Amplitude Response Curves of Frequency-Locked Rotations

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1. Introduction

Phase-locked loop (PLL) is well known as an important technique in communication engineering [1], [2]. PLL consists of phase comparator, loop filter, and voltage-controlled oscillator. Nonlinear oscillations in PLL circuit have been studied in [3]–[5]. Their understanding is important for the application of PLL to various electrical and electronic circuits.

A dynamical system that is analyzed in [3]–[5] to reveal the dynamics of a PLL circuit has two steady states. One is a stable equilibrium point which corresponds to the phase-locked steady state. The other is a stable limit cycle resulting from the periodic nature of phase space, called stable limit cycle of the second kind [6] or stable rotation [7]. The rotation represents a desynchronized steady state in the PLL circuit and is regarded as another type of self-excited oscillations with natural rotation frequencies. The rotation frequency can be locked at driving frequencies of modulation signals. This letter shows response curves for harmonic amplitude of frequency-locked rotations. They have several different features from response curves of van der Pol oscillator.

**key words:** phase-locked loop, frequency entrainment, rotation, response curve

2. Mathematical Model

This section introduces a mathematical model which describes the dynamics of a PLL circuit [4]. The PLL circuit includes a sinusoidal phase comparator, a lag-lead loop filter, and a voltage-controlled oscillator. Suppose that \( \phi \) denotes the phase error of input signals to the phase comparator, then the phase error dynamics are represented in [4] by the following system of differential equations:

\[
\begin{aligned}
\frac{d\phi}{dt} &= y, \\
\frac{dy}{dt} &= -\beta y + \beta \sigma - \sin \phi + m \sqrt{\beta^2 + \Omega^2} \cos \Omega t.
\end{aligned}
\]

(1)

\( \beta \) and \( \sigma \) are the fixed parameters, and \( m \) and \( \Omega \) are the parameters of modulation signal. The parameters are in per unit system. The driving term \( m \sqrt{\beta^2 + \Omega^2} \cos \Omega t \) is slightly modified from the original one in [4] for simplicity of the present analysis. The two variables \((\phi, y)\) in the system (1) belong to cylindrical phase plane because of the periodic restoring term. The periodic nature of phase space results in the occurrence of stable rotations. Additionally, to derive theoretical response curves in Sect. 3, a smooth function \( S(\phi, y) \) for the system (1) is defined by

\[
S(\phi, y) = \frac{1}{2} y^2 - \cos \phi - \beta \sigma \phi.
\]

(2)

The time derivative of \( S \) for any solution \((\phi(t), y(t))\) is now
given by
\[ \frac{dS}{dt}(\phi(t), y(t)) = -\beta(y(t))^2 + y(t)m\sqrt{\beta^2 + \Omega^2 \cos \Omega t}. \] (3)

By integrating the above equality from \( t = 0 \) to \( \tau \), the following equality is derived:
\[ S(\phi(\tau), y(\tau)) - S(\phi(0), y(0)) = -\int_0^\tau \beta(y(t))^2 dt + \int_0^\tau y(t)m\sqrt{\beta^2 + \Omega^2 \cos \Omega t} dt. \] (4)

The equality (4) is used for the derivation of response curves in Sect. 3.

3. Amplitude Response Curves

This letter shows amplitude response curves of frequency-locked rotations against the parameters \( \Omega \) and \( m \). The system (1) under \( m = 0 \) and \([|\beta|\sigma] > 4\beta/\pi \) has one stable rotation with angular frequency \( \Omega_0 \). The condition of \( \beta \) and \( \sigma \) is obtained via the Melnikov’s method [13]. The corresponding solution \( \phi_0(t) \) to the stable rotation is represented using periodic function \( x_0(t) \) by
\[ \phi_0(t) = \Omega_0 t + x_0(t), \quad x_0(t) \equiv x_0 \left( t + \frac{2\pi}{\Omega_0} \right). \] (5)

Figure 1 shows the waveform of stable rotation in the system (1) under \( m = 0, \beta = 0.56, \) and \( \sigma = 1.7 \). The setting of \( \beta \) is based on [4], [5]. \( \Omega_0 \) is 1.6 under the parameter setting. Within the small difference between natural rotation frequency \( \Omega_0 \) and driving frequency \( \Omega \), it is expected that we observe a locked rotation at the driving frequency \( \Omega \). The type of frequency entrainment is called harmonic entrainment.

Here, the averaging method [9], [14] is used for the derivation of response curves. At the harmonic entrained states, the corresponding solution \( \phi(t) \) can be represented using periodic function \( x(t) \) as follows:
\[ \phi(t) = \Omega t + x(t), \quad x(t) \equiv x \left( t + \frac{2\pi}{\Omega} \right). \] (6)

It is now assumed that the solution \( x(t) \) is approximated as follows:
\[ x(t) = \frac{A_0}{2} + A_1(t)\cos(\Omega t + \varphi_1(t)), \] (7)

where \( A_0 \) is constant. \( A_1 \) and \( \varphi_1 \) are assumed to be slowly varying functions of \( t \) and are constant at the entrained states. By substituting \( \phi(t) = \Omega t + A_0/2 + A_1(t)\cos(\Omega t + \varphi_1(t)) \) and \( y(t) = \Omega - \Omega A_1(t)\sin(\Omega t + \varphi_1(t)) \) in the system (1) and eliminating all oscillatory terms containing \( \cos n\Omega t \) and \( \sin n\Omega t \) for \( n = 1, 2, \ldots \), the following system is obtained:
\[ \begin{align*}
\frac{dA_1}{dt} &= \frac{1}{2\Omega} \left[ J_0(A_1) + J_2(A_1) \right] \cos \left( \frac{A_0}{2} - \varphi_1 \right) \\
&\quad - m \sqrt{\beta^2 + \Omega^2 \sin \varphi_1 - \beta \Omega A_1}, \\
A_1 \frac{d\varphi_1}{dt} &= \frac{1}{2\Omega} \left[ J_0(A_1) - J_2(A_1) \right] \sin \left( \frac{A_0}{2} - \varphi_1 \right) \\
&\quad - m \sqrt{\beta^2 + \Omega^2 \cos \varphi_1 - \Omega^2 A_1,}
\end{align*} \] (8)

where \( J_n \) for \( n = 0, 2 \) are the Bessel functions of the first kind. The system (8) approximately represents the dynamics of \( A_1 \) and \( \varphi_1 \) under the above assumption and is regarded as a first-order averaged system.

Theoretical response curves of frequency-locked rotations are derived through the above preliminary. The three variables \( A_0, A_1, \varphi_1 \) are here required for the derivation. In other words, we need to obtain independent three determining equations. Two of the required equations are induced by equating the right-hand side of the averaged system (8) to zero. The remaining equation is obtained from the equality (4). \( A_1 \) and \( \varphi_1 \) are constant at the entrained states. Then, by substituting the above \( \phi(t) \) and \( y(t) \) in the equality (4) with \( \tau = 2\pi/\Omega \), the following third equation for the derivation of response curves is derived:
\[ -2\pi \beta |\sigma| = -\pi \beta \Omega (2 + A_1^2) - \pi m \sqrt{\beta^2 + \Omega^2 A_1 \sin \varphi_1}. \] (9)

Unfortunately, the three dimensional determining equations have not any explicit solution. The solutions are numerically obtained with the Newton-Raphson method in the next paragraph.

Figure 2 shows several response curves of frequency-locked rotations under \( \beta = 0.56 \) and \( \sigma = 1.7 \). Figure 2(a), which shows the harmonic amplitude \( A_1 \), is obtained by plotting the above three equations in \( \Omega - A_1 \) plane for several values of \( m \). Figure 2(b), which describes the harmonic amplitude of \( x(t) \), is given with numerical integration of the original system (1). The response curve at \( m = 0.6 \) in Fig. 2(a) is here described in the absolute value. Each closed
The obtained response curves hence delineate some different features in the frequency-locked rotations and librations. They lead to the detailed analysis of frequency-locked rotations and librations. The response curves in van der Pol oscillator are adopted for the comparison, which are given in Fig. 2 of [8], Fig. 113.1 of [6], or Fig. 12.1 of [9]. The shape of response curves in Fig. 2 differs from that of van der Pol oscillator. In the van der Pol oscillator, stable solutions have the maximum amplitude near the natural frequency. On the other hand, in the PLL circuit, every harmonic amplitude for stable solutions attains the minimum value near the natural rotation frequency \( \Omega_0 \). Furthermore, it gradually increases to each side of the minimum value in a different manner from the response curves in van der Pol oscillator.

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**References**


