数理解析研究所講究録 1476 巻 2006 年 127-135

A Classification of Semiregular RDS's with $k=12$

熊本大学 教育学部　平峰 豊 (Yutaka Hiramine)
Kumamoto University

1. Introduction

Definition 1.1. An incidence structure $(\mathbb{P},\mathbb{B})$ is called a *square* (m,u,k,λ)-*divisible design* if the following conditions (i)-(iii) are satisfied.

(i) $|\mathbb{P}|=|\mathbb{B}|=mu$.

(ii) There exists a partition $\mathbb{P}=\mathbb{P}_1\cup\mathbb{P}_2\cup\cdots\cup\mathbb{P}_m$ of $\mathbb{P}$ satisfying $|\mathbb{P}_1|=\cdots=|\mathbb{P}_m|=u$ and

$$[p,q]=\begin{cases}0 & \text{if } p,q\in\mathbb{P}_i,\ \exists i,\\ \lambda & \text{otherwise.}\end{cases}\qquad (p\neq q\in\mathbb{P}).$$

(iii) $|B|=k\quad(\forall B\in\mathbb{B})$.

The following hold.

$$k(k-1)=(m-1)u\lambda,\qquad [p]=k\quad(\forall p\in\mathbb{P})\tag{1}$$

$$k\geq u\lambda\quad(\text{Bose}-\text{Connor}[1])\tag{2}$$

Let $p\in\mathbb{P}_1$ and $B\in\mathbb{B}$ and assume that an automorphism group G of $(\mathbb{P},\mathbb{B})$ acts regularly on both $\mathbb{P}$ and $\mathbb{B}$. Set

$$D=\{x\in G\mid px\in B\}\ \text{ and }\ U=\{x\in G\mid px\in\mathbb{P}_1\}.$$

Then $|D|=k$ and U is a subgroup of G of order u satisfying

$$DD^{(-1)}=k+\lambda(G-U).\tag{3}$$

The equation (3) is equivalent to the following.

$$|aD\cap bD|=\begin{cases}0 & \text{if } aU=bU,\\ \lambda & \text{otherwise.}\end{cases}\qquad (a\neq b\in G)\tag{4}$$

Definition 1.2. Let G be a group of order mu and U a subgroup of G of order u. A k-subset D is called *a* (m,u,k,λ)-*difference set* relative to U if D satisfies (3). D is also called a *relative difference set* (RDS) relative to U.

Conversely, given a (m, u, k, λ)-difference set D in G relative to U. Then we can show that $\text{dev}(D)$ is a square (m, u, λ)-divisible design, where

$$\text{dev}(D) := (G, \ \{Dx \mid x \in G\}).$$

Definition 1.3. A square (m, u, k, λ)-divisible design is said to be *symmetric* if its dual is also a square (m, u, k, λ)-divisible design. In other words, there is a partition $\mathbb{B} = \mathbb{B}_1 \cup \cdots \cup \mathbb{B}_m$ of $\mathbb{B}$ satisfying

$$|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathbb{B}_i, \ \exists i, \\ \lambda & \text{otherwise.} \end{cases} \qquad (B \neq C \in \mathbb{B})$$

Result 1.4. (W. S. Connor [3]) *Let* $(\mathbb{P}, \mathbb{B})$ *be a square* (m, u, k, λ)*-divisible design such that* $k > u\lambda$. *If* $(k, \lambda) = 1$, *then* $(\mathbb{P}, \mathbb{B})$ *is symmetric.*

Remark 1.5. Let D be a (m, u, k, λ)-difference set in G relative to a subgroup U. If $DD^{(-1)} = D^{(-1)}D$, then $\text{dev}(D)$ is symmetric.

Result 1.6. (D. Jungnickel [9]) *If* $G \rhd U$, *then* $DD^{(-1)} = D^{(-1)}D$.

Concerning this, we have the following results.

Proposition 1.7. $\text{dev}(D)$ *is symmetric if and only if* $D^{(-1)}D = u\lambda + \lambda(G - V)$ *for a subgroup* V *of* G.

Proof. Set $(\mathbb{P}, \mathbb{B}) = \text{dev}(D)$ and assume $(\mathbb{P}, \mathbb{B})$ is symmetric. Then, there exists a partition $\mathbb{B} = \mathbb{B}_1 \cup \cdots \mathbb{B}_{u\lambda}$ of $\mathbb{B}$ such that

$$|B \cap C| = \begin{cases} 0 & \text{if } B, C \in \mathbb{B}_i, \ \exists i, \\ \lambda & \text{otherwise.} \end{cases} \qquad (B \neq C \in \mathbb{B}).$$

Set $\mathbb{B}_1 = \{Dg_1, Dg_2, \cdots, Dg_u\}$, where $g_1 = 1\}$. As $Dd_i \cap Dg_j = \emptyset$ for any distinct $i, j \in \{1, 2, \cdots, u\}$, for each $\mathbb{B}_k$ there is an element $g \in G$ so that $\mathbb{B}_k = \{Dg_1g, Dg_2g, \cdots, Dg_ug\}$.

We note that

$$\begin{aligned} &Dg_i \cap Dg_j = \emptyset \quad (\leftrightarrows \ \{(d_1, d_2) \mid d_1, d_2 \in D, d_1g_i = d_2g_j\} = \emptyset) \\ &\iff \ \{(d_1, d_2) \mid d_1, d_2 \in D, \ d_1^{-1}d_2 = g_ig_j^{-1}\} = \emptyset \qquad (*) \end{aligned}$$

Set $V = \{g_1(= 1), g_2, \cdots, g_u\}$. Let $g_i, g_j \in V$. Then, by $(*)$, $Dg_ig_j^{-1} \cap D = \emptyset$. Hence $Dg_ig_j^{-1} = Dg_k$ for some $g_k \in V$. Thus $g_ig_j^{-1} = g_k \in V$ and so V is a subgroup of G of order u. By $(*)$, we have the lemma. □

Corollary 1.8. *Let* D *be an RDS. Then* $D^{(-1)}$ *is also an RDS if and only if* $\text{dev}(D)$ *is symmetric.*

Definition 1.9. An RDS D is called *symmetric* if $\text{dev}(D)$ is symmetric, otherwise *non-symmetric.*

If the equality in (2) holds, then $k = m = u\lambda$ and so $(m, u, k, \lambda) = (u\lambda, u, u\lambda, \lambda)$.

Definition 1.10. A square (m, u, k, λ)-divisible design $(\mathbb{P}, \mathbb{B})$ is called *a transversal design* and denoted by $\mathrm{TD}_\lambda(k; u)$ if $|B \cap \mathbb{P}_i| = 1$ for $\forall B \in \mathbb{B}$ and $\forall i \in \{1, 2 \cdots, m\}$.

Therefore, a square (m, u, k, λ)-divisible design is a transversal design iff $k = m(= u\lambda)$.

$\iff k = m(= u\lambda)$.

Definition 1.11. If $k = m = u\lambda$, then a (m, u, k, λ)-difference set D in a group G is said to be *semiregular*. Cleary $|G| = u^2\lambda$.

Remark 1.12. Under the above assumption, $DD^{(-1)} \neq D^{(-1)}D$ in general. However, every known transversal design obtained from semiregular RDS is symmetric.

In this talk we give examples of semiregular RDS's D which do not satisfy the condition of Proposition 1.7. Then it gives us examples so that $\mathrm{dev}(D)$'s are non-symmetric, and consequently non-symmetric transversal designs.

2. Known non-normal semiregular RDS's

Let D be a $(u\lambda, u, u\lambda, \lambda)$-difference set (i.e. semiregular RDS) in G relative to U. Then D is called *normal* and *non-normal* according as $U \lhd G$ and $U \ntriangleleft G$, respectively. We note that $\mathrm{dev}(D)$ is symmetric for evey normal RDS applying Jungnickel's result.

Example 2.1. ([6], [7]) *The following are all known examples of non-normal semiregular RDS's.*

(i) $(u, \lambda) = (2, 2), G = \langle x, y \mid x^4 = y^2 = 1, y^{-1}xy = x^{-1}\rangle (\simeq D_8)$,
$U = \langle y \rangle$: *(4,2,4,2)-DS*

(ii) $(u, \lambda) = (4, 4), G = \langle x, y \mid x^4 = y^2, y^4 = 1, y^{-1}xy = x^{-1}\rangle$
$(\simeq Q_{16})$, $U = \langle y \rangle$: *(4,4,4,1)-DS*

(iii) $(u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^7\rangle$
$(\simeq SD_{32})$, $U = \langle y \rangle$: *(16,2,16,8)-DS*

(iv) $(u, \lambda) = (2, 8), G = \langle x, y \mid x^{16} = y^2 = 1, y^{-1}xy = x^9\rangle$
$(\simeq M_5(2))$, $U = \langle y \rangle$: *(16,2,16,8)-DS*

(v) *Let A be a $(4m^2, 2m^2 - m, m^2 - m)$-difference set in a group N. Assume t is an automorphism of N of order 2. Then $D = A \cup (N \setminus A^t)t$ is a non-normal $(4m^2, 2, 4m^2, 2m^2)$-DS in a group $N\langle t \rangle$ relative to $\langle t \rangle$.*

3. Semiregular RDS's with $|D| = 12$

From now on we assume that D is a semiregular RDS in a group G relative to a subgroup U with $|D| = 12$. Set $u = |U|$. Then $G = 12u$ and D is a $(12, u, 12, \lambda)$-DS, where $u\lambda = 12$. Thus $|D| = 12$ and one of the following holds.

(i) $(u, \lambda) = (2, 6)$, $|G| = 24$, $|U| = 2$.

(ii) $(u, \lambda) = (3, 4)$, $|G| = 36$, $|U| = 3$.

(iii) $(u, \lambda) = (4, 3)$, $|G| = 48$, $|U| = 4$.

(iv) $(u, \lambda) = (6, 2)$, $|G| = 72$, $|U| = 6$.

(v) $(u, \lambda) = (12, 1)$, $|G| = 144$, $|U| = 12$.

Remark 3.1. Let D be a semiregular RDS in a group G relative to U and let s be an automorphism of G. Then D^s is also a semiregular RDS (with the same parameters as D) in G relative to U^s.

CASE $(u, \lambda) = (2, 6)$, $|G| = 24$, $|U| = 2$

The following lemma holds.

Lemma 3.2. ([7]) *If there exists a $(2n, 2, 2n, n)$-difference set in G relative to U such that $G = NU$ for a subgroup N of G of index 2, then $n^* = 2$.*

By Lemma 3.2 we have the following.

Lemma 3.3. *If $(u, \lambda) = (2, 6)$, then $[G, G] \geq U$.*

Lemma 3.4. N. Ito ([8]) *If a group G of order $4n (> 4)$ contains a normal $(2n, 2, 2n, n)$-DS relative to U, then a Sylow 2-subgroup of G is neither cyclic nor dihedral.*

By Remark 3.1, Lemmas 3.3, 3.4, there are five possibilities.

(1) $G = Q_8 \times \mathbb{Z}_3$, $U = Z(Q_8) \times 1$.

(2) $G = Q_{24}$, $U = Z(Q_{24})$.

(3) $G = \mathbb{Z}_2 \times A_4$ and there are three possibilities for $U (\simeq \mathbb{Z}_2)$.

(4) $G = SL(2, 3)$, $U = Z(SL(2, 3))$.

(5) $G = S_4$ and there are two possibilities for $U (\simeq \mathbb{Z}_2)$.

By a computer search, we have the following.

Lemma 3.5. *Assume $(u, \lambda) = (2, 6)$. Then, a group G of order 24 contains a $(12, 2, 12, 6)$-DS if and only if $G \simeq \mathbb{Q}_8 \times \mathbb{Z}_3$, $\mathbb{Q}_{24}$ or $SL(2, 3)$.*

CASE $(u,\lambda) = (3,4)$, $|G| = 36$, $|U| = 3$.

In this case $(m,u,k,\lambda) = (12,3,12,4)$.

Lemma 3.6. *Let G be a nonabelian group of order 36. Then there are eleven possibilities.*
(i) $G \simeq D_{36}$, (ii) $G \simeq Q_{36}$, (iii) $G \simeq D_{18} \times \mathbb{Z}_2$,
(iv) $G \simeq S_3 \times S_3$, (v) $G \simeq S_3 \times \mathbb{Z}_6$,
(vi) $G \simeq (Z_2 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)) \times \mathbb{Z}_2$, $|Z(G)| = 2$,
(vii) $G \simeq (\mathbb{Z}_4 \ltimes \mathbb{Z}_3) \times \mathbb{Z}_3$, $|Z(G)| = 6$,
(viii) $G \simeq \mathbb{Z}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$, *where an element of order 4 inverts* $O_3(G)$,
(ix) $G = \langle d\rangle\langle a,b\rangle \simeq \mathbb{Z}_4 \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)$, $a^3 = b^3 = d^4 = [a,b] = 1, a^d = b^{-1}, b^d = a$,
(x) $G \simeq A_4 \times Z_3$, (xi) $G \simeq \mathbb{Z}_9 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)$.

By a computer search we have the following.

Lemma 3.7. *Assume $(u,\lambda) = (3,4)$. Then, a nonabelian group G of order 36 contains a $(12,3,12,4)$-DS if and only if*
$G \simeq (\mathbb{Z}_4 \ltimes \mathbb{Z}_3) \times \mathbb{Z}_3$ $(|Z(G)| = 6,\ U = O_3(Z(G)))$,
$A_4 \times \mathbb{Z}_3$ $(U = 1 \times Z_3)$, *or* $S_3 \times \mathbb{Z}_6$.

The first and the second cases have been previously known. But the third is a new one and has unusual properties.

Example 3.8. Let $G = \langle a,b,c \mid a^3 = b^2 = c^6 = 1,$
$b^{-1}ab = a^{-1},\ ac = ca,\ bc = cb\rangle$
and set $D = \{1, c, c^2, c^3, a, ac, b, a^2bc^5, abc^4, a^2bc, bc^4, abc\}$.
Then D is a non-symmetric $(12,3,12,4)$-DS relative to $U = \langle ac^2\rangle \simeq Z_3$.

Let $(\mathbb{P},\mathbb{B})(= \mathrm{dev}(D))$ be the corresponding transversal design and let A be an incidence matrix of $(\mathbb{P},\mathbb{B})$. Then

$$AA^T = \begin{bmatrix} 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I & 4J \\ 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 4J & 12I \end{bmatrix}$$

where, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $J = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

However,

$$A^TA = \begin{bmatrix} 12 & 4 & 4 & 4 & \cdots & \cdots & \cdots & 5 & 4 & 5 & 4 & 2 \\ 4 & 12 & 4 & 4 & \cdots & \cdots & \cdots & 4 & 5 & 4 & 5 & 4 \\ 4 & 4 & 12 & 4 & \cdots & \cdots & \cdots & 2 & 4 & 5 & 4 & 5 \\ 4 & 4 & 4 & 12 & \cdots & \cdots & \cdots & 4 & 2 & 4 & 5 & 4 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 4 & 5 & 4 & 2 & \cdots & \cdots & \cdots & 4 & 12 & 4 & 4 & 4 \\ 5 & 4 & 5 & 4 & \cdots & \cdots & \cdots & 4 & 4 & 12 & 4 & 4 \\ 4 & 5 & 4 & 5 & \cdots & \cdots & \cdots & 4 & 4 & 4 & 12 & 4 \\ 2 & 4 & 5 & 4 & \cdots & \cdots & \cdots & 4 & 4 & 4 & 4 & 12 \end{bmatrix}$$

An RDS D is called *symmetric* if $D^{(-1)}$ is also an RDS. Since AA^T has entries $2, 5 (\notin \{0, 4, 12\})$, the dual of $\mathrm{dev}(D)$ is not a transversal design. Applying Proposition 1.7, $D^{(-1)}$ is a non-symmetric RDS. As far as I know, this is the only known non-symmetric RDS.

CASE $(u, \lambda) = (4, 3)$, $|G| = 48$, $|U| = 4$

In this case $(m, u, k, \lambda) = (12, 4, 12, 3)$.

We use the following two lemmas to settle the present case.

Lemma 3.9. *If $2 \mid u$ and $2 \nmid \lambda$, then U contains every involution of G.*

Lemma 3.10. (Elliott-Butson) *Let D be a $(u\lambda, u, u\lambda, \lambda)$-DS in G relative to U. If U contains a normal subgroup N of G of order v, then DN/N is a $(u\lambda, u/v, u\lambda, v\lambda)$-DS in G/N relative to U/N.*

Applying Lemmas 3.5 and 3.10, it suffices to check the following case.

$$G = \langle a, b, c \mid a^4 = b^4 = c^3 = 1, c^{-1}ac = ab, c^{-1}bc = ab^2 \rangle, \quad U = \langle a^2, b^2 \rangle$$

We have the following by a computer search.

Lemma 3.11. *There is no $(12, 4, 12, 3)$-DS.*

CASE $(u, \lambda) = (6, 2)$, $|G| = 72$, $|U| = 6$.

In this case $(m, u, k, \lambda) = (12, 6, 12, 2)$.

Observation. For every known semiregular RDS, $|U|$ is a power of a prime.

The smallest undecided case is $u = 6$.

Lemma 3.12. *Let G be a nonabelian group of order 72. Then there are five possibilities.*
(i) $G \simeq SL(2,3) \times \mathbb{Z}_3$, (ii) $G \simeq A_4 \times \mathbb{Z}_6$, (iii) $G \simeq A_4 \times S_3$,
(iv) $G \simeq \langle t \rangle (M \times T)$, $t^2 = 1$, $M \simeq A_4$, $T \simeq \mathbb{Z}_3$,
$\langle t \rangle M \simeq S_4$, $\langle t \rangle T \simeq S_3$,
(v) $G \rhd Q$, $|Q| = 9$.

Applying Lemmas 3.5, 3.7 and 3.10, we have the following by a computer search.

Lemma 3.13. *There is no* $(12, 6, 12, 2)$*-DS.*

CASE $(u, \lambda) = (12, 1)$, $|G| = 144$, $|U| = 12$.

In this case $(m, u, k, \lambda) = (12, 12, 12, 1)$.

By Lemma 1 of [2], the following hods.

Theorem 3.14. *Every transversal design with $\lambda = 1$ is symmetric.*

The above theorem implies that if there is a $(u, u, u, 1)$-DS in a group G, then the corresponding transversal design can be extended to a projective plane of order u which admits G as a collineation group of order u^2. Thus $u \neq 12$ by Baumert-Hall [4] and the following holds.

Lemma 3.15. *There is no* $(12, 12, 12, 1)$*-DS.*

By Lemmas 3.5, 3.7, 3.11, 3.13 and 3.15, we have the following.

Theorem 3.16. *A group G contains a $(u\lambda, u, u\lambda, \lambda)$-DS D with $|D| = 12$ if and only if G is isomorphic to one of the following.*

(i) $(u, \lambda) = (2, 6)$, $G = \mathbb{Q}_8 \times \mathbb{Z}_3$, $U = Z(Q_8) \times 1 \simeq \mathbb{Z}_2$.

(ii) $(u, \lambda) = (2, 6)$, $G = \mathbb{Q}_{24}$, $U = Z(Q_{24}) \simeq \mathbb{Z}_2$.

(iii) $(u, \lambda) = (2, 6)$, $G = SL(2,3)$, $U = Z(SL(2,3)) \simeq \mathbb{Z}_2$.

(iv) $(u, \lambda) = (3, 4)$, $G = S_3 \times \mathbb{Z}_6$, $U \simeq \mathbb{Z}_3$ (*a non-symmetric RDS*).

(v) $(u, \lambda) = (3, 4)$, $G = (\mathbb{Z}_4 \ltimes \mathbb{Z}_3) \times \mathbb{Z}_3$, $|Z(G)| = 6$, $U = O_3(Z(G))$.

(vi) $(u, \lambda) = (3, 4)$, $G = A_4 \times \mathbb{Z}_3$, $U = 1 \times Z_3$.

(vii) $(u, \lambda) = (3, 4)$, $G = \mathbb{Z}_6 \times \mathbb{Z}_6$, $U \simeq \mathbb{Z}_3$.

4. Construction of non-symmetric RDS's

In this section we show the following.

Theorem 4.1. *There exists a non-symmetric* $(2^{2m}3^m, 3, 2^{2m}3^m, 2^{2m}3^{m-1})$-*difference set in* $(S_3 \times Z_6) \times \overbrace{(\mathbb{Z}_6 \times \mathbb{Z}_2) \times \cdots \times (\mathbb{Z}_6 \times \mathbb{Z}_2)}^{(m-1) times}$ *relative to* $U \times 1 \times \cdots \times 1$, *where* D *is a non-symmetric* $(12, 3, 12, 4)$-*difference set in* $S_3 \times \mathbb{Z}_6$ *relative to its non-normal subgroup* U *of order 3 (see Example 3.8).*

Corollary 4.2. *There exists a non-symmetric* $TD_{2^{2m}3^{m-1}}[2^{2m}3^m; 3]$ *for every* $m \in \mathbb{N}$.

Example 4.3.

In order to prove Theorem 4.1, we need the following lemma.

Lemma 4.4. *Let* $L = G \times H$, *where* G *be a group of order* $u^2\lambda$ *and* H *is a group of order* $u\mu$. *Let* D *be a* $(u\lambda, u, u\lambda, \lambda)DS$ *in* G *relative to a subgroup* U *of* G *of order* u *and let* C *be a* $(u\mu, u, u\mu, \mu)DS$ *in* $U \times H$ *relative to* U. *Then*

(i) *CD is a* $(u^2\lambda\mu, u, u^2\lambda\mu, u\lambda\mu)DS$ *in* L *relative to* U.

(ii) *CD is symmetric if and only if D is symmetric.*

Proof. Let $c_1, c_2 \in C$ and $d_1, d_2 \in D$ and assume $c_1d_1 = c_2d_2$. Then $c_1^{-1}c_2 = d_1d_2^{-1} \in UH \cap G = U$. Thus $d_1 = d_2$ and so $c_1 = c_2$. Therefore CD is a subset of L.

By assumption, the following hold.

$$DD^{(-1)} = u\lambda + \lambda(G - U) \tag{5}$$

$$CC^{(-1)} = u\mu + \mu(UH - U) \tag{6}$$

$$G = UD, \quad UC = UH \tag{7}$$

Hence $(CD)(CD)^{(-1)} = C(DD^{(-1)})C^{(-1)} = C(u\lambda + \lambda(G - U))C^{(-1)}$ $= u\lambda CC^{(-1)} + \lambda CGC^{(-1)} - \lambda CUC^{(-1)}$ As $C, U \subset UH \rhd U$, $CU = UC$. Similarly $GC = CG$. It follows that $(CD)(CD)^{(-1)} = u\lambda(u\mu + \mu(UH - U) + \lambda GCC^{(-1)} - \lambda UCC^{(-1)} = u^2\mu\lambda + u\mu\lambda UH - u\mu\lambda U + \lambda G(u\mu + \mu UH - \mu U) - \lambda U(u\mu + \mu UH - \mu U) = u^2\mu\lambda + u\mu\lambda(L - U)$. Thus we have (i).

Since $UH \rhd U$, $C^{(-1)}C = CC^{(-1)}$. Hence $(CD)^{(-1)}CD = D^{(-1)}(CC^{(-1)})D = D^{(-1)}(u\mu + \mu UH - \mu U))D$. By (7), the following holds.

$$(CD)^{(-1)}CD = u\mu D^{(-1)}D + u\mu\lambda L - u\mu\lambda G \tag{8}$$

Assume CD is symmetric. Then $(CD)^{(-1)}CD = u^2\mu\lambda + u\mu\lambda(L - V)$ for a subgroup V of L of order u. By (8), $u\mu D^{(-1)}D - u\mu\lambda G = u^2\mu\lambda - u\mu\lambda V$. Thus $D^{(-1)}D = u\lambda + \lambda(G - V)$. In partucular, V is a subgroup of G of order u and so D is symmetric. Conversely, assume D is symmetric. Then $D^{(-1)}D = u\lambda + \lambda(G - V)$ for a subgroup V of G of order u. Then, by (8), $(CD)^{(-1)}CD = u\mu(u\lambda + \lambda(G - V)) + u\mu\lambda L - u\mu\lambda G = u^2\mu\lambda + u\mu\lambda(L - V)$. Therefore CD is symmetric. Thus we have (ii). □

We note that Lemma 4.4(i) is a modification of Result 2.4 of [11], where N is assumed to be normal in G.

Proof of Theorem

Let D be a non-symmetric $(12,3,12,4)$DS in $M = S_3 \times \mathbb{Z}_6$ relative to a non-normal subgroup U of M (see Example 3.8). Let $H = \langle a \rangle \times \langle b \rangle (\simeq \mathbb{Z}_6 \times Z_3)$. We note that an abelian group $H \times \langle c \rangle (\simeq \mathbb{Z}_6 \times \mathbb{Z}_2 \times \mathbb{Z}_3)$ contains $(12,3,12,4)$DS, say $\{1, a, a^2, a^3, a^4c, a^5c, bc^2, ab, a^2bc, a^3bc^2, a^4bc, a^5b\}$. Set $G = H \times M$ and choose $\langle c \rangle$ as a non-normal subgroup of M. Then, applying Lemma 4.4, $H \times M$ contains a non-symmetric $(2^2 3^2 \cdot 4, 3, 2^2 3^2\, 2^2 3 \cdot 4)$DS in G relative to $1 \times U (\simeq \mathbb{Z}_3)$ as D is non-symmetric. Repeating this procedure again and again we have the theorem.

References

[1] R. C. Bose and W.S. Connor, Combinatorial properties of group divisible incomplete block designs, Ann. Math. Stat. **23** (1952) 367-383.

[2] R. C. Bose, Symmetric group divisible designs with the dual property, J. Stat. Planning Inf. **1** (1977) 87-101.

[3] W.S. Connor, Some relations among the blocks of symmetric group divisible designs, *Ann. Math. Stat.* **23** (1952), 602-609.

[4] L. Baumert and M. Hall, Nonexistence of certain planes of order 10 and 12, *J. Combin. Theory, Ser. A* **14** (1973), 273-280.

[5] Y. Hiramine, Semiregular relative difference sets in 2-groups containing a cyclic subgroup of index 2, *J. Combin. Theory, Ser. A* **99** (2002), 358-370.

[6] Y. Hiramine, On $(2n, 2, 2n, n)$ relative difference sets, *J. Combin. Theory, Ser. A* **101** (2003), 281-284.

[7] N. Ito, On Hadamard Groups, *J. Algebra* **168** (1994), 981-987.

[8] D. Jungnickel, On automorphism group of divisible designs, *Canad. J. Math.* **34** (1982), 257-297.

[9] B. Schmidt, On (p^a, p^b, p^a, p^{a-b})-Relative Difference Sets, *Journal of Algebraic Combinatorics* **6** (1997), 279-297.