数理解析研究所講究録 1476 巻 2006 年 48-56

On the Erdős r-sparse conjecture and automorphisms: *some recent developments*

名古屋大学大学院・情報科学研究科
藤原　祐一郎 (Yuichiro Fujiwara)

Graduate School of Information Science, Nagoya University,
Furo-cho, Chikusa-ku, 464-8601, Japan
E-mail: fujiwara@math.cm.is.nagoya-u.ac.jp

Abstract

In 1976, Paul Erdős conjectured that there is an integer $v_0(r)$ such that for every $v > v_0(r)$ and $v \equiv 1,3 \pmod 6$, there exists a Steiner triple system of order v containing no i blocks on $i+2$ points for every $1 < i \leq r$. Such an STS is said to be r-sparse. This article briefly surveys recent developments on the existence of r-sparse trile systems with certain automorphisms. Complete proofs for unpublished results shall be provided in future papers.

1 Introduction

A *Steiner triple system* S of *order* v, briefly STS(v), is an ordered pair $(V,\mathcal{B})$, where V is a finite set of v elements called *points*, and $\mathcal{B}$ is a set of 3-element subsets of V called *blocks*, such that each unordered pair of distinct elements of V is contained in exactly one block of $\mathcal{B}$. It is well-known that an STS(v) exists if and only if $v \equiv 1,3 \pmod 6$; such orders are called *admissible*.

A (k,l)-*configuration* in an STS is a set of l blocks whose union contains precisely k points. The unique $(6,4)$-configuration, called the *Pasch configuration*, is described by six distinct points on four blocks $\{a,b,c\}$, $\{a,d,e\}$, $\{f,b,d\}$ and $\{f,c,e\}$. One of two $(7,5)$-configurations is called the *mitre*, described by seven distinct points on five blocks $\{a,b,e\}$, $\{a,c,f\}$, $\{a,d,g\}$, $\{b,c,d\}$ and $\{e,f,g\}$; a

is referred to as the *centre* or *central element* of the mitre and the unique pair of blocks with no common point, that is, $\{b,c,d\}$ and $\{e,f,g\}$, is referred to as the *parallel blocks*. The other $(7,5)$-configuration, the *mia*, is obtained by joining two noncollinear points in a Pasch configuration: $\{a,b,c\}$, $\{a,d,e\}$, $\{f,b,d\}$, $\{f,c,e\}$ and $\{g,c,d\}$. An STS is said to be *anti-Pasch* or *anti-mitre* if it contains no Pasch configuration or mitre configuration, respectively. In particular, an anti-Pasch STS does not contain a mia configuration.

In 1976, Erdős [7] conjectured that for every $r \geq 4$ there is an integer $v_0(r)$ such that for every $v > v_0(r)$, $v \equiv 1,3 \pmod 6$, there is an STS(v) containing no $(j+2,j)$-configuration for every $2 \leq j \leq r$. Such an STS is said to be *r-sparse*. Every STS is 3-sparse and an r-sparse STS is also $(r-1)$-sparse. An STS is 4-sparse if and only if it is anti-Pasch; and it is 5-sparse if and only if it is both anti-Pasch and anti-mitre.

As well as in combinatorial design theory, 4- and 5-sparse triple systems with particular properties are also important in some applications to information theory (see, for example, Chee, Colbourn and Ling [3], Johnson and Weller [16], Vasic, Kurtas and Kuznetsov [23] and Vasic and Milenkovic [24]), and hence constructions for an r-sparse STS and related designs are studied extensively from both sides (see Fujiwara [9, 10, 11], Wolfe [25] and Colbourn and Rosa [5]). Also, sparseness of triple systems has been studied from the view of extremal set theory (see Lefmann, Phelps and Rödl [17]).

Frequently, actions of a finite group on a triple system have helped us discover an r-sparse STS and develop a construction method. An *automorphism* of an STS$(v) = (V,\mathcal{B})$ is a permutation on V that maps each block in $\mathcal{B}$ to a block of $\mathcal{B}$, and the *full automorphism group* is the group of all automorphisms of the STS. A *flag* of an STS $(V,\mathcal{B})$ is a pair (x,B) with $x \in V$ and $B \in \mathcal{B}$.

An STS is said to be *point-transitive* if its full automorphism group contains a subgroup which acts transitively on the point set. Similarly, we say that an STS is *block-transitive*, *flag-transitive*, *2-transitive*, or *2-homogeneous* if its full automorphism group contains a subgroup which acts transitively on the blocks, flags, ordered pairs of points, or unordered pairs of points, respectively.

The well-known construction for STSs of Netto [20] involving regular actions of $GF(q)$ on the point set generates 4- and 5-sparse STSs. The direct product construction for 5-sparse triple systems developed by Ling [18] employs an abelian group which acts regularly on the point set.

Theorem 1.1 **(Ling [18])** *If there exist a point-transitive* 5*-sparse* STS(v) *over an abelian group,* $v \equiv 1 \pmod 6$ *and a* 5*-sparse* STS(w)*, then there exists a* 5*-sparse*

STS(vw).

Forbes, Grannell and Griggs [8] discovered a construction method for block-transitive STSs and found examples of 6-sparse STSs, which have the highest sparseness at the time of writing. They also developed a recursive construction similar to Theorem 1.1 for block-transitive 6-sparse STSs and constructed infinitely many examples of such STSs. No 6-sparse STS other than these triple systems is known.

Also, when examining properties of an STS by using computers, group actions often simplify its calculations. In fact, by checking for r-sparseness the block-transitive STSs arising from one of known constructions, Forbes, Grannell and Griggs [8] found the first examples of 6-sparse STSs. By limiting the search to point-transitive STS(v) over cyclic groups, Colbourn, Mendelsohn, Rosa and Širáň [4] found a 5-sparse STS(v) for nearly all admissible $v < 100$.

Furthermore, an r-sparse STS with certain automorphisms is of some use for LDPC codes (see, for example, Vasic, Kurtas and Kuznetsov [23] and Vasic and Milenkovic [24]).

This article briefly surveys recent developments on the existence of r-sparse trile systems with nontrivial automorphsisms. In section 2, we consider 4- and 5-sparse STSs. In section 3, we list recent results on an STS with higher sparseness. Complete proofs for unpublished results shall be provided in future papers.

2 4- and 5-sparse systems

In this section, we mainly consider sharply point-transitive 4- and 5-sparse STSs. The existence problem of 4-sparse STS was completely settled by Grannell, Griggs and Whitehead [14]:

Theorem 2.1 (Grannell, Griggs and Whitehead) [14] *There exists a* 4*-sparse* STS(v) *if and only if* $v \equiv 1, 3 \pmod 6$ *and* $v \neq 7, 13$.

Many of the construction techniques for 4-sparse STSs due to Ling, Colbourn, Grannell and Griggs [19] and Grannell, Griggs and Whitehead [14] are generalized for 5-sparse systems by the author [10] and Wolfe [26]. Recently, Wolfe [26] proved that there exists a 5-sparse STS for, in some sense, almost all admissible orders.

Let S and T be two subsets of $\boldsymbol{Z}^+ = \{1,2,3,\dots\}$. Define the arithmetic density of S as compared to T as:

$$d(S;T) = \lim_{n\to\infty} \frac{|\{x \in S \cap T : x \leq n\}|}{|\{x \in T : x \leq n\}|}.$$

Theorem 2.2 (Wolfe) [26] *The arithmetic density of the spectrum of* 5*-sparse Steiner triple systems as compared to the set of all admissible orders is* 1.

As is mentioned, 4- and 5-sparse STSs of small or prime power orders had been known to exist. The author [11] recently gave general constructions for sharply point-transitive 4- and 5-sparse STSs over an abelian group G. Often a sharply point-transitive STS is simply said to be *transitive*. Transitive STS(v) over the cyclic group of order v is said to be *cyclic*.

Theorem 2.3 (Fujiwara) [11] *There exists a cyclic* 4*-sparse* STS(v) *for* $v \equiv 3 \pmod 6$ *satisfying one of the condition (i)* $(v,27) \neq 9$, *(ii)* $v \equiv 0 \pmod 7$, *or (iii)* $v \equiv 0 \pmod 5$.

Theorem 2.4 (Fujiwara) [11] *If there exist a cyclic* 5*-sparse* STS(v) *and a cyclic* 5*-sparse* STS(w), *where* $v,w \equiv 1 \pmod 6$, *then there exists a cyclic* 5*-sparse* STS(vw).

Theorem 2.5 (Fujiwara) [11] *If there exist a transitive* 5*-sparse* STS(v) *over an abelian group* G, $v \equiv 1 \pmod 6$ *and a transitive* 5*-sparse* STS(w) *over an abelian group* G', *then there exists a transitive* 5*-sparse* STS(vw) *over* $G \times G'$.

3 Higher sparseness and automorphisms

In this section, we deal with an STS with higher sparseness.

In the previous section, we saw that the Erdős r-sparse conjecture is true for $r = 4$ and that a 5-sparse STS exists for almost all admissible orders. While the Erdős r-sparse conjecture says that for any $r \geq 4$ an r-sparse STS(v) exists for all sufficiently large admissible v, little is known about the existence of an STS with higher sparseness. In fact, no example of r-sparse systems is realized for $r \geq 7$ (and $v > 3$), and no affirmative answer to the r-sparse conjecture is known in this range. As is mentioned, the only existence result on r-sparse STSs for $r \geq 6$ is the infinite series due to Forbes, Grannell and Griggs [8].

For an STS of higher sparseness admitting a transitive automorphism group, the author [12] gave some nonexistence results. In what follows, we ignore the two trivial systems, that is, STS(1) and STS(3), unless they play a significant role.

Theorem 3.1 (Fujiwara) [12] *For every $r \geq 13$, there exists no point-transitive* STS *over an abelian group.*

This bound can be strengthened whit ceratin additional condition. A point-transitive STS $(V, \mathcal{B})$ over a group G has a *short orbit* if there exist a block $B \in \mathcal{B}$ and an element $x \in G$ such that $B^x = B$ and $x \neq 1$, the identity element. $(V, \mathcal{B})$ has a $\boldsymbol{Z}_3$*-orbit* if $\mathcal{B}$ contains a block having the form $\{a, a^x, a^{x^2}\}$, where $x^3 = 1$. $\boldsymbol{Z}_3$-orbit privent an STS from being high-sparse.

Theorem 3.2 (Fujiwara) [12] *Assume that there exists a point-transitive r-sparse* STS *over an abelian group G. Further, if the* STS *has a $\boldsymbol{Z}_3$-orbit, then $r \leq 9$.*

Following is an immediate corollary of these theorems.

Corollary 3.3 (Fujiwara) [12] *For every $r \geq 13$, there exists no cyclic r-sparse* STS(v). *In particular, when $v \equiv 3 \pmod 6$, no cyclic r-sparse* STS(v) *exists for every $r \geq 10$.*

The classification of STSs admitting other types of transitive actions and Theorem 3.1 gives further nonexistence results on an STS with higer sparseness. The details shall be presented in a future paper so we only mention the consequence.

Corollary 3.4 (Fujiwara) [12] *For every $r \geq 5$, there exists no 2-transitive r-sparse* STS.

Corollary 3.5 (Fujiwara) [12] *For every $r \geq 6$, there exists no 2-homogeneous r-sparse* STS.

Corollary 3.6 (Fujiwara) [12] *For every $r \geq 6$, there exists no flag-transitive r-sparse* STS.

Corollary 3.7 (Fujiwara) [12] *For every $r \geq 13$, there exists no block-transitive r-sparse* STS.

It is notable that the construction developed by Grannell, Griggs and Murphy [13] can generate finitely many examples of 6-sparse STSs but none of them is 7-sparse (see Forbes, Grannell and Griggs [8]).

The author [12] also gave stronger bounds on sparseness for Steiner triple systems admitting a nontrivial automorphism with fixed points.

An STS(v) is said to be *1-rotational* over a group G if it admits G as a subgroup of the full automorphism group and G fixes exactly one point and acts regularly on the other points. A 1-rotational automorphism is closely related to an involution.

An STS is said to be *reverse* if it admits an involutory automorphism fixing exactly one point. Any 1-rotational STS is reverse. Indeed, for every 1-rotational STS(v) over a group G, the order of G is $v-1$ and even. Hence, G has at least one involution.

Buratti [1] showed that there exists a 1-rotational STS(v) over an abelian group if and only if $v \equiv 3,9 \pmod{24}$ or $v \equiv 1,19 \pmod{72}$. He also gave partial answers for an arbitrary group. The combined work of Doyen [6], Rosa [21] and Teirlinck [22] established the fact that the spectrum for reverse STS is the set of all $v \equiv 1,3,9$ or $19 \pmod{24}$. An STS admitting an automorphism with more than one fixed point is known to exist (see Hartman and Hoffman [15]) and may also be considered. However, the fixed points must induce a smaller STS as a subsystem, and hence sparseness of the original Steiner system can not exceed that of the small sub-STS. Most interesting is the case when the induced subsystem is a trivial STS, that is, one point and no block, or three points and one block. The following theorem shows that such an STS is at most 4-sparse.

Theorem 3.8 (Fujiwara) [12] *For every $r \geq 5$, there exists no r-sparse* STS *admitting an involutory automorphism fixing exactly one or three points.*

The following is an immediate corollary of the theorem above.

Corollary 3.9 (Fujiwara) [12] *For every $r \geq 5$, there exists no reverse r-sparse* STS.

Since a 1-rotational STS is also reverse, we have:

Corollary 3.10 (Fujiwara) [12] *For every $r \geq 5$, there exists no 1-rotaional r-sparse* STS.

It is well known that the points and lines of $AG(n,3)$ forms the elements and triples of a 1-rotaional, and thus reverse, 4-sparse STS(3^n). In this sense, the bounds of Theorem 3.8, Corollary 3.9 and 3.10 are best possible.

Corollary 3.10 limits the sparseness of a 1-rotational STS over any finite group even if it is nonabelian. The same bound for a rotational group action fixing three points inducing the other trivial subsystem follows from the same argument. However, if groups are restricted to abelian ones, we can easily obtain much stronger theorem. In fact, sparseness is limited to the lowest.

Theorem 3.11 (Fujiwara) [12] *If the full automorphism group of an* STS *S contains an abelian subgroup which fixes more than one point and acts transitively on the other points, then S is not* 4*-sparse.*

In the remainder of this paper, we give a sporadic result on automorphisms, similar to those we have discussed.

An STS is said to be *bicyclic* if it admits a permutation on points consisting of a pair of cycles of length k and $v-k$ as an automorphism. Calahan and Gardner [2] proved that there exists a bicyclic STS(v) for $k>1$ if and only if $v \equiv 1,3 \pmod 6$, $k \mid v$, and either $k \equiv 1 \pmod 6$ and $3k \mid v$; or $k \equiv 3 \pmod 6$ and $k \neq 9$.

Theorem 3.12 (Fujiwara) [12] *Let S be a bicyclic r-sparse* STS *and l be length of the smaller cycle of its bicyclic automorphism. Then,*

$$r \leq \begin{cases} 4 & \textit{when} \quad l = 1,3, \\ 9 & \textit{when} \quad l \equiv 3 \pmod 6, \\ 12 & \textit{when} \quad l \equiv 1 \pmod 6. \end{cases}$$

References

[1] M. Buratti, 1-rotational Steiner triple systems over arbitrary groups, J. Combin. Des. **9** (2001), 215–226.

[2] R. Calahan-Zijlstra and R. B. Gardner, Bicyclic Steiner triple systems, Discrete Math. **128** (1994), 35–44.

[3] Y. M. Chee, C. J. Colbourn, and A. C. H. Ling, Asymptotically optimal erasure-resilient codes for large disk arrays, Discrete Appl. Math. **102** (2000), 3–36.

[4] C. J. Colbourn, E. Mendelsohn, A. Rosa, and J. Širáň, Anti-mitre Steiner triple systems, Graphs Combin. **10** (1994), 215–224.

[5] C. J. Colbourn and A. Rosa, Triple systems, Oxford University Press, New York, 1999.

[6] J. Doyen, A note on Steiner triple systems, Discrete Math. **1** (1972), 315–319.

[7] P. Erdős, Problems and results in combinatorial analysis, Creation in Math. **9** (1976), 25.

[8] A. D. Forbes, M. J. Grannell, and T. S. Griggs, On 6-sparse Steiner triple systems, preprint.

[9] Y. Fujiwara, Constructions for anti-mitre Steiner triple systems, J. Combin. Des. 13 (2005), 286–291.

[10] Y. Fujiwara, Infinite classes of anti-mitre and 5-sparse Steiner triple systems, J. Combin. Des. to appear.

[11] Y. Fujiwara, cyclic 4- and 5-sparse Steiner triple systems, submitted.

[12] Y. Fujiwara, Nonexistence of sparse triple systems over abelian groups and involutions, submitted.

[13] M. J. Grannell, T. S. Griggs, and J. P. Murphy, Some new perfect Steiner triple systems, J. Combin. Des. **7** (1999), 327–330.

[14] M. J. Grannell, T. S. Griggs, and C. A. Whitehead, The resolution of the anti-Pasch conjecture, J. Combin. Des. **8** (2000), 300–309.

[15] A. Hartman and D. G. Hoffman, Steiner triple systems with an involution, European J. Combin. **8** (1987), 371–378.

[16] S. J. Johnson and S. R. Weller, Resolvable 2-designs for regular low-density parity-check codes, IEEE Trans. Comm. **51** (2003), 1413–1419.

[17] H. Lefmann, K. T. Phelps, and V. Rödl, Extremal problems for triple systems, J. Combin. Des. **1** (1993), 379–394.

[18] A. C. H. Ling, A direct product construction for 5-sparse triple systems, J. Combin. Des. **5** (1997), 443–447.

[19] A. C. H. Ling, C. J. Colbourn, M. J. Grannell, and T. S. Griggs, Construction techniques for anti-Pasch Steiner triple systems, J. London Math. Soc. (2) **61** (2000), 641–657.

[20] E. Netto, Zur Theorie der Triplesyteme, *Math. Ann.* **42** (1893) 143-152.

[21] A. Rosa, On reverse Steiner triple systems, Discrete Math. **2** (1972), 61–71.

[22] L. Teirlinck, The existence of Steiner triple systems, Discrete Math. **6** (1973), 301–302.

[23] B. Vasic, E. M. Kurtas, and A. V. Kuznetsov, Kirkman systems and their application in perpendicular magnetic recording, IEEE Trans. Magn. **38** (2002), 1705–1710.

[24] B. Vasic and O. Milenkovic, Combinatorial constructions of low-density parity-check codes for iterative decoding, IEEE Trans. Inform. Theory **50** (2004), 1156–1176.

[25] A. Wolfe, The resolution of the anti-mitre Steiner triple system conjecture, J. Combin. Des. to appear.

[26] A. Wolfe, private communication, (2005).