mathbb{Z}_6 \times \mathbb{Z}_3)$ contains $(12, 3, 12, 4)\text{DS}$, say \{1, $a$, $a^2$, $a^{3}$, $a^4c$, $a^5c$, $bc^2$, $ab$, $a^2bc$, $a^3bc^2$, $a^4\theta c$, $a^5b$\}. Set $G = H \times M$ and choose $\langle c \rangle$ as a non-normal subgroup of $M$. Then, applying Lemma 4.4, $H \times M$ contains a non-symmetric $(2^23^2 \cdot 4, 3, 2^23^22^23 \cdot 4)\text{DS}$ in $G$ relative to $1 \times U(\simeq \mathbb{Z}_3)$ as $D$ is non-symmetric. Repeating this procedure again and again we have the theorem.

**References**


