数理解析研究所講究録 1476 巻 2006 年 57-64

ON THE ZEROS OF EISENSTEIN SERIES ASSOCIATED WITH $\Gamma_0^*(2)$, $\Gamma_0^*(3)$, AND SOME GROUPS

TSUYOSHI MIEZAKI, HIROSHI NOZAKI, JUNICHI SHIGEZUMI

Graduate school of Mathematics Kyushu University
Hakozaki 6-10-1 Higashi-ku, Fukuoka, 812-8581 Japan

1. Introduction

Let $k \geqslant 4$ be an even integer, for $z \in \mathbb{H} := \{z \in \mathbb{C} \,;\, Im(z) > 0\}$, let

(1) $$E_k(z) := \frac{1}{2} \sum_{(c,d)=1} (cz+d)^{-k}$$

be the *Eisenstein series* associated with $\mathrm{SL}_2(\mathbb{Z})$. Then,

$$\mathbb{F} := \left\{|z| \geqslant 1,\ -\frac{1}{2} \leqslant Re(z) \leqslant 0\right\} \bigcup \left\{|z| > 1,\ 0 \leqslant Re(z) < \frac{1}{2}\right\}$$

is a *fundamental domain* of $\mathrm{SL}_2(\mathbb{Z})$.

In [RSD], F. K. C. Rankin and H. P. F. Swinnerton-Dyer considered the problem of locating the zeros of $E_k(z)$ in $\mathbb{F}$. They proved that for $k = 12n + s$ ($s = 4, 6, 8, 10, 0,$ *and* 14), then n zeros are in $A := \{z \in \mathbb{C} \,;\, |z| = 1,\ \pi/2 < Arg(z) < 2\pi/3\}$. They also said in the last part of the paper, "This method can equally well be applied to Eisenstein series associated with subgroup of the modular group." However, it seems unclear how widely this claim holds.

Here, we consider the same problem for Fricke groups $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$(See [K], [Q]), which are commensurable groups of $\mathrm{SL}_2(\mathbb{Z})$. For a fixed prime p, we define the following;

(2) $$\Gamma_0^*(p) := \Gamma_0(p) \cup \Gamma_0(p)\, W_p,$$

where

(3) $$\Gamma_0(p) := \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,;\, c \equiv 0 \pmod p\right\}, \quad W_p := \begin{pmatrix} 0 & -1/\sqrt{p} \\ \sqrt{p} & 0 \end{pmatrix}.$$

Let $k \geqslant 4$ be an even integer, for $z \in \mathbb{H}$, let

(4) $$E_{k,p}^*(z) := \frac{1}{p^{k/2}+1}\left(p^{k/2}E_k(pz) + E_k(z)\right)$$

be the Eisenstein series associated with $\Gamma_0^*(p)$. Then the next regions

$$\mathbb{F}^*(2) := \left\{|z| \geqslant 1/\sqrt{2},\ -\frac{1}{2} \leqslant Re(z) \leqslant 0\right\} \bigcup \left\{|z| > 1/\sqrt{2},\ 0 \leqslant Re(z) < \frac{1}{2}\right\},$$

$$\mathbb{F}^*(3) := \left\{|z| \geqslant 1/\sqrt{3},\ -\frac{1}{2} \leqslant Re(z) \leqslant 0\right\} \bigcup \left\{|z| > 1/\sqrt{3},\ 0 \leqslant Re(z) < \frac{1}{2}\right\}$$

are fundamental domains of $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$, respectively.

Define $A_2^* := \{z \in \mathbb{C} \,;\, |z| = 1/\sqrt{2},\ \pi/2 < Arg(z) < 3\pi/4\}$, and $A_3^* := \{z \in \mathbb{C} \,;\, |z| = 1/\sqrt{3},\ \pi/2 < Arg(z) < 5\pi/6\}$. Then we have $\overline{A_2^*} = A_2^* \cup \{i/\sqrt{2}, e^{3\pi/4}/\sqrt{2}\}$, and $\overline{A_3^*} = A_3^* \cup \{i/\sqrt{3}, e^{5\pi/6}/\sqrt{3}\}$.

In this paper, we will apply the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer (RSD Method) to the Eisenstein series associated with $\Gamma_0^*(2)$ and $\Gamma_0^*(3)$. We will prove the next theorems.

Theorem 1. *Let $k \geqslant 4$ be an even integer. $E_{k,2}^*(z)$ has all zeros on $\overline{A_2^*}$.*

Theorem 2. *Let $k \geqslant 4$ be an even integer. $E_{k,3}^*(z)$ has all zeros on $\overline{A_3^*}$.*

Date: October 3, 2005.

2. $\Gamma_0^*(2)$ (Proof of Theorem1)

2.1. **Preliminaries.** We give the next definition;

$$F_{k,2}^*(\theta) := e^{ik\theta/2} E_{k,2}^*\left(e^{i\theta}/\sqrt{2}\right). \tag{5}$$

Before proving Theorem1, we consider an expansion of $F_{k,2}^*(\theta)$.

By the definition of $E_k(z)$, $E_{k,2}^*(z)$ (*cf.* (1),(4)), we have

$$\begin{aligned} &2(2^{k/2}+1)e^{ik\theta/2}E_{k,2}^*\left(e^{i\theta}/\sqrt{2}\right) \\ &= 2^{k/2}\sum_{(c,d)=1}(ce^{-i\theta/2}+\sqrt{2}de^{i\theta/2})^{-k} + 2^{k/2}\sum_{(c,d)=1}(ce^{i\theta/2}+\sqrt{2}de^{-i\theta/2})^{-k}. \end{aligned}$$

Now, we consider the case if c is *even*. We have

$$\begin{aligned} 2^{k/2}\sum_{\substack{(c,d)=1\\ c:even}}(ce^{-i\theta/2}+\sqrt{2}de^{i\theta/2})^{-k} &= 2^{k/2}\sum_{\substack{(c,d)=1\\ d:odd}}(2c'e^{-i\theta/2}+\sqrt{2}de^{i\theta/2})^{-k} \quad (c=2c') \\ &= \sum_{\substack{(c,d)=1\\ d:odd}}(\sqrt{2}c'e^{-i\theta/2}+de^{i\theta/2})^{-k} = \sum_{\substack{(c,d)=1\\ c:odd}}(ce^{i\theta/2}+\sqrt{2}de^{-i\theta/2})^{-k}. \end{aligned}$$

Thus we can write as follows;

$$F_{k,2}^*(\theta) = \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd}}(ce^{i\theta/2}+\sqrt{2}de^{-i\theta/2})^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1\\ c:odd}}(ce^{-i\theta/2}+\sqrt{2}de^{i\theta/2})^{-k}. \tag{6}$$

Hence we use this expression as a definition.

In the last part of this section, we compare the two series in this expression. Note that for any pair (c,d), $(ce^{i\theta/2}+\sqrt{2}de^{-i\theta/2})^{-k}$ and $(ce^{-i\theta/2}+\sqrt{2}de^{i\theta/2})^{-k}$ are conjugates of each other. The next lemma follows.

Lemma 2.1. *$F_{k,2}^*(\theta)$ is real, for $\forall\theta\in\mathbb{R}$.*

2.2. **Application of the RSD Method.** We will apply the method of F. K. C. Rankin and H. P. F. Swinnerton-Dyer (RSD Method) to the Eisenstein series associated with $\Gamma_0^*(2)$. We note that $N := c^2+d^2$.

Firstly, we consider the case $N=1$. Because c is odd, there are two cases, $(c,d)=(1,0)$ and $(c,d)=(-1,0)$. Then

$$F_{k,2}^*(\theta) = 2\cos(k\theta/2) + R_2^*, \tag{7}$$

where R_2^* is the summation of the rest terms.

Let $v_k(c,d,\theta) := |ce^{i\theta/2}+\sqrt{2}de^{-i\theta/2}|^{-k}$, then $v_k(c,d,\theta) = 1/\left(c^2+2d^2+2\sqrt{2}cd\cos\theta\right)^{k/2}$, and $v_k(c,d,\theta) = v_k(-c,-d,\theta)$.

Now we will consider the next three cases, namely $N=2,5$, and $N\geqslant 10$. Note that $\theta\in[\pi/2,3\pi/4]$. When $N=2$, $v_k(1,1,\theta)\leqslant 1$, $v_k(1,-1,\theta)\leqslant(1/3)^{k/2}$. When $N=5$, $v_k(1,2,\theta)\leqslant(1/5)^{k/2}$, $v_k(1,-2,\theta)\leqslant(1/3)^k$. When $N\geqslant 10$, $|ce^{i\theta/2}\pm\sqrt{2}de^{-i\theta/2}|^2\geqslant(c^2+d^2)/3 = N/3$, and the rest of the question is about the number of terms with $c^2+d^2=N$. Because c is odd, $|c|=1,3,...,2N'-1\leqslant N^{1/2}$, so the number of $|c|$ is not more than $(N^{1/2}+1)/2$. Thus the number of terms with $c^2+d^2=N$ is not more than $2(N^{1/2}+1)\leqslant 3N^{1/2}$, for $N\geqslant 5$. Then we get the upper bound $\frac{162}{k-3}\left(\frac{1}{3}\right)^{k/2}$.

Thus

$$|R_2^*| \leqslant 2+2\left(\frac{1}{3}\right)^{k/2}+2\left(\frac{1}{5}\right)^{k/2}+2\left(\frac{1}{3}\right)^{k}+\frac{162}{k-3}\left(\frac{1}{3}\right)^{k/2}. \tag{8}$$

Recalling "RSD Method", we want to show that $|R_2^*|<2$. But the right-hand side is greater than 2. The point is the case $(c,d)=\pm(1,1)$. We will consider the expansion of the method.

2.3. **Expansion of the RSD Method (1).** In the previous subsection, the point was the case $(c,d) = \pm(1,1)$. Notice that "$v_k(1,1,\theta) < 1 \Leftrightarrow \theta < 3\pi/4$". So we can easily expect that we get a good bound for $\theta \in [\pi/2, 3\pi/4 - x]$ for small $x > 0$. But if $k = 8n$, we need $|R_2^*| < 2$ for $\theta = 3\pi/4$ in this method. We will consider the case when $k = 8n, \theta = 3\pi/4$ in the next section.

Let $k = 8n + s$ ($n = m(k)$, $s = 4, 6, 0,$ *and* 10). If $k < 8$, then $n < 1$. Consequently, $F_{k,2}^*(\theta)$ has at least 0 zeros, which does not make sense. So we may assume that $k \geqslant 8$.

The first problem is how small x should be. We consider each of the cases $s = 4, 6, 0,$ *and* 10.

When $s = 4$, $(2n+1)\pi \leqslant k\theta/2 \leqslant (3n+1)\pi + \pi/2$. So the last integer point(*i.e.* ± 1) is $k\theta/2 = (3n+1)\pi$, then $\theta = 3\pi/4 - \pi/k$. Similarly, when $s = 6$, *and* 10, we have $\theta = 3\pi/4 - \pi/2k$, $3\pi/4 - 3\pi/2k$, respectively. When $s = 0$, the second to the last integer point is $\theta = 3\pi/4 - \pi/k$.

Thus we need $x \leqslant \pi/2k$.

Lemma 2.2. *Let $k \geqslant 8$. For $\forall\theta \in [\pi/2, 3\pi/4 - x]$ $(x = \pi/2k)$, $|R_2^*| < 2$.*

Before proving the above lemma, we need the following preliminaries.

Proposition 2.1.

(1) *If $0 \leqslant x \leqslant \pi/2$, then $\sin x \geqslant 1 - \cos x$.*
(2) *If $0 \leqslant x \leqslant \pi/16$, then $1 - \cos x \geqslant \frac{31}{64}x^2$.*

Proof of Lemma 2.2. Let $k \geqslant 8$ and $x = \pi/2k$, then $0 \leqslant x \leqslant \pi/16$.

$$|e^{i\theta/2} + \sqrt{2}e^{-i\theta/2}|^2 \geqslant 1 + \frac{31}{16}x^2. \quad (Prop. 2.1)$$

$$|e^{i\theta/2} + \sqrt{2}e^{-i\theta/2}|^k \geqslant 1 + \frac{k}{2}\frac{31}{16}x^2 \geqslant 1 + \frac{31}{4}x^2. \quad (k \geqslant 8)$$

$$v_k(1,1,\theta) \leqslant 1 - \frac{(31/4)}{1 + (31/4)x^2}x^2 \leqslant 1 - \frac{31 \times 256}{31\pi^2 + 1024}x^2.$$

Thus

$$2v_k(1,1,\theta) \leqslant 2 - \frac{31 \times 512}{31\pi^2 + 1024}\left(\frac{\pi}{2k}\right)^2 \leqslant 2 - \frac{265}{9}\frac{1}{k^2}.$$

In inequality(15), replace 2 with the bound $2 - \frac{265}{9}\frac{1}{k^2}$. Then

$$|R_2^*| \leqslant 2 - \frac{265}{9}\frac{1}{k^2} + 35\left(\frac{1}{3}\right)^{k/2} \quad (k \geqslant 8).$$

Finally, we can show that $35\left(\frac{1}{3}\right)^{k/2} < \frac{265}{9}\frac{1}{k^2}$. So, the proof is complete. □

2.4. **Expansion of the RSD Method (2).** For the case "$k = 8n, \theta = 3\pi/4$", we need the next lemma.

Lemma 2.3. *Let k be an integer such that $k = 8n$ for $\exists n \in \mathbb{N}$. If n is even, then $F_{k,2}^*(3\pi/4) > 0$. On the other hand if n is odd, then $F_{k,2}^*(3\pi/4) < 0$.*

Before proving this lemma, recall that $E_k(z)$ is the *modular form* of weight k for $\mathrm{SL}_2(\mathbb{Z})$ for $k \geqslant 4$: *even*. Then

$$E_k(z+1) = E_k(z), \quad E_k(-1/z) = z^k E_k(z). \tag{9}$$

Proof of Lemma 2.3. Let $k = 8n$ $(n \geqslant 1)$. By the definition of $E_{k,2}^*(z), F_{k,2}^*(z)$ (*cf.* (4),(12)), we have

$$F_{k,2}^*(3\pi/4) = \frac{e^{i3(k/8)\pi}}{2^{k/2} + 1}\left(2^{k/2}E_k(-1+i) + E_k\left(\frac{-1+i}{2}\right)\right).$$

By using the equations (9), $E_k(-1+i) = E_k(i)$, $E_k((-1+i)/2) = 2^{k/2}E_k(i)$. Then

$$F_{k,2}^*(3\pi/4) = 2e^{i(k/8)\pi}\frac{2^{k/2}}{2^{k/2}+1}F_k(\pi/2).$$

The next question is: "Which one holds; $F_k(\pi/2) < 0$ or $F_k(\pi/2) > 0$?".

In [RSD], they showed $F_k(\theta) := e^{ik\theta/2}E_k(\theta) = 2\cos(k\theta/2) + R_1$. Then they proved $|R_1| < 2$ for $k \geqslant 12$. Moreover, for $k = 8$, $|R_1|$ is not more than $1.29658... < 2$. It is monotonically decreasing in k. Thus we can show

$$|R_1| < 2 \quad for \quad \forall k \geqslant 8. \tag{10}$$

When $k = 8n$,

$$F^*_{8n,2}(3\pi/4) = 2e^{in\pi}\frac{2^{4n}}{2^{4n}+1}F_{8n}(\pi/2),$$

where $\frac{2^{4n}}{2^{4n}+1} > 0$, $F_{8n}(\pi/2) = 2\cos(2n\pi) + R_1 > 0$. So the sign($\pm$) of $F^*_{k,2}(3\pi/4)$ is that of $e^{in\pi}$. Thus the proof is complete. □

2.5. **Valence formula for $\Gamma_0^*(2)$.** In order to decide the locating of all zeros of $E^*_{k,2}(z)$, we need the valence formula for $\Gamma_0^*(2)$:

Proposition 2.2. *Let f be a modular function of weight k for $\Gamma_0^*(2)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{2}}(f) + \frac{1}{4}v_{\rho_2}(f) + \sum_{\substack{p\in\Gamma_0^*(2)\backslash\mathbb{H} \\ p\neq i/\sqrt{2},\ \rho_2}} v_p(f) = \frac{k}{8}, \tag{11}$$

where $v_p(f)$ is the order of f at p, and $\rho := e^{2\pi/3}$, and $\rho_2 := e^{3\pi/4}/\sqrt{2}$.

Remark 1. *Let $k \geqslant 4$ be an even integer. We have*

$$v_{i/\sqrt{2}}(E^*_{k,2}) = s_k \quad (s_k = 0,1 \text{ such that } 2s_k \equiv k \pmod 4),$$
$$v_{\rho_2}(E^*_{k,2}) = t_k \quad (t_k = 0,1,2,3 \text{ such that } -2t_k \equiv k \pmod 8).$$

3. $\Gamma_0^*(3)$ (Proof of Theorem2)

3.1. **Preliminaries.** We give the next definition;

$$F^*_{k,3}(\theta) := e^{ik\theta/2}E^*_{k,3}\left(e^{i\theta}/\sqrt{3}\right). \tag{12}$$

By the definition of $E_k(z), E^*_{k,3}(z)$ (*cf.* (1),(4)), we have

$$2(3^{k/2}+1)e^{ik\theta/2}E^*_{k,3}\left(e^{i\theta}/\sqrt{3}\right)$$
$$= 3^{k/2}\sum_{(c,d)=1}(ce^{-i\theta/2}+\sqrt{3}de^{i\theta/2})^{-k} + 3^{k/2}\sum_{(c,d)=1}(ce^{i\theta/2}+\sqrt{3}de^{-i\theta/2})^{-k}.$$

We consider the case if 3 is divisible by c. Then we can write as follows;

$$F^*_{k,3}(\theta) = \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 3\nmid c}}(ce^{i\theta/2}+\sqrt{3}de^{-i\theta/2})^{-k} + \frac{1}{2}\sum_{\substack{(c,d)=1 \\ 3\nmid c}}(ce^{-i\theta/2}+\sqrt{3}de^{i\theta/2})^{-k}. \tag{13}$$

The next lemma follows.

Lemma 3.1. *$F^*_{k,3}(\theta)$ is real, for $\forall\theta\in\mathbb{R}$.*

3.2. **Application of the RSD Method.** We note that $N := c^2 + d^2$, and consider the case $N = 1$. Then we can write;

$$F^*_{k,3}(\theta) = 2\cos(k\theta/2) + R^*_3. \quad (\exists R^*_3 \in \mathbb{R}) \tag{14}$$

Let $v_k(c,d,\theta) := |ce^{i\theta/2}+\sqrt{3}de^{-i\theta/2}|^{-k}$. Now we will consider the next cases, namely $N = 2,5,10,13,17$, and $N \geqslant 25$. Considering $\theta \in [\pi/2, 5\pi/6]$, we calculate $v_k(c,d,\theta)$ for $N = 2,5,10,13,17$. Furthermore, for $N \geqslant 25$, we get the upper bound $\frac{352\sqrt{6}}{k-3}\left(\frac{1}{2}\right)^k$. Thus

$$|R^*_3| \leqslant 4 + 176\left(\frac{1}{2}\right)^k \tag{15}$$

Now, we want to show that $|R^*_3| < 2$. But the right-hand side is much greater than 2. The points are the cases $(c,d) = \pm(1,1),\ \pm(2,1)$.

3.3. **Expansion of the RSD Method (1).** In this subsection, we will prove following lemma.

Lemma 3.2. *Let $k \geqslant 8$. For $\forall\theta \in [\pi/2, 5\pi/6 - x]$ $(x = \pi/3k)$, $|R_3^*| < 2$.*

Before proving the above lemma, we need the following preliminaries.

Proposition 3.1.

(1) *For $k \geqslant 8$, $\left(\frac{3}{2}\right)^{2/k} \leqslant 1 + \left(2\log\frac{3}{2}\right)\frac{1}{k} + \frac{1}{2}\left(2\log\frac{3}{2}\right)^2\left(\frac{3}{2}\right)^{2/k}\frac{1}{k^2}$.*
(2) *For $k \geqslant 8$, $3 + 2\sqrt{3}\cos\left(\frac{5\pi}{6} - \frac{\pi}{3k}\right) \geqslant \frac{\pi}{\sqrt{3}}\frac{1}{k}$.*
(3) *For $k \geqslant 8$, and let $x = \pi/3k$, then $4 + 2\sqrt{3}\cos\left(\frac{5\pi}{6} - x\right) \geqslant \left(\frac{3}{2}\right)^{2/k}\left(1 + \frac{256\times7\times13}{3\times127\times k}x^2\right)$.*

Proposition 3.2.

(1) *For $k \geqslant 8$, $3^{2/k} \leqslant 1 + (2\log 3)\frac{1}{k} + \frac{1}{2}(2\log 3)^2 3^{2/k}\frac{1}{k^2}$.*
(2) *For $k \geqslant 8$, $6 + 4\sqrt{3}\cos\left(\frac{5\pi}{6} - \frac{\pi}{3k}\right) \geqslant \frac{2\pi}{\sqrt{3}}\frac{1}{k}$.*
(3) *For $k \geqslant 8$, and let $x = \pi/3k$, then $7 + 4\sqrt{3}\cos\left(\frac{5\pi}{6} - x\right) \geqslant 3^{2/k}\left(1 + \frac{256\times7\times13}{3\times127\times k}x^2\right)$.*

Proof of Lemma 3.2. Let $k \geqslant 8$ and $x = \pi/3k$, then $0 \leqslant x \leqslant \pi/24$.
By Proposition 3.1

$$|e^{i\theta/2} + \sqrt{3}e^{-i\theta/2}|^2 \geqslant \left(\frac{3}{2}\right)^{2/k}\left(1 + \frac{256\times7\times13}{3\times127\times k}x^2\right). \quad (Prop.3.1(3))$$

$$v_k(1,1,\theta) \leqslant \frac{2}{3} - \frac{107}{8}x^2.$$

Similarly, by Proposition 3.2

$$|2e^{i\theta/2} + \sqrt{3}e^{-i\theta/2}|^2 \geqslant 3^{2/k}\left(1 + \frac{256\times7\times13}{3\times127\times k}x^2\right). \quad (Prop.3.1(3))$$

$$v_k(2,1,\theta) \leqslant \frac{1}{3} - \frac{107}{16}x^2.$$

In inequality(15), replace 4 with these bounds. Then

$$|R_3^*| \leqslant 2 - \frac{107\pi^2}{24}\frac{1}{k^2} + 176\left(\frac{1}{2}\right)^k.$$

We can show that $176\left(\frac{1}{2}\right)^k < \frac{107\pi^2}{24}\frac{1}{k^2}$. □

3.4. **Expansion of the RSD Method (2).** For the case "$k = 12n, \theta = 5\pi/6$", we need the next lemma.

Lemma 3.3. *Let k be the integer such that $k = 12n$ for $\exists n \in \mathbb{N}$. If n is even, then $F_{k,3}^*(5\pi/6) > 0$. On the other hand, if n is odd, then $F_{k,3}^*(5\pi/6) < 0$.*

Proof. Let $k = 12n$ $(n \geqslant 1)$. By the definition of $E_{k,3}^*(z), F_{k,3}^*(z)$ (*cf.* (4),(12)), we have

$$F_{k,3}^*(5\pi/6) = \frac{e^{i5(k/12)\pi}}{3^{k/2}+1}\left(3^{k/2}E_k\left(\frac{-3+\sqrt{3}i}{2}\right) + E_k\left(\frac{-\sqrt{3}+i}{2\sqrt{3}}\right)\right).$$

By using the equations (9), for $k = 12n$,

$$F_{12n,3}^*(5\pi/6) = 2e^{in\pi}\frac{3^{6n}}{3^{6n}+1}F_{12n}(2\pi/3),$$

where $\frac{3^{6n}}{3^{6n}+1} > 0$, $F_{12n}(2\pi/3) = 2\cos(4n\pi) + R_1 > 0$(*cf.* (10)). So the sign($\pm$) of $F_{k,3}^*(5\pi/6)$ is that of $e^{in\pi}$. Thus the proof is complete. □

3.5. **Valence formula for $\Gamma_0^*(3)$.**

Proposition 3.3. *Let f be a modular function of weight k for $\Gamma_0^*(3)$, which is not identically zero. We have*

$$v_\infty(f) + \frac{1}{2}v_{i/\sqrt{3}}(f) + \frac{1}{6}v_{\rho_3}(f) + \sum_{\substack{p\in\Gamma_0^*(3)\backslash\mathbb{H} \\ p\neq i/\sqrt{3},\ \rho_3}} v_p(f) = \frac{k}{6}, \tag{16}$$

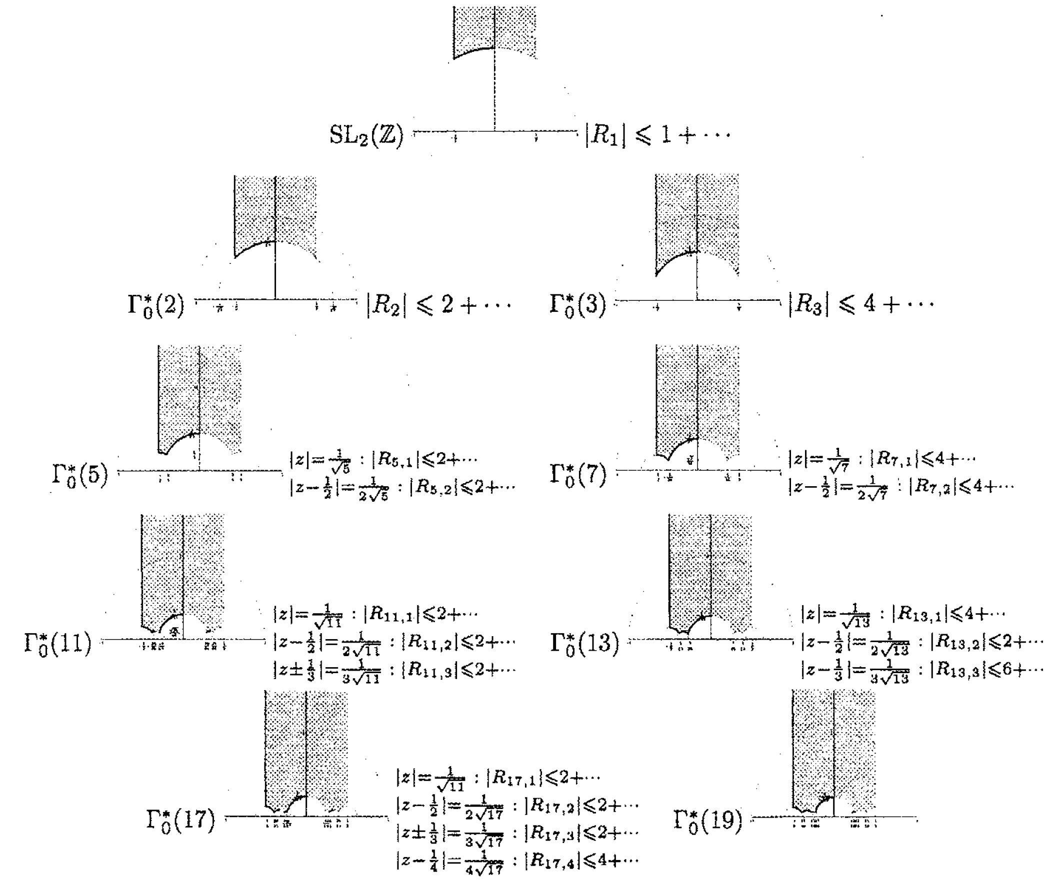
where $\rho_3 := e^{5\pi/6}/\sqrt{3}$.

Remark 2. *Let $k \geqslant 4$ be an even integer. We have*

$$v_{i/\sqrt{3}}(E_{k,3}^*) = s_k \quad (s_k = 0, 1 \text{ such that } 2s_k \equiv k \pmod 4),$$
$$v_{\rho_3}(E_{k,3}^*) = t_k \quad (t_k = 0, 1, 2, 3, 4, 5 \text{ such that } -2t_k \equiv k \pmod{12}).$$

4. On some other groups

4.1. **Frick group $\Gamma_0^*(p)$.** For $\mathrm{SL}_2(\mathbb{Z})$ and $\Gamma_0^*(p)$ for prime p such that $2 \leqslant p \leqslant 19$, their fundamental domains and bounds for Eisenstein series are following:



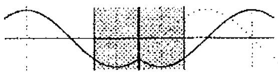
4.1.1. $\Gamma_0^*(5)$. We can write:

$$|z| = \frac{1}{\sqrt{5}} \ : \ e^{\frac{ik\theta_1}{2}} E_{k,5}^*\left(\frac{e^{i\theta_1}}{\sqrt{5}}\right) = 2\cos\left(\frac{k\theta_1}{2}\right) + R_{5,1}.$$

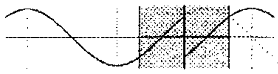
$$\left|z - \frac{1}{2}\right| = \frac{1}{2\sqrt{5}} \ : \ e^{\frac{ik\theta_2}{2}} E_{k,5}^*\left(\frac{e^{i\theta_2}}{2\sqrt{5}} - \frac{1}{2}\right) = 2\cos\left(\frac{k\theta_2}{2}\right) + R_{5,2}.$$

We have $\theta_2 = \theta_1 - \pi/2$, and $e^{\frac{ik\theta_2}{2}} = (-i)^{k/2} e^{\frac{ik\theta_1}{2}}$.

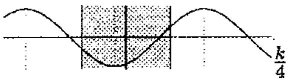
We consider the bound for $\theta_1 \in [\pi/2, \pi - \alpha - \pi/k]$ and $\theta_2 \in [\pi/2 - \alpha - k/2, \pi/2]$, where $\tan\alpha = 1/2$. If $4 \nmid p$, then we want to show $\frac{k-2}{4} - 1$ zeros are on the arcs. Also, if $4 \mid p$, then we want $\frac{k}{4}$ zeros to be there.



$\frac{k-2}{4} - 1 \rightarrow \frac{k-2}{4} - 3$



$\frac{k-2}{4} - 2$

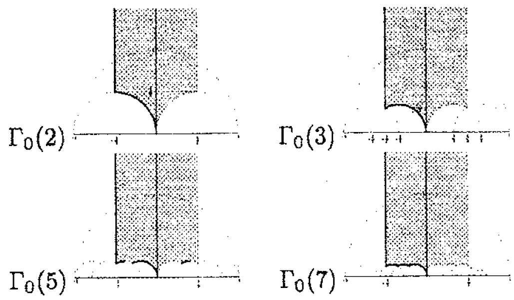


$\frac{k}{4} \rightarrow \frac{k}{4} - 2$

We proved $E^*_{k,5}$ has almost all zeros on the arcs $|z| = 1/\sqrt{5}$ and $|z - 1/2| = 1/2\sqrt{5}$ for $k \geqslant 12$, except for at most 2 points.

Similarly, we proved $E^*_{k,11}$ has almost all zeros on the arcs $|z| = 1/\sqrt{11}$, $|z - 1/2| = 1/2\sqrt{11}$, and $|z - 1/3| = 1/3\sqrt{11}$ for $k \geqslant 16$, except for at most 4 points.

4.2. **Congruence subgroup $\Gamma_0(p)$.** For $\Gamma_0^*(p)$ for prime p, their fundamental domains are following:



$\Gamma_0^*(p)$ has two cusps ∞ and 0. Thus we define Eisenstein Series for $\Gamma_0^*(p)$ as follows:

$$E^{\infty}_{k,p}(z) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p|c}} (cz+d)^{-k}, \quad \text{for cusp } \infty.$$

$$E^{0}_{k,p}(z) = \frac{1}{2} \sum_{\substack{(c,d)=1 \\ p\nmid c}} (cz+d)^{-k}, \quad \text{for cusp } 0.$$

4.2.1. $\Gamma_0(2)$. Considering that $\begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ -\sqrt{2} & 0 \end{pmatrix} : z \mapsto -1 - \frac{1}{2z}$, we have

$$E^{\infty}_{k,2}(\frac{e^{i\theta}}{2} - \frac{1}{2}) = \left(-\frac{1}{2} - \frac{i}{2}\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)\right)^k E^0_{k,2}(-\frac{1}{2} - \frac{i}{2}\tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right))$$

Furthermore, we have

$$e^{\frac{ik\theta}{2}} E^{\infty}_{k,2}(\frac{e^{i\theta}}{2} - \frac{1}{2}) = 2\cos(k\theta/2) + R : \text{real}.$$

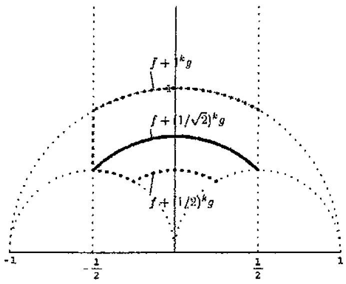
However, for $\theta \in [\pi/2, 0]$, we have $|R| \leqslant \infty$. This problem is much more difficult than $\Gamma_0^*(p)$. We proved

$$|R| < 2 \quad \text{for } \theta \in [\pi/2, \pi/6] \text{ and } k \geqslant 8.$$

We also proved

$$|R| < 2 \quad \text{for } \theta \in [\pi/2, \pi/12] \text{ and } k \geqslant 16,$$
$$|R| < 2 \quad \text{for } \theta \in [\pi/2, \pi/18] \text{ and } k \geqslant 20.$$

Finally, let $f = E^{\infty}_{k,2}(z)$, and $g = E^0_{k,2}(z)$. The locations of zeros of some linear combinations of f and g are following:



It is interesting question to consider the location of zeros of $f + r^k g$.

Remark 3. *Getz*[G] *considered a similar problem for the zeros of extremal modular forms of* $\mathrm{SL}_2(\mathbb{Z})$. *It seems that similar results do not hold for extremal modular forms of* $\Gamma_0^*(2)$ *and* $\Gamma_0^*(3)$. *We plan to look into this in the near future.*

Acknowledgement.

We thank Professor Eiichi Bannai for suggesting these problems as a master course project for us.

REFERENCES

[G] J. Getz, *A generalization of a theorem of Rankin and Swinnerton-Dyer on zeros of modular forms.*, Proc. Amer. Math. Soc., 132(2004), No. 8, 2221-2231.

[K] A. Krieg, *Modular Forms on the Fricke Group.*, Abh. Math. Sem. Univ. Hamburg, 65(1995), 293-299.

[Q] H. -G. Quebbemann, *Atkin-Lehner eigenforms and strongly modular lattices.*, Enseign. Math. (2), 43(1997), No. 1-2, 55-65.

[RSD] F. K. C. Rankin, H. P. F. Swinnerton-Dyer, *On the zeros of Eisenstein Series*, Bull. London Math. Soc., 2(1970), 169-170.

[S] J. -P. Serre, *A Corse in Arithmetic*, Graduate Texts in Mathematics, No. 7, Springer-Verlag, New York-Heidelberg, 1973. (Translation of *Cours d'arithmétique (French)*, Presses Univ. France, Paris, 1970.)