

Strong desynchronizing effects of weak noise in globally coupled systems

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In assemblies of globally coupled dynamical units, weak noise perturbing independently the individual units can cause anomalous dispersion in the synchronized cloud of the units in the phase space. When the noise-free dynamics of the synchronized assembly is nonperiodic, various moments of the linear dimension of the cloud as a function of the noise strength exhibit multiscaling properties with parameter-dependent scaling exponents. Some numerical evidence of this peculiar behavior as well as its interpretation in terms of a multiplicative stochastic process with small additive noise is provided. Universality of the phenomenon is also discussed.

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I. INTRODUCTION

Oscillatory behavior, periodic or nonperiodic, observable on a macroscopic scale may often be a result of collective synchronization of a population of micro-oscillators [1–3]. Consider, for simplicity, a population of identical oscillators with global coupling. Systems of globally and uniformly coupled oscillators in fact serve as a natural model for Josephson junction arrays [4,5], multimode lasers [6,7], and various biological systems such as flashing swarms of fireflies [8–10] and pacemaker cells responsible for circadian rhythms [8]. Under suitable conditions, the whole population will exhibit complete synchrony when the oscillators are free from intrinsic or extrinsic randomness.

In practical situations, however, the oscillators would be more or less noisy or nonidentical, so that, instead of exhibiting perfect synchrony, they will be distributed in the phase space to form a cloud of a finite extension. It seems quite natural to expect that, as far as the noise remains sufficiently weak, the long-time average of the linear dimension of this cloud, denoted by $\langle r \rangle$, should be proportional to the noise strength. This is actually the case when the oscillators are of the limit cycle type. We shall find that, when the oscillators are chaotic, in contrast, such linear dependence breaks down under broad conditions, and is replaced by a more general power-law dependence. The main goal of the present paper is to explain theoretically why such peculiar behavior is possible for globally coupled chaotic oscillators. This will be verified with some numerical evidence for a number of population models. Our theory, which is quite similar to the one developed earlier to explain a certain unique feature of turbulence in nonlocally coupled systems [11–15], suggests that the general moments $\langle r^q \rangle$ as a function of the noise strength exhibit a simple multiscaling law. As a natural consequence of the same theory, similar multiscaling behavior is expected to occur even for the populations of limit cycle oscillators provided that, apart from the random noise, the oscillators are subjected to another random force that is common over the whole population. The crucial point here is that the average motion has to be stochastic for the multiscaling behavior to occur.

In Sec. II, we start with a general class of differential-equation models for globally coupled elements with additive noise. Then, as a specific model belonging to this class, we

study numerically a population of the chaotic Rössler oscillators [16], and show the anomaly in the size of the synchronized cloud of the oscillators in the phase space. In Sec. III, we develop a theory to explain the origin of a general scaling law including the results of Sec. II as a special case. Our theory becomes almost identical with the one developed in Ref. [12] if the effects of the external noise are replaced by the effects of spatial nonuniformity. The arguments in Sec. III suggest that the phenomenon of concern should be so universal that a number of restrictions imposed on our model could be removed. We provide some evidence for this in Sec. V, where three different types of population model will be discussed. The final section summarizes our main conclusions.

II. MODEL AND NUMERICAL SIMULATION

Consider a population of N identical units with the intrinsic dynamics given by $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$, where N is an arbitrary number. Typically, the dynamical units considered are chaotic oscillators. Introducing all-to-all type linear coupling and also external additive noise driving the oscillators individually, we have the following system of coupled differential equations:

$$\frac{d\mathbf{X}_i}{dt} = \mathbf{F}(\mathbf{X}_i) + K \cdot [\bar{\mathbf{X}}(t) - \mathbf{X}_i(t)] + f \cdot \boldsymbol{\eta}_i(t) \quad (i = 1, \dots, N). \quad (2.1)$$

Here $\bar{\mathbf{X}}$ is the simple average of \mathbf{X}_i over the population, i.e.,

$$\bar{\mathbf{X}}(t) \equiv \frac{1}{N} \sum_{i=1}^N \mathbf{X}_i(t), \quad (2.2)$$

and K is a positive coupling constant, thus working in favor of synchronous behavior of the population; $\boldsymbol{\eta}_i(t)$ represents the external noise with suitably normalized intensity applied independently on the oscillators, so that the coefficient f measures the intensity of the noise. For the sake of simplicity, the coupling term has been arranged so that the dynamics of a given unit that is in perfect synchrony with the average motion would be identical with its own dynamics without

coupling. The coupling strength K may be generalized to a coupling matrix, without qualitatively affecting the whole arguments that follow.

In the absence of noise, the population will be attracted to a perfectly synchronized motion for sufficiently large K . In the one-oscillator phase space, the dynamics could then be imagined as an evolution of a single point. When a small random noise is introduced, the resulting dynamics can no longer be represented by a point in the phase space. Instead, we have a cloud of a finite extension whose shape will be changing variously in time. The effects of noise on the collective dynamics could conveniently be characterized by the average linear dimension of the oscillator cloud in the phase space defined by

$$r(t) \equiv \frac{1}{N} \sum_{i=1}^N |\bar{X}(t) - X_i(t)|. \quad (2.3)$$

We define its various moments by a long-time average of $\langle r^q \rangle$:

$$\langle r^q \rangle \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T r^q(t) dt. \quad (2.4)$$

Our main concern below is the dependence of $\langle r^q \rangle$ on f , K , and q . Our intuition may tell that $\langle r^q \rangle$ will simply be proportional to f^q , which seemed to be true from some numerical simulation [17]. However, we see below that this is not always valid. Instead, $\langle r^q \rangle$ behaves under broad conditions like

$$\langle r^q \rangle \propto f^{\alpha(q)} \quad (2.5)$$

for sufficiently small f , where the exponent $\alpha(q)$ is not linear in q and also changes continuously with K and other system parameters.

The above power-law dependence can be confirmed by a numerical analysis of globally coupled Rössler oscillators in the chaotic regime. By assuming the coupling matrix to be diagonal, the system is governed by the equations

$$\begin{aligned} \dot{X}_i &= -(Y_i + Z_i) + K(\bar{X} - X_i) + f\eta_{i,X}, \\ \dot{Y}_i &= X_i + aY_i + K(\bar{Y} - Y_i) + f\eta_{i,Y}, \\ \dot{Z}_i &= b - cZ_i + X_iZ_i + K(\bar{Z} - Z_i) + f\eta_{i,Z}, \end{aligned} \quad (2.6)$$

where the noise components $\eta_{i,\nu}$ ($\nu = X, Y, Z$) are assumed to be mutually independent and white Gaussian with vanishing mean and unit variance. We will choose the standard parameter values $a = 0.3$, $b = 0.2$, and $c = 5.7$. It turns out that in the absence of noise, the state of perfect synchrony is stable above some critical value K_c of K , where $K_c \approx 0.4$. We will work with this condition throughout, and concentrate on the size of the oscillator cloud when the noise sets in. Some numerical results for the first moment $\langle r \rangle$ are summarized in Figs. 1 and 2. In Fig. 1, log-log plots of $\langle r \rangle$ versus f are displayed for different values of K . The power-law behavior as given by Eq. (2.5) with $q = 1$ seems to hold well with the exponent $\alpha(1)$ changing with K . Figure 2 shows the depen-

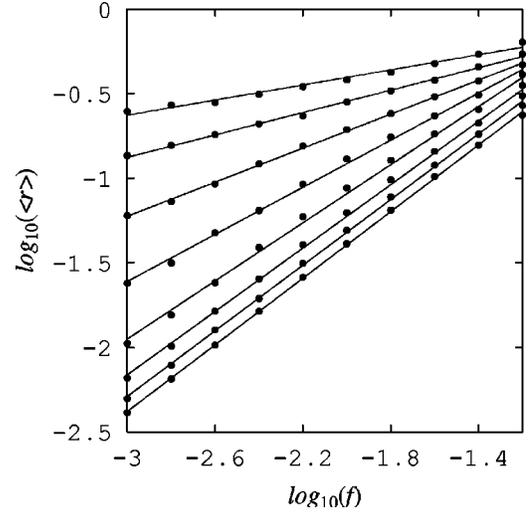


FIG. 1. $\log_{10}\langle r \rangle$ vs $\log_{10}f$ calculated numerically from Eq. (2.6) with $N = 64$ for several values of K . The top line corresponds to $K = 0.142$ and the bottom one to $K = 0.170$, with the uniform interval of 0.004.

dence of the estimated values of $\alpha(1)$ on K . For sufficiently strong coupling, the exponent saturates to the normal value $\alpha(1) = 1$, while for weaker coupling $\alpha(1)$ varies between 0 and 1. Since the noise is sufficiently small, the value of $\alpha(1)$ less than 1 implies anomalous amplification in the size of the oscillator cloud as compared with the normal case.

III. ORIGIN OF ANOMALOUS FLUCTUATIONS

In this section, we present a theory on the origin of anomalous size fluctuations of a synchronized cluster found numerically in the preceding section. Our theory can be developed quite in parallel with the theory developed before on the multiscaled turbulence in nonlocally coupled systems [11–15]. Let us start with Eq. (2.1). Note that the average motion of the population obeys the equation

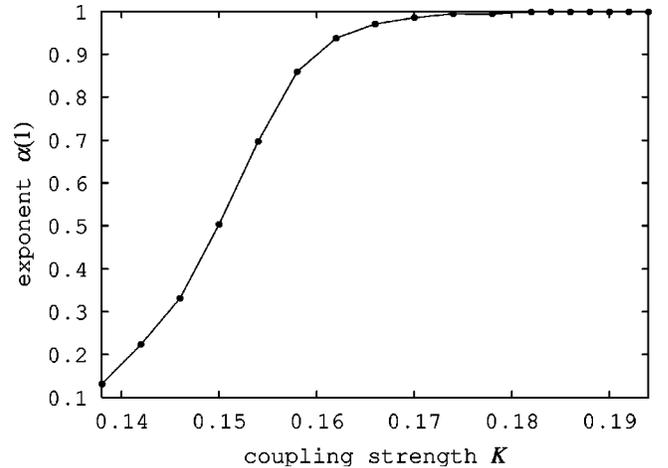


FIG. 2. Exponent $\alpha(1)$ vs coupling strength K , obtained from the data of Fig. 1 using the method of least-squares.

$$\frac{d\bar{X}}{dt} = F(\bar{X}) + O\left(\frac{1}{N} \sum_{i=1}^N |X_i - \bar{X}|^2\right). \quad (3.1)$$

Thus, the deviation defined by $x_i(t) \equiv X_i(t) - \bar{X}(t)$ is governed by

$$\frac{dx_i}{dt} = [DF(\bar{X}(t)) - K]x_i + f\eta_i(t) + O(|x_i|^2), \quad (3.2)$$

where $DF(\bar{X}(t))$ is the Jacobian of $F(X)$ at $X = \bar{X}(t)$. For sufficiently large t , the vector $x_i(t)$ is expected to become parallel with the Lyapunov eigenvector of $DF(\bar{X}(t)) - K$ corresponding to the largest Lyapunov eigenvalue. By denoting the amplitude of this eigencomponent of $x_i(t)$ as $x_i(t)$, and dropping the suffix i for the sake of simplicity, Eq. (3.2) is reduced to a scalar equation

$$\frac{dx}{dt} = \lambda(t)x + f\eta(t) + O(x^2), \quad (3.3)$$

where $\lambda(t)$ is the local Lyapunov exponent, and $\eta_i(t)$ is the projection of $\eta_i(t)$ onto the corresponding Lyapunov eigenvector. For the coupled Rössler oscillators given by Eq. (2.6), we have

$$\lambda(t) = \lambda_r(t) - K, \quad (3.4)$$

where $\lambda_r(t)$ is the local Lyapunov exponent of the individual Rössler oscillator. Since the sign of the long-time average of $\lambda(t)$, which will be denoted by λ_0 , determines the stability of the perfect synchrony of the population in the absence of noise, the critical coupling strength for the coupled Rössler oscillators is simply given by $K_c = \lambda_0$.

Since we are working with the condition that the long-time average of $\lambda(t)$ is negative, and also with sufficiently weak noise, there should exist a range of x where the nonlinear term of $O(x^2)$ as well as the noise term $f\eta(t)$ are negligible. Let this range be specified by $x_{\min} \ll x \ll x_{\max}$, where x_{\min} should be proportional to f . In this range, we have a linear multiplicative stochastic equation

$$\frac{dx}{dt} = \lambda(t)x. \quad (3.5)$$

Note that the stochasticity here is of a deterministic origin. The stationary distribution from Eq. (3.5) can be discussed analytically in the following way. Introducing a new variable y by

$$y = \ln|x|, \quad (3.6)$$

we rewrite Eq. (3.5) as

$$\frac{dy}{dt} = \lambda(t). \quad (3.7)$$

This is integrated to give

$$y(t) - y(0) = \int_0^t \lambda(t) dt \equiv \Lambda(t). \quad (3.8)$$

The time-dependent probability distribution function (PDF) of y , denoted by $Q(y, t)$, satisfies

$$Q(y, t) = \int_{-\infty}^{\infty} \omega(\Lambda, t) Q(y - \Lambda, 0) d\Lambda, \quad (3.9)$$

where $\omega(\Lambda, t)$ is the PDF of $\Lambda(t)$ and is equivalent to the transition probability between the states at times 0 and t . From the definition of Λ , $\omega(\Lambda, t)$ is expected to converge to a Gaussian of mean $t(\lambda_0 - K)$ [$= t(K_c - K)$] and variance tD_λ as $t \rightarrow \infty$, where

$$D_\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} \left\langle \left\{ \int_0^T [\lambda(t) - \lambda_0] dt \right\}^2 \right\rangle. \quad (3.10)$$

Thus, the stationary PDF of y becomes

$$Q(y) \propto e^{-\beta t}, \quad (3.11)$$

where

$$\beta = \frac{2\lambda_0}{D_\lambda} \quad (> 0). \quad (3.12)$$

Note that for the coupled Rössler oscillators, β is given by

$$\beta = \frac{2(K - K_c)}{D_\lambda}. \quad (3.13)$$

The stationary PDF for the original variable x , denoted by $P(x)$, must satisfy the relation $2P(x)dx = Q(y)dy$. Thus, from Eq. (3.11), we obtain

$$P(x) \propto x^{-(\beta+1)} \quad (x_{\min} \ll x \ll x_{\max}). \quad (3.14)$$

Coming back to the original stochastic equation (3.3), the effects of additive noise and nonlinearity must now be incorporated. There exists a characteristic value of x below which the additive noise term dominates the other terms. This value is what we denoted by x_{\min} . For $x \ll x_{\min}$, the power-law divergence of $P(x)$ will be saturated to a constant. On the other hand, there exists the second characteristic value of x , denoted by x_{\max} , above which the nonlinear term is the most dominant. If the nature of the nonlinearity is such that it decelerates rather than accelerates the growth of x , which we assume, the power-law decay of $P(x)$ will be replaced by a much sharper decay above x_{\max} . As far as their dependence on f is concerned, x_{\min} and x_{\max} may be specified as

$$x_{\min} = f, \quad (3.15)$$

$$x_{\max} = 1. \quad (3.16)$$

Thus, the entire profile of $P(x)$ will be such that it is nearly constant below $x = x_{\min}$, obeys the above-mentioned power law, and decays quickly above $x = x_{\max}$. On further idealization, we represent $P(x)$ by the analytic form

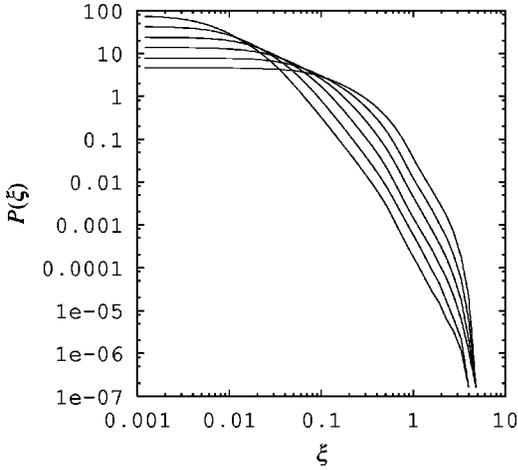


FIG. 3. Stationary PDFs of $P(\xi)$ calculated numerically from Eq. (2.6) for several values of noise strength f , where $K=0.16$ is fixed. The curves correspond to $f=10^{-1}$ (lowest level at lower ξ), $10^{-1.25}$, $10^{-1.5}$, $10^{-1.75}$, $10^{-2.00}$, and $10^{-2.25}$ (highest).

$$P(x) = \begin{cases} Cf^{-(\beta+1)} & (0 < x < f) \\ Cx^{-(\beta+1)} & (f < x < 1) \\ 0 & (1 < x), \end{cases} \quad (3.17)$$

where C is a normalization constant.

Figures 3 and 4 show log-log plots of $P(\xi)$ for the deviation ξ of the first component X obtained numerically from the coupled Rössler oscillators [Eq. (2.6)] for some values of f and K . It is seen that each curve is approximately composed of three parts corresponding to the three characteristic regimes in Eq. (3.17). The range of ξ where $P(\xi)$ is nearly constant extends proportionally to f , which is consistent with Eq. (3.17), while the sharp drop of $P(\xi)$ seems to occur at some ξ independent of f , which is again consistent with Eq. (3.17) with x replaced by ξ .

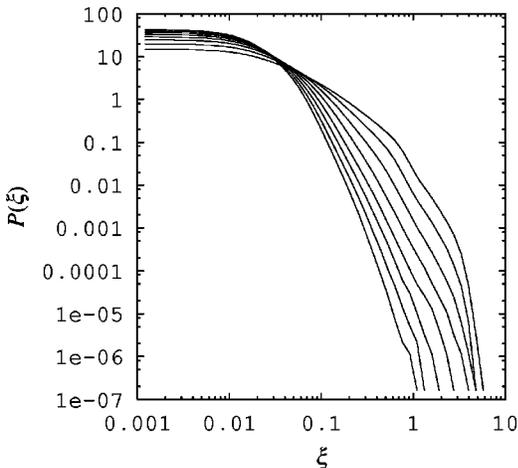


FIG. 4. Stationary PDFs $P(\xi)$ calculated numerically from Eq. (2.6) for several values of coupling strength K , where $f=0.01$ is fixed. The curves correspond to $K=0.150$ (slowest decay), 0.155, 0.160, 0.165, 0.170, 0.175, and 0.185 (fastest decay).

Up to this point, we have discussed the statistics of the deviation x of a single element from the mean motion of the population. In what follows, we will identify the statistics of $|x|$ with that of r , i.e., the deviation averaged over the whole population. As far as some qualitative features such as the power-law dependence of $\langle r \rangle$ on f are concerned, this assumption seems to be justified because the dynamical units are driven by a common multiplier $\lambda(t)$ by virtue of the global nature of the coupling.

By using Eq. (3.17), it is thus straightforward to calculate various moments of r . As for the first moment, we have

$$\langle r \rangle \approx \int_{-\infty}^{\infty} |x| P(x) dx \propto \begin{cases} f^\beta & (0 < \beta < 1) \\ f^1 & (\beta > 1). \end{cases} \quad (3.18)$$

Comparing the above equation with the expression in Eq. (3.13) for the coupled Rössler oscillators, one may now understand the reason for the observed anomalous power-law dependence of $\langle r \rangle$ on the external noise when the system is not too far from the critical point $K=K_c$.

The observed change in the scaling exponent with K shown in Fig. 2 still deviates considerably from Eq. (3.18) with β given by Eq. (3.13). Specifically, the numerical results do not exhibit sharp changes near $\beta=0$ and $\beta=1$. Such discrepancy seems to be due to the fact that the range of validity of Eq. (3.18) in terms of f shrinks to zero as β approaches 0 or 1. A little more careful analysis shows that under fixed f , we have $\langle r \rangle \sim 1/|\ln f|$ as $\beta \rightarrow 0$ and $\langle r \rangle \sim |f^\beta \ln f|$ as $\beta \rightarrow 1$.

Similar calculations for the higher moments $\langle r^q \rangle$ are also straightforward, and the results are simply given by

$$\langle r^q \rangle \propto \begin{cases} f^\beta & (0 < \beta < q) \\ f^q & (\beta > q), \end{cases} \quad (3.19)$$

except for the weak logarithmic singularity mentioned above. Thus, the anomalous fluctuations could be visible through higher moments in the range of stronger coupling where no anomaly is visible through lower moments. Specifically, for the coupled Rössler oscillators, the range of K where the q th moment behaves anomalously is given by

$$K_c < K < K_c + \frac{qD_\lambda}{2}. \quad (3.20)$$

Near the critical point at which the average Lyapunov exponent vanishes, Eq. (3.3) is of the same form as the equation employed for discussing the so-called *on-off intermittency* [18–21] with noise [22,23]. Since the dynamics of $r(t)$ would qualitatively be the same as that of $x(t)$ of a representative oscillator, this implies that noisy on-off intermittency could also be observed in $r(t)$. This was confirmed numerically for the coupled Rössler oscillators, though we will not report its details. The only thing to be remarked is that the origins of the power law in the on-off intermittency and that of our present concern are completely independent.

IV. UNIVERSALITY OF ANOMALOUS FLUCTUATIONS

Up to the preceding section, our discussion has been based on the following three assumptions: (i) The system is described by a set of ordinary differential equations; (ii) the source of randomness working against complete synchrony is represented by external additive noise; (iii) the constituents of the population are intrinsically chaotic. In what follows, we will show that neither of these assumptions is necessary, which implies that the anomalous behavior of concern would be quite general.

A. Case of globally coupled maps

The discrete-time analog to Eq. (2.1) is the system of globally coupled maps [24,25] with noise. Assuming for the sake of simplicity that the individual map is one-dimensional, we are concerned with the model equation of the form

$$X_i(n+1) = (1-K)M(X_i(n)) + KM(\bar{X}(n)) + f\eta_i \quad (i = 1, \dots, N). \quad (4.1)$$

One may develop arguments similar to the case of continuous-time dynamics, and derive a discretized version of Eq. (3.3):

$$x_{n+1} = e^{\lambda_n} x_n + f\eta_n + O(x_n^2). \quad (4.2)$$

Furthermore, the arguments leading to a stationary PDF similar to the form of Eq. (3.17) are almost the same as before. As an example, let us consider a generalized tent map for $M(x)$ defined by

$$M(x) = \begin{cases} x/a & (0 \leq x \leq a) \\ (1-x)/(1-a) & (a \leq x \leq 1). \end{cases} \quad (4.3)$$

The crucial condition for the occurrence of an anomaly of $\langle r \rangle$ is that the local Lyapunov exponent in Eq. (4.2) fluctuates between positive and negative values. Thus, the conventional tent map ($a=0.5$) for which $\lambda = \ln 2$ identically is ruled out. In a suitable range of a and K , the system given by Eq. (4.1) is confirmed to exhibit power-law fluctuations in the form of Eq. (3.18). The estimated exponent $\alpha(1)$ of the first moment of r as a function of K is displayed in Fig. 5.

B. Effects of heterogeneity

Any kind of heterogeneity present in the population could give a source of incompleteness in synchrony. We focus on the following inhomogeneous system of elements *without* additive noise:

$$\frac{dX_i}{dt} = F(X_i, a_i) + K \cdot [\bar{X}(t) - X_i(t)]. \quad (4.4)$$

Here the heterogeneity is represented by a weak distribution in a parameter a , its value for the i th element being denoted by a_i . The dynamics of the deviation x of a representative oscillator is described by

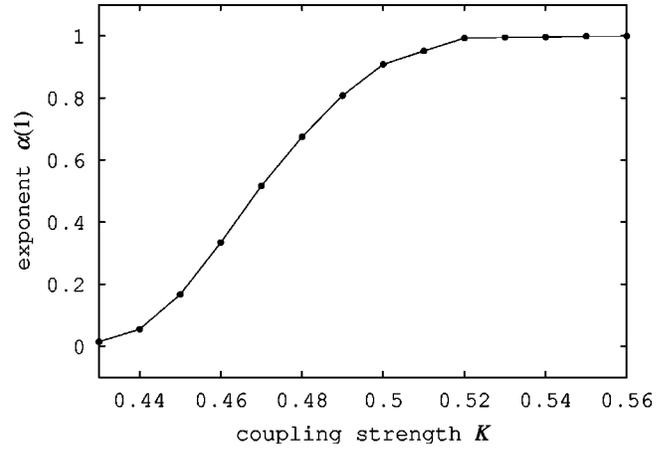


FIG. 5. Exponent $\alpha(1)$ vs coupling strength K calculated numerically from the generalized tent map [Eq. (4.3)] with $a=0.75$.

$$\frac{dx}{dt} = [DF(\bar{X}(t), \bar{a}) - K]x + \delta a \cdot \frac{\partial F(\bar{X}(t), \bar{a})}{\partial a} + O(|x|^2, \delta a^2), \quad (4.5)$$

where the suffix i is dropped and δa denotes the deviation of a from its population average \bar{a} . The effect of the second term on the right-hand side is essentially the same as that of the additive noise in Eq. (3.2), thus resulting in the anomalous power-law dependence of $\langle r^q \rangle$ on the strength of inhomogeneity.

C. Populations of nonchaotic units

We have seen that the anomalous fluctuations exhibited by the synchronized cluster can arise only if the local Lyapunov exponent associated with the individual unit fluctuates randomly. This means that we can never expect such an anomaly for populations of *nonchaotic* dynamical units. Even for this class of systems, however, the anomalous fluctuations may arise provided the whole population is driven externally by a common random force apart from the random noise considered previously. The system of this class will take the form

$$\frac{dX_i}{dt} = F(X_i) + K \cdot [\bar{X}(t) - X_i(t)] + G(t) + f \cdot \eta_i(t), \quad (4.6)$$

where the individual dynamics $\dot{X} = F(X)$ is assumed to be nonchaotic and $G(t)$ represents the coherent random force independent of i . By virtue of the $G(t)$ term, the local Lyapunov exponent now fluctuates randomly, so that the anomalous fluctuation in r could be recovered. This can be demonstrated with a population of oscillatory/excitable units of the FitzHugh-Nagumo type [26,27]. The specific form of the model is given by

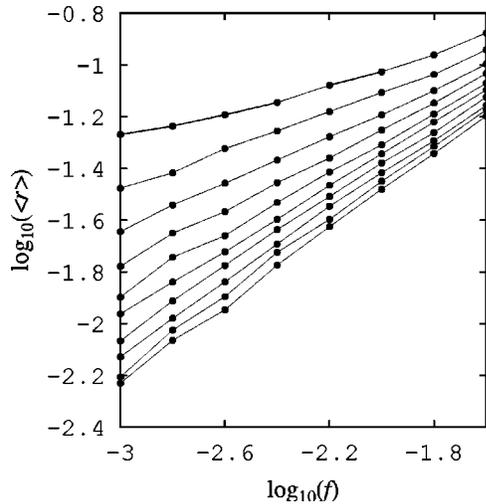


FIG. 6. $\log\langle r \rangle$ vs $\log f$ for several values of K , from 0.01 (top) to 0.19 (bottom) with the uniform increment of 0.002, calculated numerically from Eqs. (4.7) and (4.8); $N=64$, $(a,b,\epsilon) = (1.0, 0.58, 0.1)$, $(G_0, \gamma, g) = (0.5, 1.0, 5.0)$.

$$\frac{dX_i}{dt} = \frac{1}{\epsilon} (X_i - X_i^3 - Y_i) + K(\bar{X} - X_i) + G_0 \sin \phi(t) + f \eta_{i,x}(t),$$

$$\frac{dY_i}{dt} = aX_i + b + K(\bar{Y} - Y_i) + f \eta_{i,y}(t), \quad (4.7)$$

where $G_0 \sin \phi(t)$ represents external random force with ϕ generated from the dynamics of a random walker:

$$\frac{d^2 \phi}{dt^2} = -\gamma \frac{d\phi}{dt} + g \eta. \quad (4.8)$$

Some numerical results of the above model are shown in Figs. 6 and 7, where the parameter values are chosen such that the individual unit is nonoscillatory but excitable. A nontrivial power-law dependence of $\langle r \rangle$ on f with a parameter-dependent exponent is again confirmed. Similar

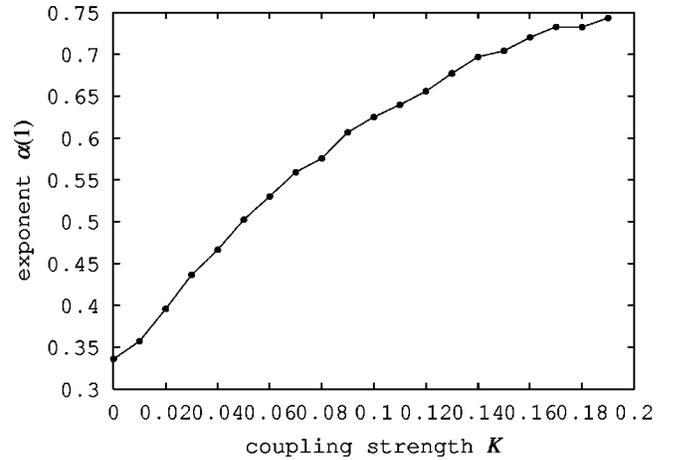


FIG. 7. Exponent $\alpha(1)$ vs coupling strength K obtained from the data of Fig. 6 using the method of least-squares.

behavior of $\langle r \rangle$ of course persists when the individual dynamics becomes oscillatory.

V. SUMMARY

In the present paper, we argued that in globally coupled systems of nonlinear dynamical units, small noise or imposed heterogeneities can generally cause anomalously strong dispersion of the synchronized clusters in the phase space. This was demonstrated numerically with a number of population models with chaotic dynamical units. The numerical results were explained theoretically in terms of a multiplicative stochastic process with additive noise. It turned out that the crucial condition for the occurrence of such an anomaly is the random fluctuations of the local Lyapunov exponent associated with the individual units. This fact suggested some possible generalizations of the class of systems capable of exhibiting similar behavior. In particular, the chaotic nature of the individual dynamics seemed unnecessary provided the population is subjected to a common random drive, and this was actually demonstrated with the population of the FitzHugh-Nagumo-type excitable units.

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- [1] A.T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980).
 - [2] Y. Kuramoto, *Chemical Oscillation, Waves, and Turbulence* (Springer-Verlag, Tokyo, 1984).
 - [3] J.F. Heagy, T.L. Carroll, and L.M. Pecora, *Phys. Rev. E* **50**, 1874 (1994).
 - [4] P. Hadley, M.R. Beasley, and K. Wiesenfeld, *Phys. Rev. B* **38**, 8712 (1988).
 - [5] K. Wiesenfeld, P. Colet, and S.H. Strogatz, *Phys. Rev. Lett.* **76**, 404 (1996).
 - [6] S.S. Wang and H.G. Winful, *Appl. Phys. Lett.* **52**, 1774 (1988).
 - [7] K. Wiesenfeld, C. Bracikowski, G. James, and R. Roy, *Phys. Rev. Lett.* **65**, 1749 (1990).
 - [8] A.T. Winfree, *J. Theor. Biol.* **16**, 15 (1967).
 - [9] J. Buck, E. Buck, F. Hanson, J.F. Case, L. Mets, and G.J. Atta, *J. Comp. Physiol.* **144**, 277 (1981).
 - [10] J. Buck, *Q. Rev. Biol.* **63**, 265 (1988).
 - [11] Y. Kuramoto, *Prog. Theor. Phys.* **94**, 321 (1990).
 - [12] Y. Kuramoto and H. Nakao, *Phys. Rev. Lett.* **76**, 4352 (1996).
 - [13] Y. Kuramoto and H. Nakao, *Phys. Rev. Lett.* **78**, 4039 (1997).
 - [14] Y. Kuramoto and H. Nakao, *Physica D* **103**, 294 (1997).
 - [15] Y. Kuramoto, D. Battogtokh, and H. Nakao, *Phys. Rev. Lett.* **81**, 3543 (1998).
 - [16] O.E. Rössler, *Phys. Lett.* **57A**, 97 (1976).
 - [17] D.H. Zanette and A.S. Mikhailov, *Phys. Rev. E* **57**, 276 (1998).
 - [18] H. Fujisaka and H. Yamada, *Prog. Theor. Phys.* **74**, 918 (1985).

- [19] H. Fujisaka and H. Yamada, *Prog. Theor. Phys.* **75**, 1087 (1986).
- [20] N. Platt, E.A. Spiegel, and C. Tresser, *Phys. Rev. Lett.* **70**, 279 (1993).
- [21] J.F. Heagy, N. Platt, and S.M. Hammel, *Phys. Rev. E* **49**, 1140 (1994).
- [22] N. Platt, S.M. Hammel, and J.F. Heagy, *Phys. Rev. Lett.* **72**, 3498 (1994).
- [23] P.W. Hammer, N. Platt, S.M. Hammel, J.F. Heagy, and B.D. Lee, *Phys. Rev. Lett.* **22**, 1095 (1994).
- [24] K. Kaneko, *Phys. Rev. Lett.* **63**, 219 (1989).
- [25] K. Kaneko, *Physica D* **41**, 137 (1990).
- [26] R.A. FitzHugh, *Biophys. J.* **1**, 445 (1961).
- [27] J. Nagumo, S. Arimoto, and S. Yoshizawa, *Proc. IRE* **50**, 2061 (1962).