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The shape of a tridiagonal pair

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We announce that the shape conjecture is proved in general for the case where the parameter q is not a root of unity.

Let V be a finite dimensional vector space over $\mathbf{C}$ and A, A^* semisimple linear transformations of V. Let $V=\bigoplus_{i=0}^{d} V_i=\bigoplus_{i=0}^{d^*} V_i^*$ denote the eigenspace decompositions of V for A, A^*, respectively. The pair A, A^* is called a TD-pair (tridiagonal pair) if there exist an ordering of V_i and that of V_i^* such that

(1) $AV_i^* \subset V_{i-1}^* + V_i^* + V_{i+1}^* \quad (0 \leq i \leq d^*)$,
$A^*V_i \subset V_{i-1} + V_i + V_{i+1} \quad (0 \leq i \leq d)$,

where $V_{-1}=V_{d+1}=V_{-1}^*=V_{d^*+1}^*=0$, and

(2) V is irreducible as an $< A, A^* >$-module.

A TD-pair arises from a P- and Q-polynomial association scheme with A, A^* the standard generators of the Terwilliger algebra that describe the P- and Q-polynomial structures of the association scheme, and with V an irreducible submodule of the standard module of the Terwilliger algebra.

In [1], it is shown that

(i) $d=d^*$,

(ii) $\rho_i = dimV_i = dimV_{d-i} = dimV_i^* = dimV_{d-i}^* \quad (0 \leq i \leq d)$,

and the sequence ρ_i is conjectured to satisfy

(iii) (the shape conjecture) $\rho_i \leq \binom{d}{i}$.

We announce that the shape conjecture is true in the case where the parameter q is not a root of unity. The proof will be published in a subsequent paper.

A TD-pair satisfies the TD-relations (tridiagonal relations) [1]:

$$\text{(TD)} \quad \begin{aligned} &[A,[A,[A,A^*]_{q^{-1}}]_q] = [\gamma A^2 + \delta A, A^*], \\ &[A^*,[A^*,[A^*,A]_{q^{-1}}]_q] = [\gamma^* A^{*2} + \delta^* A^*, A] \end{aligned}$$

for some $\gamma, \gamma^*, \delta, \delta^* \in \mathbf{C}$, where $[x,y] = xy - yx$, $[x,y]_q = qxy - q^{-1}yx$. By applying an affine transformation to A and another one to A^*, we can normalize the TD-relations to have $\gamma = \gamma^* = 0$ when $q \neq 1$. Such TD-relations are regarded as a q-analogue of the Dolan-Grady relations that define the Onsager algebra. The shape conjecture is proved by analysing the infinite dimensional algebra generated by two symbols over $\mathbf{C}$ suject to the q-Dolan-Grady relations. In the case of $\delta = \delta^* = 0$, the q-Dolan-Grady relations are the q-Serre relations and the shape conjecture is already proved in [2].

References

[1] T. Ito, K. Tanabe, and P. Terwilliger. Some algebra related to P- and Q-polynomial association schemes, in: *Codes and Association Schemes (Piscataway NJ, 1999)*, Amer. Math. Soc., Providence RI, 2001, pp. 167–192; `arXiv:math.CO/0406556`.

[2] T. Ito and P. Terwilliger. The shape of a tridiagonal pair. *J. Pure Appl. Algebra* **188** (2004) 145–160; `arXiv:math.QA/0304244` .