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Five-dimensional black strings in Einstein-Gauss-Bonnet gravity

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We consider black-string-type solutions in five-dimensional Einstein-Gauss-Bonnet gravity. Numerically constructed solutions under static, axially symmetric and translationally invariant metric ansatz are presented. The solutions are specified by two asymptotic charges: mass of a black string and a scalar charge associated with the radion part of the metric. Regular black string solutions are found if and only if the two charges satisfy a fine-tuned relation, and otherwise the spacetime develops a singular event horizon or a naked singularity. We can also generate bubble solutions from the black strings by using a double Wick rotation.

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I. INTRODUCTION

Recently there has been a growing attention to gravity in higher dimensions, and one of the central interests is directed toward understanding the nature of higher dimensional black objects. The terminology "black objects" stems from the curious fact that in $d(\geq 5)$ dimensions various horizon topologies are allowed [1] as well as a spherical topology S^{d-2} [2]. For example, there are stringlike solutions in five dimensions, which are constructed by simply extending four-dimensional black holes into one extra dimension, and thus they have horizon topology $S^2 \times \mathrm{R}$. Black string solutions which are not uniform in the direction of an extra dimension were also found numerically [3,4]. Another example is a five-dimensional rotating black ring with horizon topology $S^2 \times S^1$ [1]. This solution gives clear evidence against black hole uniqueness in higher dimensions. As for stability, while it is well known that a uniform black string suffers from the Gregory-Laflamme instability at long wavelengths [5], stability of the other black solutions has not been fully understood yet and is now being studied actively [6–10].

Motivated by the suggestion from particle physics that our observable world may be a brane embedded in a higher dimensional bulk spacetime, configurations composed of a black object-brane system have also been an interesting issue in the past few years. A black string solution in the Randall-Sundrum braneworld was first studied in [11]. This solution is an extension of the Schwarzschild black string in flat space to a solution in an anti-de Sitter bulk bounded by a brane with a tuned tension. Much more nontrivial is a black hole localized on a brane. No exact solutions describing such black holes have been known so far, but numerical solutions were worked out in [12], and an analytic approximation method was developed in [13]. Black holes in braneworld scenarios have a phenomenological aspect as well. For instance, an exciting possibility was pointed out that black holes may be produced at accelerators [14,15], and a characteristic gravitational wave signal from a black string was discussed [16].

In this paper we investigate black string solutions of five-dimensional Einstein gravity with a "Gauss-Bonnet" higher curvature correction term. The Gauss-Bonnet combination of curvature invariants is relevant in five dimensions or higher, although it reduces to just a total derivative in four dimensions. The Gauss-Bonnet term is of importance in string theory since it arises as the next-to-leading order correction in the heterotic string effective action. This fact motivates several past studies [17–20] which considered black hole solutions in a gravitational theory with a Gauss-Bonnet correction term. (Though they have focused on a *four-dimensional* case, it does not contradict the above remark since in these works a scalar field coupled dilatonically to the Gauss-Bonnet term is introduced.) Very recently the Gauss-Bonnet correction is frequently discussed in the braneworld context as an attempt to extend the Randall-Sundrum model in a natural way [21,22]. Five-dimensional, static and spherically symmetric black holes were systematically examined in the presence of a Gauss-Bonnet term [23–25], and its consequences on the brane cosmology were discussed [25,26]. Our study here addresses the simplest setup: the Einstein-Hilbert plus the Gauss-Bonnet curvature term in five dimensions without a cosmological constant or branes.

The structure of the present paper is as follows. In the next section we outline Einstein-Gauss-Bonnet gravity, and we consider static, axially symmetric, and translationally invariant black string metric ansatz and its asymptotic boundary conditions in the framework of Einstein-Gauss-Bonnet gravity. In Sec. III we briefly discuss our problem from a viewpoint of the reduced theory in four dimensions. In Sec. IV, first we provide an approximate analytic solution valid for large mass and then present our numerical results. We also discuss bubble solutions generated from the black string solutions. Section V contains some remarks.

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II. PRELIMINARIES

A. Einstein-Gauss-Bonnet gravity

The gravitational theory that we consider in this paper is defined by the action

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}[\mathcal{R} + \alpha\mathcal{L}_{\rm GB}], \tag{1}$$

where G_5 is the five-dimensional Newton's constant, and the Gauss-Bonnet Lagrangian is given by

$$\mathcal{L}_{\rm GB} = \mathcal{R}^2 - 4\mathcal{R}_{cd}\mathcal{R}^{cd} + \mathcal{R}_{cdef}\mathcal{R}^{cdef}. \tag{2}$$

A parameter α has dimension of (length)^2. Variation of this action gives the following field equations for the metric:

$$\mathcal{G}_{ab} = \frac{\alpha}{2}\mathcal{H}_{ab}, \tag{3}$$

where $\mathcal{G}_{ab}$ is the Einstein tensor and $\mathcal{H}_{ab}$ is the Lanczos tensor,

$$\mathcal{H}_{ab} = \mathcal{L}_{\rm GB}g_{ab} - 4(\mathcal{R}\mathcal{R}_{ab} - 2\mathcal{R}_{ac}\mathcal{R}^c{}_b - 2\mathcal{R}_{acbd}\mathcal{R}^{cd} + \mathcal{R}_{acde}\mathcal{R}_b{}^{cde}). \tag{4}$$

Although the action involves the terms higher order in curvature, the field equations (3) are second order. This is the special feature of the Gauss-Bonnet Lagrangian (2). For a nice review on Einstein-Gauss-Bonnet gravity (and its Lovelock generalization), see, e.g., [27].

B. Two coordinate systems for black string spacetime

We are interested in a black-string-type solution of the field equations (3). We write a general metric of a static black string spacetime as

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2d\Omega^2 + e^{2\Phi(r)}dy^2, \tag{5}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the line element on a unit 2-sphere, r is the circumferential radius, and y is a coordinate in the direction of an extra dimension. A straightforward calculation gives explicit forms of the Einstein and Lanczos tensor components, which are presented in the Appendix.

As pointed out in [28], the radion part $g_{yy} = e^{2\Phi(r)}$ is crucial for constructing a black string solution in the framework of Einstein-Gauss-Bonnet gravity. In fact, if we assume $\Phi(r) = \text{const}$, then it follows from the (t, t) and (r, r) components of the field equations that $e^{-2B(r)} = e^{2A(r)} = 1 - 2m/r$ where m is an integration constant. This is incompatible with the (y, y) component of the field equations unless $\alpha = 0$.

The event horizon of a black string is located at some finite radius in the coordinate system of (5), $e^{2A(r_h)} = 0$. Such singularity in the metric component might cause difficulty when integrating the field equations numerically, and thus it will be better to use the so-called "tortoise" coordinate defined by $d\rho^2 = e^{2B(r)-2A(r)}dr^2$. Using this coordinate system, the metric is expressed as

$$ds^2 = N^2(\rho)(-dt^2 + d\rho^2) + r^2(\rho)d\Omega^2 + C^2(\rho)dy^2. \tag{6}$$

Here we write $e^{A(r)} = N(\rho)$ and $e^{\Phi(r)} = C(\rho)$. The Einstein and Lanczos tensors in this coordinate system are also presented in the Appendix. Now the horizon $r = r_h$ is mapped to $\rho = -\infty$, and the metric components are expected to be slowly varying functions of ρ in this limit, which makes a numerical investigation easier. Also, there is another advantage of using this coordinate system. As will be seen, $r(\rho)$ is not a monotonic function for a class of solutions. Because of this unusual behavior, g_{tt} and g_{yy} are two-valued functions of r, while they are still single valued in ρ, and hence ρ is an appropriate coordinate for describing such a class of solutions.

C. Asymptotic boundary conditions

Let us investigate the asymptotic behavior of the metric functions. First we work in the coordinate system of (5). For sufficiently large r, we may neglect $\mathcal{H}_a{}^b$ because it asymptotically decays in higher powers of $1/r$, and the field equations become $\mathcal{G}_a{}^b \approx 0$. Then we have

$$\mathcal{G}_t{}^t + \mathcal{G}_r{}^r + \mathcal{G}_\theta{}^\theta + \mathcal{G}_\varphi{}^\varphi - 2\mathcal{G}_y{}^y = 3e^{-2B}\left[\Phi'' + (\Phi')^2 + \left(A' - B' + \frac{2}{r}\right)\Phi'\right] \approx 0, \tag{7}$$

where the prime denotes differentiation with respect to r. This equation can be integrated to give

$$(e^\Phi)' \approx -Q\frac{e^{B-A}}{r^2}, \tag{8}$$

$$e^\Phi \approx 1 + Q\int_r^\infty \frac{e^{B-A}}{r^2}dr \approx 1 + \frac{Q}{r}, \tag{9}$$

where Q is an integration constant. From Eq. (9), we obtain

$$-\frac{1}{4\pi}\int \partial_r\Phi \cdot r^2 d\Omega = Q, \tag{10}$$

where the integral is over a 2-sphere at infinity. Thus, Q is regarded as a charge associated with the "scalar field" Φ, which we called radion earlier.

When the Gauss-Bonnet term is turned off ($\alpha = 0$), the relation (8) is *exact* for all values of r. Then, it can be easily seen that $(e^\Phi)'$ and e^Φ diverge at the horizon, $r = r_h$, because $e^{A(r_h)} = 0$. To avoid this, we must set $Q = 0$, and hence a black string solution in Einstein gravity cannot possess a "scalar charge." In Einstein-Gauss-Bonnet gravity, however, there is no reason for prohibiting a nonzero scalar charge.

Another set of field equations yields

$$2\mathcal{G}_\theta{}^\theta + \mathcal{G}_r{}^r - \mathcal{G}_t{}^t = 2e^{-2B}\left[A'' + (A')^2 + \left(\Phi' - B' + \frac{2}{r}\right)A'\right] \approx 0, \tag{11}$$

where we have used Eq. (7). From this we similarly obtain

$$(e^A)' \approx M\frac{e^{B-\Phi}}{r^2}, \tag{12}$$

$$e^A \approx 1 - M\int_r^\infty \frac{e^{B-\Phi}}{r^2}dr \approx 1 - \frac{M}{r}, \tag{13}$$

where M is an integration constant. Since $-g_{tt} \approx 1 - 2M/r$ near infinity, M is indeed the Arnowitt-Deser-Misner mass. Finally, the asymptotic behavior of g_{rr} is determined by the equation $\mathcal{G}_r{}^r \approx 0$ as

$$e^{2B(r)} \approx 1 + \frac{2(M-Q)}{r}. \tag{14}$$

Thus the asymptotic form of the metric functions is specified by two parameters, M and Q.

Now let us move on to the second coordinate system (6), which we will use for actual numerical calculations. Asymptotic forms of $N(\rho)$ and $C(\rho)$ are, of course,

$$N(\rho) \approx 1 - \frac{M}{r(\rho)}, \tag{15}$$

$$C(\rho) \approx 1 + \frac{Q}{r(\rho)}. \tag{16}$$

Since $r' = e^{A-B}$ by definition, we have for a large value of ρ

$$r'(\rho) \approx 1 - \frac{2M}{r(\rho)} + \frac{Q}{r(\rho)}. \tag{17}$$

Although Einstein-Gauss-Bonnet gravity has a parameter of dimension of $(\text{length})^2$, α, it does not appear in the above asymptotic analysis. This parameter can always be set equal to unity in the field equations (provided that $\alpha > 0$) by redefining the radial coordinate as $r \to r/\alpha^{1/2}$ (or $\rho \to \rho/\alpha^{1/2}$) and the asymptotic charges as $M \to M/\alpha^{1/2}$, $Q \to Q/\alpha^{1/2}$. In this sense, free parameters that specify black string solutions are two dimensionless combinations, $M/\alpha^{1/2}$ and $Q/\alpha^{1/2}$.

III. FOUR-DIMENSIONAL POINT OF VIEW

Translational invariance in the y direction of black string spacetime allows us to analyze such spacetime from a point of view of the reduced theory in four dimensions. Before proceeding to solve the field equations, let us take a brief look at the problem from this viewpoint.

Substituting the metric ansatz

$$g_{ab}dx^a dx^b = e^{-\Phi(\hat{x})}q_{\mu\nu}(\hat{x})d\hat{x}^\mu d\hat{x}^\nu + e^{2\Phi(\hat{x})}dy^2, \tag{18}$$

into the five-dimensional action (1), we obtain the following four-dimensional action:

$$S = \frac{1}{16\pi G_4}\int d\hat{x}^4\sqrt{-q}\Big\{R - \frac{3}{2}(\nabla\Phi)^2 + \alpha e^\Phi L_{\text{GB}} + \alpha e^\Phi[4G_{\mu\nu}\nabla^\mu\Phi\nabla^\nu\Phi - 3\Box\Phi(\nabla\Phi)^2]\Big\}, \tag{19}$$

where G_4 is the four-dimensional Newton's constant, R, $G_{\mu\nu}$, and L_{GB} are constructed from the metric $q_{\mu\nu}$, ∇_μ is the derivative operator associated with $q_{\mu\nu}$, and we have used a notation $(\nabla\Phi)^2 := \nabla_\mu\Phi\nabla^\mu\Phi$, $\Box\Phi := \nabla_\mu\nabla^\mu\Phi$. With some algebra we have omitted the total derivative terms in Eq. (19).

The reduced action consists of the Einstein-Hilbert term and a scalar field Φ, which has a nontrivial kinetic term and is coupled to the four-dimensional Gauss-Bonnet term as well as to the Einstein tensor. Unfortunately, this action itself does not seem to bring us much insight on the properties of the black strings. However, black hole solutions of the rather simpler action

$$S = \frac{1}{16\pi G_4}\int dx^4\sqrt{-q}\left[R - \frac{3}{2}(\nabla\Phi)^2 + \alpha e^\Phi L_{\text{GB}}\right], \tag{20}$$

which contains only the terms in the first line in the action (19), are extensively studied in Refs. [17–20]. It is worth noting beforehand that our black string solutions share some of the properties with four-dimensional black hole solutions of the action (20). For example, the same exotic structure of the metric near $r = r_m$ (which is to be mentioned in the next section) is addressed in Ref. [18].

IV. BLACK STRING SOLUTIONS IN EINSTEIN-GAUSS-BONNET GRAVITY

A. Perturbative solution

Before presenting numerical results, we shall construct a regular black string solution analytically by taking the Gauss-Bonnet term as a perturbation from pure Einstein gravity.

We start from the Schwarzschild black string in Einstein gravity,

$$ds_0^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega^2 + dy^2, \tag{21}$$

with $f(r) := 1 - 2M/r$. We solve the field equations iteratively by taking α as a small expansion parameter around this background. Substituting the background solution (21) into the Lanczos tensor, we have

$$\mathcal{H}_y{}^y = \frac{48M^2}{r^6}, \qquad \mathcal{H}_t{}^t = \mathcal{H}_r{}^r = \mathcal{H}_\theta{}^\theta = \mathcal{H}_\varphi{}^\varphi = 0. \tag{22}$$

We make the ansatz

$$e^{2A(r)} = f(r) + 2\alpha A_1(r) + \alpha^2 A_2(r) + \cdots, \tag{23}$$

$$e^{-2B(r)} = f(r) - 2\alpha B_1(r) - \alpha^2 B_2(r) + \cdots, \tag{24}$$

$$e^{\Phi(r)} = 1 + \alpha\Phi_1(r) + \alpha^2\Phi_2(r) + \cdots, \tag{25}$$

and then the equation for $\Phi_1(r)$ is obtained as

$$[r(r-2M)\Phi_1{}']' = -16\frac{M^2}{r^4}. \tag{26}$$

A solution which vanishes at infinity and is regular at the horizon is given by

$$\Phi_1(r) = \frac{2}{3}\frac{1}{M}\left(\frac{1}{r} + \frac{M}{r^2} + \frac{4M^2}{3r^3}\right). \tag{27}$$

Comparing this with Eq. (9), we see that the asymptotic charge Q is not independent of mass M for this solution:

$$Q = \frac{2\alpha}{3M} + \mathcal{O}(\alpha^2). \tag{28}$$

The other two sets of the field equations of $\mathcal{O}(\alpha)$ give

$$A_1(r) = \frac{2}{3}\left(\frac{1}{r^2} + \frac{5M}{6r^3} + \frac{2M^2}{r^4}\right), \tag{29}$$

$$B_1(r) = -\frac{2}{3M}\frac{1}{r} - \frac{1}{3}\left(\frac{1}{r^2} + \frac{M}{r^3} - \frac{20M^2}{3r^4}\right). \tag{30}$$

Here one of the two integration constants is determined by the condition that A_1 and B_1 vanish at infinity, and the other, which appears in the form of $+c/r$ (respectively, $-c/r$) in A_1 (respectively, B_1), is set to be zero since we are considering a solution with a fixed mass M. The event horizon is located at

$$r_h = 2M - \frac{23}{18}\frac{\alpha}{M} + \mathcal{O}(\alpha^2), \tag{31}$$

with which it can be seen that $e^{2A(r_h)} = e^{-2B(r_h)} = \mathcal{O}(\alpha^2)$.

A comment is now in order. Mignemi and Stewart [17] investigated black hole solutions of the action (20) in the same way as above. Namely, they perturbatively obtained a solution valid at first order in α, assuming the Schwarzschild black hole solution to be a background. Their solution for the scalar field coincides with ours (27) up to the overall factor [see Eq. (5) of Ref. [17]]. This is not a surprise because $G_{\mu\nu}$ and $\nabla_\mu\Phi$ vanish for the background solution and hence the second line of the action (19) does not contribute to the quadratic order of perturbation.

B. Numerical results

The numerical integration is performed by using the coordinate system (6). For given values of M and Q, we impose the asymptotic boundary conditions following Eqs. (15)–(17), and then integrate the (t,t), (θ,θ), and (y,y) components of the field equations starting from sufficiently large ρ towards $\rho \to -\infty$. The constraint equation [the (r,r) component] is used to make sure that numerical errors are sufficiently small.

1. Regular black string

If and only if a relation between M and Q is fine-tuned, we can obtain a regular black string solution, an example of which is presented in Fig. 1. For a large mass black string, the fine-tuned relation $Q = Q_c(M)$ is well approximated by Eq. (28),

$$Q_c(M) \simeq \frac{2\alpha}{3M}, \qquad M^2 \gg \alpha, \tag{32}$$

and the solution is of course well approximated by the perturbative solution given in the previous subsection. On the other hand, for smaller M the regular solutions deviate from the perturbative solution due to the higher-order terms in α, though their effect is not so large. The circumferential radius r is a monotonic function of ρ for this class of solutions.

This solution has a regular event horizon, and near the horizon the metric components are expressed as [18,19]

$$e^{2A(r)} = a_1 x + \frac{a_2}{2}x^2 + \cdots, \tag{33}$$

$$e^{-2B(r)} = b_1 x + \frac{b_2}{2}x^2 + \cdots, \tag{34}$$

$$\Phi(r) = \Phi_h + \phi_1 x + \frac{\phi_2}{2}x^2 + \cdots, \tag{35}$$

where $x = r - r_h$. Substituting these into the field equations and solving the resultant algebraic equations, we can determine the coefficients order by order. From the leading order equations [the (t,t) and (r,r) components give the same equation], we have

$$\phi_1 = \frac{-r_h \pm \sqrt{r_h^2 - 8\alpha}}{r_h^2 + 4\alpha}, \tag{36}$$

$$b_1 = \frac{2}{r_h \pm \sqrt{r_h^2 - 8\alpha}}, \tag{37}$$

and

$$3\frac{a_2}{a_1} + \frac{b_2}{b_1} = -\frac{8r_h}{r_h^2 + 4\alpha}. \tag{38}$$

Only in the upper "+" sign case, b_1 is well behaved in the

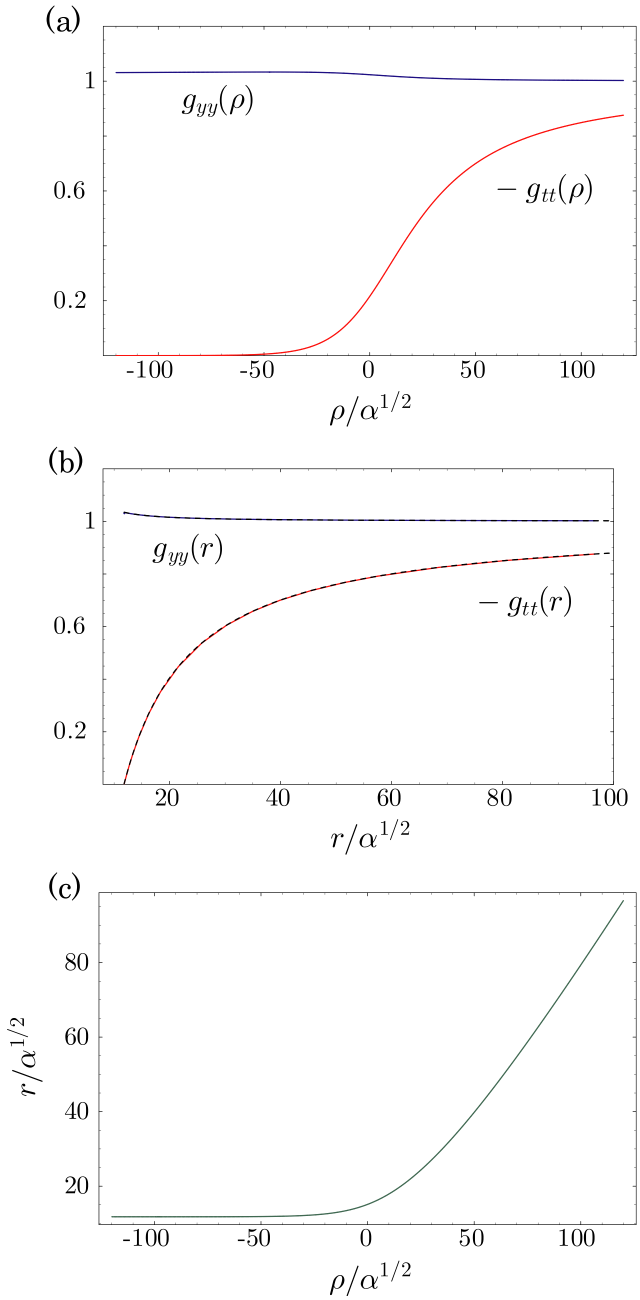

FIG. 1 (color online). Metric components g_{tt} (red lines) and g_{yy} (blue lines) as functions of the tortoise coordinate ρ (a) and the circumferential radius r (b), and the relation between ρ and r (c) for a black string with rather large mass $M = 6\alpha^{1/2}$ and $Q = 2\alpha/3M \approx Q_c$. The two parameters are fine-tuned so that the solution shows a regular behavior. The perturbative solution is plotted as a reference [dashed line in (b)].

$\alpha \to 0$ limit. Black string solutions are found only in this branch, and one can easily confirm that the perturbative solution indeed chooses this branch. Since ϕ_1 and b_1 must be real, for given α there is a minimum horizon size:

$$r_h \geq \sqrt{8\alpha}. \tag{39}$$

From the next-to-leading order equations and Eq. (38) we can determine the second derivatives of the metric functions at the horizon. For the + sign case they are given by

$$\frac{a_2}{a_1} = -\frac{3r_h}{r_h^2 + 4\alpha} - \frac{1}{2}\left[\frac{1}{r_h} - \frac{3r_h^2 + 4\alpha}{(r_h^2 + 4\alpha)\sqrt{r_h^2 - 8\alpha}}\right], \tag{40}$$

$$\frac{b_2}{b_1} = \frac{r_h}{r_h^2 + 4\alpha} + \frac{3}{2}\left[\frac{1}{r_h} - \frac{3r_h^2 + 4\alpha}{(r_h^2 + 4\alpha)\sqrt{r_h^2 - 8\alpha}}\right], \tag{41}$$

$$\phi_2 = \frac{4\alpha}{r_h(r_h^2 + 4\alpha)^2} \times \frac{r_h(3r_h^2 - 4\alpha)\sqrt{r_h^2 - 8\alpha} + (3r_h^2 + 4\alpha)(r_h^2 - 4\alpha)}{r_h(r_h^2 - 8\alpha) + (r_h^2 - 4\alpha)\sqrt{r_h^2 - 8\alpha}}. \tag{42}$$

It can be seen from these equations that taking $r_h \to \sqrt{8\alpha}$, the coefficients a_2, b_2, and ϕ_2 diverge, which leads to a singular solution in the minimum horizon size limit.

Repeating the same procedure, we obtain all the coefficients b_i and ϕ_i ($i = 1, 2, \ldots$) and the ratio a_j/a_1 ($j = 2, 3, \ldots$), but a_1 itself and Φ_h are left undetermined. The values of these two coefficients are shifted by rescaling t and y coordinates as $t \to \lambda_t t$ and $y \to \lambda_y y$ with constant λ_t and λ_y and are to be fixed so that $e^{2A(\infty)} = e^{2\Phi(\infty)} = 1$. Thus, the only parameter we can choose freely at the regular horizon is the value of the horizon radius r_h, and this degree of freedom corresponds to one freely chosen charge at infinity.

Now we mention the implication of the minimum mass. Let us consider a length scale l where the Einstein-Hilbert term and the Gauss-Bonnet term become comparable. Since $\mathcal{R} \sim M/r^3$ and $\mathcal{L}_{\rm GB} \sim \mathcal{R}^2$, l is roughly given by $\sim(\alpha M)^{1/3}$. For large mass we have $l \ll r_h \sim M$, so that $\mathcal{R} \gg \alpha\mathcal{L}_{\rm GB}$ outside the horizon. Even for the minimum mass, we have $M_0 \sim \alpha^{1/2}$ and thus the scale l is of order the horizon size: $l \sim \alpha^{1/2} \sim \sqrt{8\alpha}$. In other words, *the Gauss-Bonnet term can never dominate the Einstein-Hilbert term outside the horizon.*

The horizon radius is turned out to be monotonically decreasing with decreasing mass and reaches the minimum at $M = M_0$ (Fig. 2). We numerically find $M_0 \simeq 1.9843\alpha^{1/2}$ and $Q_c(M_0) \simeq 0.48930\alpha^{1/2}$. For the minimum mass black string, the quantitative difference from the Schwarzschild black string in Einstein gravity with the same mass stands out well (Fig. 3).

We can relate the expansion coefficients and the asymptotic charges by using the field equation $\mathcal{G}_t{}^t - \mathcal{G}_y{}^y = \alpha(\mathcal{H}_t{}^t - \mathcal{H}_y{}^y)/2$. This combination reduces to the total derivative form

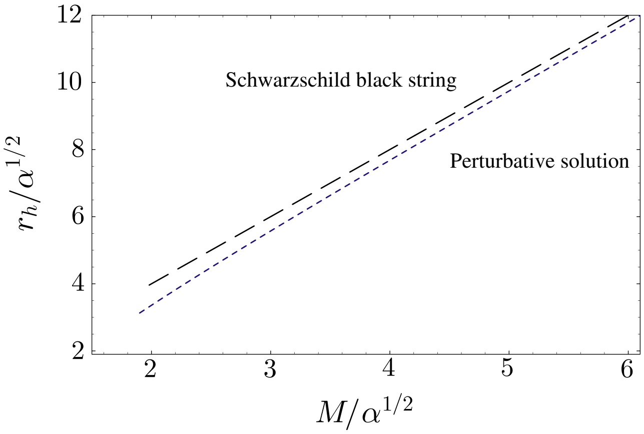

FIG. 2 (color online). Horizon radius as a function of M (red circles). The blue dashed line indicates the horizon radius of the perturbative solution, $r_h = 2M - 23\alpha/(18M)$, which approximates the regular black string quite well for $M^2 \gg \alpha$.

$$\frac{d}{dr}[r^2 e^{A-B}(e^{\Phi})' - r^2 e^{\Phi-B}(e^{A})'] = 4\alpha \frac{d}{dr}\{(e^{-3B} - e^{-B})[e^{A}(e^{\Phi})' - e^{\Phi}(e^{A})']\},$$

and hence

$$\mathcal{F}(r) := r^2 e^{\Phi-B}(e^{A})' - r^2 e^{A-B}(e^{\Phi})' + 4\alpha\{(e^{-3B} - e^{-B})[e^{A}(e^{\Phi})' - e^{\Phi}(e^{A})']\} \quad (43)$$

is a constant. Evaluating this at the horizon and at infinity, we obtain

$$M + Q_c(M) = \frac{1}{2}(r_h^2 + 4\alpha)e^{\Phi_h}\sqrt{a_1 b_1}. \quad (44)$$

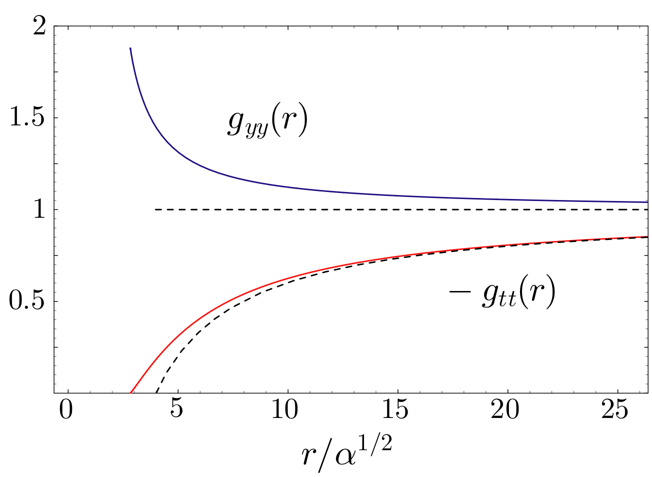

FIG. 3 (color online). Quantitative difference between the minimum mass regular black string in Einstein-Gauss-Bonnet gravity (red and blue solid lines) and the Schwarzschild black string in Einstein gravity with the same mass (dashed lines).

2. *Null singularity*

If one does not impose the fine-tuned relation between the asymptotic charges, other types of solutions can be found. For the asymptotic charges satisfying $Q < Q_c(M)$,

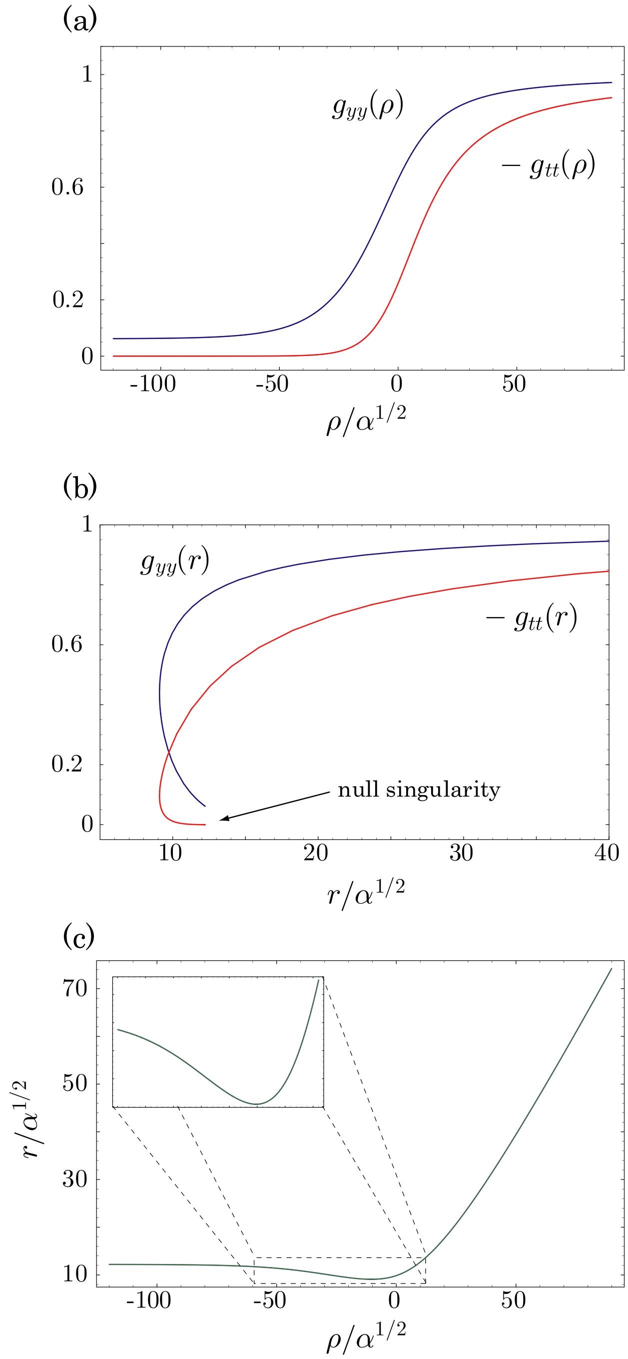

FIG. 4 (color online). Metric components g_{tt} (red lines) and g_{yy} (blue lines) as functions of the tortoise coordinate ρ (a) and the circumferential radius r (b), and the relation between ρ and r (c) for a black string with $M = 3\alpha^{1/2}$ and $Q = -\alpha^{1/2}$. The two parameters are not fine-tuned. The metric components are two-valued functions of r, and there is a null singularity at the Killing horizon ($g_{tt} = 0$).

the solution shows rather exotic behavior, as is presented in Figs. 4 and 5. A particular feature is that the circumferential radius takes the minimum at a finite value of ρ (denoted as ρ_m), where $g_{rr} = (N/\partial_\rho r)^2$, $\partial_r g_{tt}$, and $\partial_r g_{yy}$ diverge.

We can find a solution of this class for M smaller than the minimum mass of the regular black string M_0, which is not contradictory because, as will be seen shortly, this solution does not have a regular horizon. The critical value $Q_c(M)$ is not defined for $M < M_0$, but we can extend the definition of $Q_c(M)$ to the smaller mass region as the largest value of Q for which the solution falls in this class. Let us denote the extended critical value for $M < M_0$ by $\bar{Q}_c(M)$. Namely, the parameter region in which solutions of this class exist is bounded above by Q_c for $M > M_0$ and by $\bar{Q}_c$ for $M < M_0$. For fixed M, ρ_m is decreasing with increasing Q, and in the fine-tuned limit we have

$$\lim_{Q\to Q_c} \rho_m \to -\infty. \tag{45}$$

Near the minimum circumferential radius, $r = r_m := r(\rho_m)$, the metric components behave as $r - r_m \sim [g_{tt}(r) - g_{tt}(r_m)]^2$, $r - r_m \sim [g_{yy}(r) - g_{yy}(r_m)]^2$, and $r - r_m \sim (\rho - \rho_m)^2$. Noticing these, we can expand the metric components as [18,19]

$$\begin{aligned} A(r) &= \bar{A}_m + \bar{A}_* x^{1/2} + \bar{A}_1 x + \cdots, \\ e^{-2B(r)} &= \bar{b}_1 x + \cdots, \\ \Phi(r) &= \bar{\Phi}_m + \bar{\phi}_* x^{1/2} + \bar{\phi}_1 x + \cdots, \end{aligned} \tag{46}$$

where $x = r - r_m$. The coefficients are constrained by

$$\begin{aligned} \bar{b}_1 &= \frac{2}{r_m}, \qquad \bar{A}_1 = -\frac{1}{2}\bar{A}_*^2 - \frac{r_m}{r_m^2 + 4\alpha}, \\ \bar{\phi}_* \bar{A}_* &= \frac{2r_m}{r_m^2 + 4\alpha}, \qquad \bar{\phi}_1 = -\frac{1}{2}\bar{\phi}_*^2 - \frac{r_m}{r_m^2 + 4\alpha}. \end{aligned} \tag{47}$$

In contrast to the regular horizon case, there are two free parameters in this expansion. Namely, given the values of r_m and either $\bar{A}_*$ or $\bar{\phi}_*$, we can determine all the remaining coefficients order by order. These 2 degrees of freedom correspond to the two freely chosen asymptotic charges. Note that $\bar{A}_m$ and $\bar{\Phi}_m$, of which values are shifted by rescaling t and y coordinates, should be adjusted to satisfy $e^{2A(\infty)} = e^{2\Phi(\infty)} = 1$.

Using the expansion (47), one can easily confirm that the curvature invariants do not diverge at $r = r_m$, though, of course, regularity at this point is rather obvious if one uses the tortoise coordinate. A similar structure of metric near the minimum circumferential radius was reported in Ref. [18] in the framework of dilatonic Gauss-Bonnet gravity (20). However, since in the analysis of Ref. [18] they did not use the tortoise coordinate, and they integrated the field equations from inside toward infinity using the radial coordinate r, "the structure inside ρ_m" was not recognized.

For $\rho < \rho_m$, the circumferential radius increases as ρ decreases, until g_{tt} vanishes at $r(-\infty) =: r_s$. The metric components exponentially approach their values at $r = r_s$ as $\rho \to -\infty$: $g_{tt} \sim e^{c_1\rho}$ and $x := r_s - r \sim e^{c_2\rho}$ with c_1, $c_2 > 0$. Then, the (r, r) component behaves like

$$e^{-2B} = \left(\frac{\partial_\rho r}{N}\right)^2 \sim e^{(c_1 - 2c_2)\rho} \sim x^{c_1/c_2 - 2}. \tag{48}$$

We numerically confirmed that g_{rr} neither vanishes nor diverges at $r = r_s$ (see Fig. 5). Thus, the only possibility is $c_1/c_2 = 2$ and we have $g_{tt} \sim x^2$ at the leading order. From this argument we obtain

$$\begin{aligned} e^{2A(r)} &= \frac{\tilde{a}_2}{2}x^2 - \frac{\tilde{a}_3}{6}x^3 + \cdots, \\ e^{-2B(r)} &= \tilde{b}_s - \tilde{b}_1 x + \cdots, \\ \Phi(r) &= \tilde{\Phi}_s - \tilde{\phi}_1 x + \frac{\tilde{\phi}_2}{2}x^2 + \cdots. \end{aligned} \tag{49}$$

The coefficients are constrained by

$$\tilde{b}_1 = \frac{r_s}{4\alpha}\frac{r_s^2 + 4\alpha + 4\alpha\tilde{b}_s}{r_s^2 + 4\alpha - 4\alpha\tilde{b}_s}, \tag{50}$$

$$\tilde{\phi}_1 = -\frac{2r_s}{r_s^2 + 4\alpha - 12\alpha\tilde{b}_s}, \tag{51}$$

$$2\frac{\tilde{a}_3}{\tilde{a}_2} + 3\frac{\tilde{b}_1}{\tilde{b}_s} = -\frac{6r_s(r_s^2 + 4\alpha - 12\alpha\tilde{b}_s)}{(r_s^2 + 4\alpha - 4\alpha\tilde{b}_s)^2}, \tag{52}$$

$$\tilde{\phi}_2 + \tilde{\phi}_1^2 = \frac{(1 + 3\tilde{b}_s)r_s^2 + 4\alpha(1 - \tilde{b}_s)(1 - 3\tilde{b}_s)}{\tilde{b}_s(r_s^2 + 4\alpha - 4\alpha\tilde{b}_s)(r_s^2 + 4\alpha - 12\alpha\tilde{b}_s)}. \tag{53}$$

Using this expansion, we can show that the Kretschmann scalar, $\mathcal{R}_{abcd}\mathcal{R}^{abcd}$, diverges as

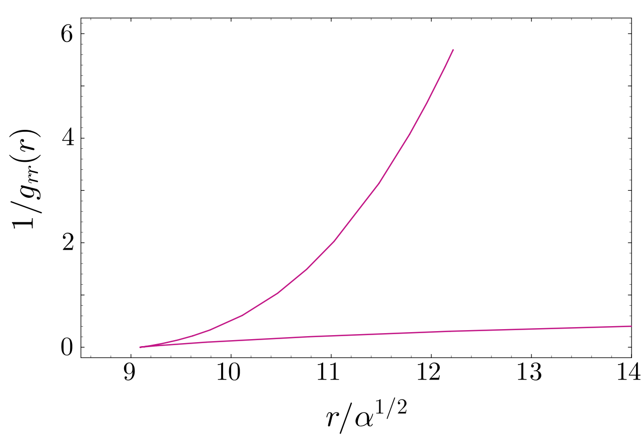

FIG. 5 (color online). Behavior of $1/g_{rr}(r) = e^{-2B(r)}$ for $M = 3\alpha^{1/2}$ and $Q = -\alpha^{1/2}$ black string. It vanishes at $r = r_m \approx 9.1\alpha^{1/2}$, but is finite at $r = r_s \approx 12\alpha^{1/2}$.

$$\mathcal{R}_{abcd}\mathcal{R}^{abcd} \sim \mathcal{O}\left(\frac{1}{(r_s - r)^2}\right), \tag{54}$$

which can be confirmed numerically as well. Thus, the solution has a null singularity at the Killing horizon. Note that here again there are two free parameters, r_s and $\tilde{b}_s$, and this is consistent with the analysis near $r = r_m$.

For the same mass black strings, we find that

$$r_h(M, Q_c) < r_s(M, Q), \tag{55}$$

namely, the radius of the singular horizon is larger than that of the event horizon of the regular black string with the same mass.

The constancy of $\mathcal{F}$ implies that

$$M + Q = \mathcal{F}(r_m) = \mathcal{F}(r_s), \tag{56}$$

with

$$\mathcal{F}(r_m) = \frac{1}{2}(r_m^2 + 4\alpha)e^{\bar{\Phi}_m}\sqrt{\bar{a}_m\bar{b}_1}\left(\frac{\bar{a}_*}{2\bar{a}_m} - \bar{\phi}_*\right), \tag{57}$$

and

$$\mathcal{F}(r_s) = (r_s^2 + 4\alpha - 4\alpha\tilde{b}_s)e^{\tilde{\Phi}_s}\sqrt{\frac{\tilde{a}_2\tilde{b}_s}{2}}. \tag{58}$$

From Eq. (50) we see that $r_s^2 + 4\alpha - 4\alpha\tilde{b}_s > 0$ because both $\tilde{b}_1$ and $\tilde{b}_s$ are positive, which yields the positivity of $\mathcal{F}(r_s)$. This means that the asymptotic charge Q is restricted in the region $-M < Q < Q_c(M)$ for this class of solutions. The limit point $M + Q = 0$ can only be achieved by setting $e^{2\tilde{\Phi}_s} = g_{yy}(r_s) = 0$. One can easily deduce that if one chooses the asymptotic charges satisfying $M + Q < 0$, the metric component g_{yy} vanishes at finite ρ, namely, spacetime has a bubblelike structure. This is in fact the case, and such a class of solutions will be discussed in the next subsection.

3. Naked singularity

For larger values of the scalar charge, $Q > Q_c(M)$ [or $\bar{Q}_c(M)$], we find that $N''(\rho)$, $r''(\rho)$, and $C''(\rho)$ [or $C'(\rho)$] diverge at a finite ρ and the Kretschmann scalar also diverges there; the solution has a naked singularity. Thus critical values Q_c and $\bar{Q}_c$ give the boundary of the two classes of solutions, with a null singularity and with a naked singularity.

C. Bubble dual of black strings

As was already mentioned, we obtain bubble solutions, rather than black strings, if $M + Q$ is negative (Fig. 6). The behavior of the solutions in the parameter region $M + Q < 0$ can be understood by the following "dual" picture [29]. Bubble-type spacetime can be constructed from the black string solutions of the form of (5) via a double Wick rotation,

$$t \mapsto iy, \qquad y \mapsto it, \tag{59}$$

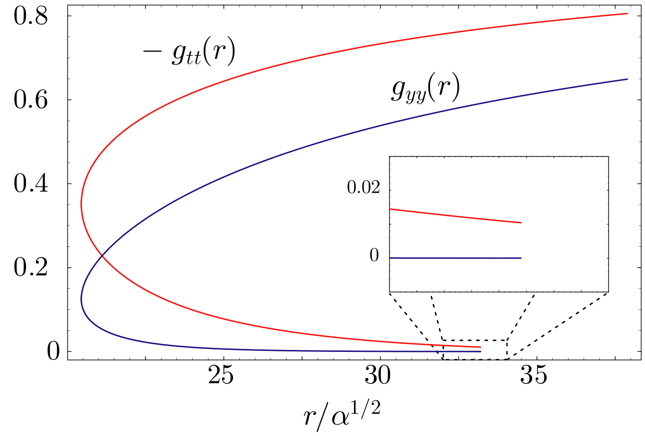

FIG. 6 (color online). Behavior of the metric components for $M = 3\alpha^{1/2}$ and $Q = -6\alpha^{1/2} < -M$. The spacetime has a bubblelike structure.

with a simultaneous exchange of the charges,

$$(M, Q) \mapsto (-Q, -M). \tag{60}$$

The transformation rule (60) follows from the asymptotic behavior of g_{tt} and g_{yy} [see Eqs. (9) and (13)]. Under this "duality" transformation, any black strings with asymptotic charges M and $Q(> -M)$ are mapped to bubble configurations with $M' = -Q$ and $Q' = -M$, but now the two charges satisfy $M' + Q' < 0$. A solution with a null singularity is dual to a bubblelike solution that is singular at the point where g_{yy} vanishes, but this singularity is no longer null. A naked singularity in the solutions for $Q > Q_c(M)$ remains naked under the duality transformation. Only for the fine-tuned values of M and Q we obtain a regular bubble solution.

D. Summary

The results obtained in this section are summarized in Fig. 7 as a "phase diagram." Although the asymptotic analysis allows any values of charges M and Q, we have seen that a vast area of the parameter space is covered by string-type configurations with a singular Killing horizon, corresponding bubble-type configurations with negative $M + Q$ which is generated by a double Wick rotation, and solutions with a naked singularity. We have shown that regular solutions (and their bubble counterparts) form a one-parameter family whose asymptotic charges are given by $(M, Q_c(M))$. We have obtained an approximate solution that fits quite well the regular black string with $M^2 \gg \alpha$ by a perturbative expansion with respect to α. In order for the horizon to be regular, its radius must be bounded below: $r_h \geq \sqrt{8\alpha}$, leading to the existence of the minimum mass black string.

The fine-tuned curve $Q = Q_c(M)$ that generates the one-parameter family of the regular black strings can be extended to the small mass region in the sense that it still

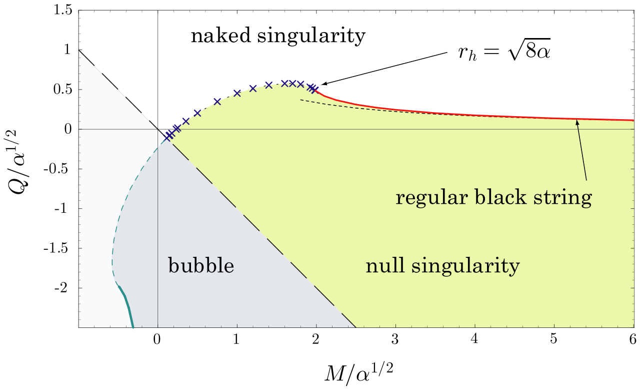

FIG. 7 (color online). Classification of solutions with M and Q. Regular black string solutions can be found only for fine-tuned values of M and Q (red solid line), and the large one is well approximated by the perturbative solution (dashed line just below the red solid one). The fine-tuned curve is a boundary of two regions in which solutions with a null singularity and a naked singularity, respectively, reside. This boundary curve can be extended down to $M = -Q \approx 0.112\alpha^{1/2}$. The nature of the bubble-type solutions below the $M + Q = 0$ line can be understood by the dual picture.

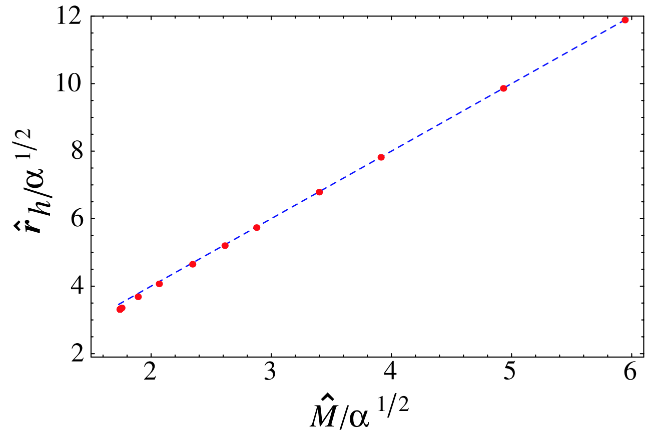

FIG. 8 (color online). Horizon radius in the Einstein frame $\hat{r}_h$ as a function of $\hat{M}$ (red circles). The blue dashed line shows the horizon radius of the perturbative solution, $\hat{r}_h = 2\hat{M}$. [It is easy to show that $q_{tt} = 1 - 2\hat{M}/\hat{r} + \mathcal{O}(\alpha^2)$ and $q_{\hat{r}\hat{r}} = (1 - 2\hat{M}/\hat{r})^{-1} + \mathcal{O}(\alpha^2)$ for the perturbative solution [17].]

represents a boundary of the classes of the solutions with a null singularity and a naked singularity, though regular solutions do not exist on the extended curve $Q = \bar{Q}_c(M)$. The boundary curve truly terminates at $(M, Q) \approx (0.112\alpha^{1/2}, -0.012\alpha^{1/2})$, where it encounters the $M + Q = 0$ line. This line is an invariant set of the transformation $(M, Q) \to (-Q, -M)$, and solutions reside below this line have a bubblelike structure.

V. FINAL REMARKS

We have seen aspects of black string configurations in the framework of five-dimensional Einstein-Gauss-Bonnet gravity. In the present paper we focused on static black-string-type metric respecting spherical symmetry on the transverse hypersurface and did not consider (self-gravitating) branes nor a cosmological constant that warps spacetime. Consequently our metric ansatz is very simple, possessing translational invariance in the extra y direction, but a nontrivial radion part $g_{yy} = e^{2\Phi(r)}$ is assumed, without which black-string-type solutions are not allowed in Einstein-Gauss-Bonnet gravity [28]. From a phenomenological point of view, it would be interesting to generalize the black string solutions in this paper to afford brane boundary conditions and a warp factor due to a bulk cosmological constant.

Stability of solutions is an important issue in black hole physics. A stability analysis for the four-dimensional black hole solutions of dilatonic Gauss-Bonnet gravity (20) [17–20] was done in Refs. [30,31]. Here, following Refs. [31,32], we briefly comment on the stability of our (regular) black string solution within translationally invariant and spherically symmetric configurations. Under the restriction of translationally invariant perturbations we may work in the reduced action (19). In the similar action of dilatonic Gauss-Bonnet gravity (20), the minimum mass solution does not correspond to the end of a sequence of solutions, and for masses near the minimum we have two solutions with different horizon radii [31]. With the aid of a catastrophe theory,[1] one can argue that an instability mode sets in when the mass is reduced to the minimum value. If we attempt to apply the same argument to our situation by seeing whether there exists such a "turning point" in the mass-horizon radius diagram, we have to use mass and a horizon radius defined in the Einstein frame [32], not M and r_h that we have used throughout this paper. The Einstein frame metric $q_{\mu\nu}$ is conformally related to the Jordan frame metric g_{ab} via Eq. (18),

$$q_{\mu\nu}d\hat{x}^\mu d\hat{x}^\nu = e^{\Phi}(-e^{2A}dt^2 + e^{2B}dr^2 + r^2 d\Omega^2) = -e^{\Phi+2A}dt^2 + \frac{e^{2B}}{(1 + r\partial_r\Phi/2)^2}d\hat{r}^2 + \hat{r}^2 d\Omega^2, \quad (61)$$

from which mass $\hat{M}$ and a horizon radius $\hat{r}_h$ in the Einstein frame can be read off as

$$\hat{M} = M - \frac{Q}{2}, \qquad \hat{r}_h = e^{\Phi(r_h)/2}r_h. \quad (62)$$

The relation between $\hat{M}$ and $\hat{r}_h$ is plotted in Fig. 8, and we see the monotonic behavior, with no turning point in this diagram. Therefore, it is strongly indicated that there is no instability under translationally invariant and spherically symmetric, linear perturbations. If there arises no instability mode down to the minimum mass, the sequence of

[1] It is shown that a stability analysis within *linear* perturbations is consistent with the analysis via a catastrophe theory [32].

solutions suddenly hits singularity when the mass is reduced, say, by Hawking radiation. However, another type of instability may arise before reaching the minimum mass. For example, something similar to the Gregory-Laflamme instability of black strings [5] may happen. Since the radius of the extra dimension can be arranged to be arbitrary small in the present context, the usual criteria for the Gregory-Laflamme instability can be avoided easily. However, the Gauss-Bonnet correction will bring extra terms to the perturbation equation because of the presence of the Weyl tensor in the field equations, and such terms may stabilize or destabilize the black string solutions. This is an open issue that we hope to return to in a future publication.

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APPENDIX: NONZERO COMPONENTS OF THE EINSTEIN AND LANCZOS TENSORS

(i) The Einstein and Lanczos tensors in the coordinate system of (5):

$$\mathcal{G}_t{}^t = \frac{e^{-2B}}{r^2}[r^2\Phi'' + 2r\Phi' + r^2(\Phi')^2 - r^2B'\Phi' - 2rB' + 1 - e^{2B}], \tag{A1}$$

$$\mathcal{G}_r{}^r = \frac{e^{-2B}}{r^2}[r^2A'\Phi' + 2r(A' + \Phi') + 1 - e^{2B}], \tag{A2}$$

$$\mathcal{G}_y{}^y = \frac{e^{-2B}}{r^2}[r^2A'' + 2rA' + r^2(A')^2 - r^2A'B' - 2rB' + 1 - e^{2B}], \tag{A3}$$

$$\mathcal{G}_\theta{}^\theta = \mathcal{G}_\varphi{}^\varphi = \frac{e^{-2B}}{r^2}[r^2A'' + 2rA' + r^2(A')^2 - r^2A'B' + r^2\Phi'' + 2r\Phi' + r^2(\Phi')^2 - r^2B'\Phi' - r(A' + \Phi' + B') + r^2A'\Phi'], \tag{A4}$$

and

$$\frac{1}{8}\mathcal{H}_t{}^t = \frac{e^{-2B}}{r^2}\{(1 - 3e^{-2B})B'\Phi' - (1 - e^{-2B}) \times [(\Phi')^2 + \Phi'']\}, \tag{A5}$$

$$\frac{1}{8}\mathcal{H}_r{}^r = \frac{e^{-2B}}{r^2}[-(1 - 3e^{-2B})A'\Phi'], \tag{A6}$$

$$\frac{1}{8}\mathcal{H}_y{}^y = \frac{e^{-2B}}{r^2}\{(1 - 3e^{-2B})A'B' - (1 - e^{-2B}) \times [(A')^2 + A'']\}, \tag{A7}$$

$$\frac{1}{8}\mathcal{H}_\theta{}^\theta = \frac{1}{8}\mathcal{H}_\varphi{}^\varphi = \frac{e^{-4B}}{r}\{\Phi'[A'' + (A')^2] + A'[\Phi'' + (\Phi')^2] - 3A'B'\Phi'\}, \tag{A8}$$

where $' := \partial_r$.

(ii) The Einstein and Lanczos tensors in the coordinate system of (6):

$$N^2\mathcal{G}_t{}^t = \frac{C''}{C} - \frac{N'C'}{NC} + 2\frac{r''}{r} - 2\frac{N'r'}{Nr} + 2\frac{C'r'}{Cr} + \left(\frac{r'}{r}\right)^2 - \frac{N^2}{r^2}, \tag{A9}$$

$$N^2\mathcal{G}_\rho{}^\rho = \frac{N'C'}{NC} + 2\frac{N'r'}{Nr} + 2\frac{C'r'}{Cr} + \left(\frac{r'}{r}\right)^2 - \frac{N^2}{r^2}, \tag{A10}$$

$$N^2\mathcal{G}_y{}^y = \frac{N''}{N} - \left(\frac{N'}{N}\right)^2 + 2\frac{r''}{r} + \left(\frac{r'}{r}\right)^2 - \frac{N^2}{r^2}, \tag{A11}$$

$$N^2\mathcal{G}_\theta{}^\theta = N^2\mathcal{G}_\varphi{}^\varphi = \frac{N''}{N} - \left(\frac{N'}{N}\right)^2 + \frac{C''}{C} + \frac{r''}{r} + \frac{C'r'}{Cr}, \tag{A12}$$

and

$$\frac{N^2}{8}\mathcal{H}_t{}^t = -\frac{1}{r^2}\left(\frac{C''}{C} - \frac{N'C'}{NC}\right) + \frac{1}{N^2}\left[\frac{C''}{C}\left(\frac{r'}{r}\right)^2 + 2\frac{C'r'}{Cr}\frac{r''}{r} - 3\frac{N'C'}{NC}\left(\frac{r'}{r}\right)^2\right], \tag{A13}$$

$$\frac{N^2}{8}\mathcal{H}_\rho{}^\rho = -\frac{1}{r^2}\frac{N'C'}{NC} + \frac{3}{N^2}\frac{N'C'}{NC}\left(\frac{r'}{r}\right)^2, \tag{A14}$$

$$\frac{N^2}{8}\mathcal{H}_y{}^y = -\frac{1}{r^2}\left[\frac{N''}{N} - \left(\frac{N'}{N}\right)^2\right] + \frac{1}{N^2}\left[\frac{N''}{N}\left(\frac{r'}{r}\right)^2 + 2\frac{N'r'}{Nr}\frac{r''}{r} - 3\left(\frac{N'}{N}\right)^2\left(\frac{r'}{r}\right)^2\right], \tag{A15}$$

$$\frac{N^2}{8}\mathcal{H}_\theta{}^\theta = \frac{N^2}{8}\mathcal{H}_\varphi{}^\varphi = \frac{1}{N^2}\left[\frac{N''C'r'}{NCr} + \frac{N'C''r'}{NCr} + \frac{N'C'r''}{NCr} - 3\frac{C'r'}{Cr}\left(\frac{N'}{N}\right)^2\right], \tag{A16}$$

where $' := \partial_\rho$.

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