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Kyoto University
Dislocation formation in two-phase alloys

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A phase field model is presented to study dislocation formation (coherency loss) in two-phase binary alloys. In our model the elastic energy density is a periodic function of the shear and tetragonal strains, which allows multiple formation of dislocations. The composition is coupled to the elastic field twofold via lattice misfit and via composition dependence of the elastic moduli. By numerically integrating the dynamic equations in two dimensions, we find that dislocations appear in pairs in the interface region and grow into slips. One end of each slip glides preferentially into the softer region, while the other end remains trapped at the interface. Under uniaxial stretching at deep quenching, slips appear in the softer region and do not penetrate into the harder domains, giving rise to a gradual increase of the stress with increasing applied strain in plastic flow.

I. INTRODUCTION

In crystalline solids many kinds of phase transformations are strongly influenced by the elastic field. Since the work by Cahn most theoretical studies have been focused on the coherent case in which the lattice planes are continuous through the interfaces. In the incoherent case, on the other hand, dislocations appear around the interfaces and the continuity is lost partially or even completely. Such incoherent microstructures emerge in various alloys when the lattice constants or the crystalline structures of the two phases are not close. Moreover, they are produced in plastic flow because dislocations generated tend to be trapped at the interfaces. In particular, coherency loss has been extensively studied in the presence of precipitates of the L12 structure. Theory for the incoherent case is much more difficult than for the coherent case, obviously because the effects cannot be adequately described within the usual linear elasticity theory. The aim of this paper is hence to present a simple mathematical model reasonably describing the incoherent effects in binary alloys. Use will be made of a recent nonlinear elasticity theory of plastic flow by one of the present authors. A number of authors have studied composition changes around dislocations fixed in space and time, which lead to a compositional Cottrell atmosphere or preferential nucleation around a dislocation. As recent numerical work in two dimensions, phase separation has been studied by Léonard and Desai and by Hu and Chen using a continuum Ginzburg-Landau or phase field model in the presence of fixed dislocations. In these papers, dislocations preexist as singular objects before composition changes. We also mention atomistic simulations of dislocation motion influenced by diffusing solutes or by precipitated domains.

Mechanical properties of two-phase solids are very different from those of one-phase solids. In the presence of precipitated domains, dislocations can be pinned at the interface regions and networks of high-density dislocations can be formed preferentially in softer regions after deformations. These effects are very complex but important in technology. Our simulations will give some insights on the behavior of dislocations in two-phase states.

II. FREE ENERGY FUNCTIONAL

We consider a binary alloy consisting of two components A and B neglecting vacancies and interstitials. The compositions \( c_A \) and \( c_B \) of the two components satisfy \( c_A + c_B = 1 \). In real metallic alloys undergoing a phase transition, there can be a change in the atomic configuration within unit cells as well as in the overall composition, resulting in ordered domains with the so-called L12 or L1$_2$ structure. However, in this paper, the composition difference is the sole order parameter

\[
\psi = c_A - c_B. \tag{2.1}
\]

for simplicity. The other variables representing the order-disorder phase transition are neglected. Then \( \psi \) is in the range \(-1 \leq \psi \leq 1\) and

\[
c_A = \frac{1}{2}(1 + \psi), \quad c_B = \frac{1}{2}(1 - \psi). \tag{2.2}
\]

In our free energy \( f = \int d\mathbf{r} f \) the order parameter \( \psi \) and the elastic displacement vector \( \mathbf{u} = (u_x, u_y) \) are coupled. The free energy density \( f \) is of the form

\[
f = f_{\text{BW}}(\psi) + \frac{c}{2} | \nabla \psi |^2 + a \epsilon_1 \psi + f_{\text{el}}. \tag{2.3}
\]

The first term is the Bragg-Williams free energy density expressed as

\[
f_{\text{BW}}(\psi) = \frac{1}{2} \psi^2 - c_1 \epsilon_2 \psi + c_2 \epsilon_2. \tag{2.4}
\]
where \( v_0 \) is the volume of a unit cell representing the atomic volume, \( T_0 \) is the mean-field critical temperature in the absence of the coupling to the elastic field. If \(|\psi| \ll 1\), we obtain the Landau expansion
\[
\frac{v_0}{k_B T} f_{\text{BW}} = \frac{1 + \psi}{2} \ln(1 + \psi) + \frac{1 - \psi}{2} \ln(1 - \psi) - T_0 \psi^2/2T,
\]
(2.4)
where the coupling between the composition and the dilation strain \( \epsilon_i \). This coupling arises in the presence of a difference in the atomic sizes of the two species and is consistent with the empirical fact that the lattice constant changes linearly as a function of the average composition in many one-phase alloys (Vegard law). It gives rise to a difference in the lattice constants of the two phases in phase separation (lattice misfit). It also explains a composition inhomogeneity (Cottrell atmosphere in one-phase states or precipitate in two-phase states) around a dislocation.

In two dimensions \( f_{\text{el}} \) depends on the following strains
\[
e_1 = \nabla_x u_x + \nabla_y u_y,
\]
\[
e_2 = \nabla_x u_y - \nabla_y u_x,
\]
\[
e_3 = \nabla_y u_x - \nabla_x u_y,
\]
(2.5)
where \( \nabla_x = \partial/\partial x \) and \( \nabla_y = \partial/\partial y \). The elastic displacement \( u \) is measured in a reference one-phase state at the critical composition. We call \( e_1 \) the tetragonal strain and \( e_3 \) the shear strain. In this paper we use a nonlinear elastic energy density of the form
\[
f_{\text{el}} = \frac{1}{2} Ke_1^2 + \Phi(\psi, e_2, e_3).
\]
(2.6)
The first term represents the elastic energy due to dilation with \( K \) being the bulk modulus. The second term arises from anisotropic shear deformations defined for arbitrary values of \( e_2 \) and \( e_3 \). Assuming a square lattice structure, we set
\[
\Phi = \frac{\mu_2}{4\pi^2} [1 - \cos(2\pi e_2)] + \frac{\mu_3}{4\pi^2} [1 - \cos(2\pi e_3)].
\]
(2.7)
The principal crystal axes are along or make angles of \( \pm \pi/4 \) with respect to the \( x \) or \( y \) axis. In Fig. 1 we plot \( \Phi \) as a function of \( e_2 \) and \( e_3 \) for the case \( \mu_2 = \mu_3 = \mu_0 \) in units of \( \mu_0 \). If the system is homogeneous, elastic stability is attained for \( \partial^2 \Phi/\partial e_2^2 > 0 \) and \( \partial^2 \Phi/\partial e_3^2 > 0 \) or in the regions \( |e_2 - m| < 1/4 \) and \( |e_3 - m| < 1/4 \) with \( n \) and \( m \) being integer values.

For small strains \( |e_2| \ll 1 \) and \( |e_3| \ll 1 \), it follows the usual standard form
\[
\Phi \approx \frac{1}{2} \mu_2 e_2^2 + \frac{1}{2} \mu_3 e_3^2,
\]
(2.8)
in the linear elasticity theory. Therefore,
\[
\mu_2 = \frac{1}{2}(C_{11} - C_{12}), \quad \mu_3 = C_{44},
\]
(2.9)
in terms of the usual elastic moduli \( C_{11} \), \( C_{12} \), and \( C_{44} \). In the original theory\(^3\) the isotropic linear elasticity with constant \( \mu_2 = \mu_3 \) was assumed. Subsequent theories treated the case of the cubic linear elasticity with constant \( \mu_2 \) and \( \mu_3 \).\(^{2,4,6,23}\) In the present paper, while \( K \) is a constant, \( \mu_2 \) and \( \mu_3 \) depend on the composition as
\[
\mu_2 = \mu_2(\psi), \quad \mu_3 = \mu_3(\psi).
\]
(2.10)
If \( |e_2| > 1 \) and \( |e_3| > 1 \), the regions with larger (smaller) \( \psi \) are harder (softer) than those with smaller (larger) \( \psi \). It is known that this elastic inhomogeneity gives rise to asymmetric elastic deformations in two-phase structures and eventual pinning of the domain growth.\(^{4,23,24}\)

In our theory \( \Phi(\psi, e_2, e_3) \) in Eq. (2.7) is the simplest period function of \( e_2 \) and \( e_3 \) with period 1. The periodicity arises from the fact that the square lattice is invariant with respect to a slip of the crystal structure by a unit lattice constant along a line parallel to the \( x \) or \( y \) axis. Notice that, under rotation of the reference frame by \( \theta \), \( e_2 \) and \( e_3 \) are changed to \( e_2' \) and \( e_3' \), respectively, with
\[
e_2' = e_2 \cos 2\theta + e_3 \sin 2\theta,
\]
\[
e_3' = -e_2 \sin 2\theta + e_3 \cos 2\theta.
\]
(2.11)
For \( \theta = \pi/2 \) we have \( e_2' = -e_2 \) and \( e_3' = -e_3 \), so \( f_{\text{el}} \) in Eq. (2.7) remains invariant. For \( \theta = \pi/4 \) we have \( e_2' = e_3 \) and \( e_3' = -e_2 \) and recognize that the roles of tetragonal and shear strains are exchanged. For \( \mu_2 = \mu_3 \), the linear elasticity in Eq. (2.8) becomes isotropic, but the nonlinear elasticity is still anisotropic [from the fourth-order terms in the expansion of \( \Phi \) in Eq. (2.7) in powers of \( e_2 \) and \( e_3 \)].

The elastic stress tensor \( \sigma = \{\sigma_{ij}\} \) is expressed as
\[
\sigma_{xx} = Ke_1 + \alpha \psi + \mu_2 \sin(2\pi e_2)/2\pi,
\]
\[
\sigma_{yy} = Ke_1 + \alpha \psi - \mu_2 \sin(2\pi e_2)/2\pi,
\]
\[
\sigma_{xy} = \sigma_{yx} = \mu_3 \sin(2\pi e_3)/2\pi.
\]
(2.12)
In the linear elasticity, \( \sin(2\pi e_2)/2\pi \) and \( \sin(2\pi e_3)/2\pi \) are replaced by \( e_2 \) and \( e_3 \), respectively. Notice the relation
where $\psi$ is fixed in the functional derivative $\delta F / \delta u$.

The mechanical equilibrium condition $\nabla \cdot \sigma = 0$ is equivalent to the extremum condition $\delta F / \delta u = 0$. In the coherent case this condition may be assumed even in dynamics. In fact, using this condition in the linear elasticity, the elastic field has been expressed in terms of $\psi$ in the previous theories (see the appendix). We then find the following. (i) The typical strain around domains is given by

$$ e_0 = a \Delta c / L_0, $$

where $\Delta c = \Delta \psi / 2$ is the composition difference between the two phases and

$$ L_0 = K + \mu_20 $$

is the longitudinal elastic modulus. This strain needs to be small ($e_0 < 1/4$ approximately) as long as the system stays in the coherent regime. (ii) As will be shown in the the Appendix, in the limit of weak cubic elasticity and weak elastic inhomogeneity, one-phase states become linearly unstable for $k_B T - T_0 + T(\psi^2/2)v_0 < a^2 / L_0$. At the critical composition $\langle \psi \rangle = 0$ this condition becomes $T < T_s$ with

$$ T_s = T_0 + v_0 a^2 / L_0 k_B. $$

(iii) Furthermore, Eq. (A.5) suggests that the typical domain size in steady pinned states is a decreasing function of the quench depth $T - T_s$.

### III. DYNAMIC EQUATIONS

In the incoherent case the mechanical equilibrium does not hold around dislocation cores when dislocations are created and when they are moving. We thus need to set up the dynamic equation for the elastic displacement $u$. In this paper the lattice velocity $v = \partial u / \partial t$ obeys the momentum equation

$$ \rho \frac{\partial v}{\partial t} = \eta_0 \nabla^2 v + \nabla \cdot \sigma. $$

The mass density $\rho$ and the shear viscosity $\eta_0$ are constants. We neglect the bulk viscosity term ($\propto \nabla \cdot v$) in Eq. (3.1) for simplicity. In our model the sound waves relax owing to this viscous damping and the mechanical equilibrium $\nabla \cdot \sigma = 0$ is rapidly attained unless $\eta_0$ is very small. Note that the nonlinear terms in Eq. (3.1) are only those in $\sigma_{ij}$ in Eq. (2.12).

The composition obeys the diffusive equation

$$ \frac{\partial \psi}{\partial t} = \nabla \cdot \lambda(\psi) \nabla \frac{\delta F}{\delta \psi}. $$

The kinetic coefficient depends on $\psi$ as

$$ \lambda(\psi) = \lambda_0 (1 - \psi^2) = 4 \lambda_0 c_A c_B, $$

where $\lambda_0$ is a constant. Here $u$ is fixed in the chemical potential difference $\delta F / \delta \psi$, so

\[ \text{FIG. 2. Displacement vector for a slip (dislocation pair) making an angle of 3}\pi/4 \text{ (upper plate) and 0 (lower plate) with respect to the x (horizontal) axis in a one-phase steady state at } T/T_0 = 2.5. \text{ The arrows are from the initial position in a perfect crystal to the deformed position. The degree of darkness represents the composition.} \]

\[ \text{The last two terms arise from the elastic inhomogeneity. If } \lambda(\psi) \text{ is of the form of Eq. (3.3), the diffusion equation } \frac{\partial c_A}{\partial t} = D_0 \nabla^2 c_A(\partial c_A / \partial t) = D_0 \nabla^2 c_B \text{ follows in the dilute limit } c_A \rightarrow 0 \text{ (} c_B \text{ fixed)} \text{ with} \]

$$ D_0 = \lambda_0 k_B T u_0^2, $$

where the coupling to the elastic field becomes negligible. In usual solid mixtures the diffusion is very slow and vacancies are in many cases crucial for a microscopic description of diffusion. Effects of such point defects are not treated in the present theory.
The total free energy \( F_{\text{tot}} = F + e_d r r^2 / 2 \) including the kinetic energy then changes in time as
\[
\frac{d}{dt} F_{\text{tot}} = - \int dr \left[ \sum_{ij} \eta_0 (\nabla \psi_j)^2 + \lambda(\psi) \left| \nabla \frac{\delta F}{\delta \psi_i} \right|^2 \right].
\]

Here the surface integrals have been omitted, which vanish if the boundaries are fixed and there is no flux of the atoms from outside (and also if the periodic boundary condition is imposed in simulations). The above time-derivative is nonpositive-definite. As a result, the equilibrium is attained when \( \psi = \nabla \cdot \vec{r} = 0 \) and \( \delta F / \delta \psi = \text{const} \).

If the lattice is deformed significantly, we should add the convective term \(-\nabla (\psi \vec{v})\) on the right-hand side of Eq. (3.2), treating Eq. (3.2) as the equation in the Euler description. If its presence is assumed, another term of the form \(-\psi \nabla \delta F / \delta \psi\) becomes also needed on the right-hand side of Eq. (3.1). With these two terms we again have \( dF_{\text{tot}} / dt \equiv 0 \). These two terms are well known in critical dynamics of fluids. However, in our solid case, the magnitude of the displacement \( \Delta u = \int_0^t dt' \vec{v}(r, t') \) remains small and these two terms give rise to no essential differences in our results at not large applied strains.

Also note that the dynamic Eqs. (3.2) and (3.3) may be treated as Langevin equations with addition of the random noise terms related to the kinetic coefficients \( \lambda(\psi) \) and \( \eta_0 \) via the fluctuation-dissipation relations. In this paper, however, we neglect the random noise, because the thermal energy \( k_B T \) will be assumed to be much smaller than the typical energy of elastic deformations.

**IV. NUMERICAL RESULTS**

**A. Method**

We integrated Eqs. (3.1) and (3.2) in two dimensions on a 256 \( \times \) 256 square lattice. The mesh size \( \Delta x \) was set equal to the lattice constant \( a \) in the reference state with \( \vec{u} = 0 \), so the system length is \( L_0 = 256a \). The vectors \( \vec{u} \) and \( \vec{v} \) are defined at the lattice points \( (n, m) \), while the strains, the tensors, and the composition are defined on the middle points \( (n+1/2, m+1/2) \). These are needed to realize well-defined microscopic slips in our numerical scheme. The periodic boundary condition was imposed except the simulation of applying uniaxial deformation (Figs. 13 and 14). Because the time scale of \( \vec{u} \) is shorter than that of \( \psi_0 \), we integrated Eq. (3.1) using an implicit Crank-Nicolson method. Space and time will be measured in units of \( a \) and
\[
\tau_0 = (\rho \mu_{20})^{1/2} a, \quad (4.1)
\]
respectively, where \( \mu_{20} \) is defined by Eq. (2.10) and \( (\mu_{20}/\rho)^{1/2} \) is the transverse sound velocity propagating in
the \( \cdot \) direction. The free energies and the free energy densities are measured in units of \( \mu_{20}^{-\alpha^2} \) and \( \mu_{20}^{-\alpha^2} \), respectively. For simplicity, the scaled time \( \tau_0^{-1} \), position vector \( a^{-1} r \), and displacement vector \( a^{-1} u \) will be written as \( t \), \( r \), and \( u \), respectively, in the same notation.

In this paper we set \( K/\mu_{20} = 4.5 \), \( \alpha/\mu_{20} = 0.6 \), \( C/\alpha^2/\mu_{20} = 0.05 \) and \( k_B T_0/\mu_{20} = 0.05 \), where \( T_0 \) is the mean-field critical temperature in Eq. (2.4). Since \( T \sim T_0 \) hereafter, the elastic energy to create a single slip \( \sim \mu_{20} \psi \) is much larger than \( k_B T \) in our simulations. Furthermore, we assume weak cubic elastic anisotropy with \( \mu_{31}/\mu_{20} = 1.1 \) and moderate elastic inhomogeneity with \( \mu_{21} = \mu_{31} = 0.6 \).

The dimensionless kinetic coefficients are given by

\[
\lambda_0^* = \lambda_0 \tau_0 \mu_{20} \alpha^{-2}, \quad \eta_0^* = \eta_0 \tau_0 \mu_{20}.
\]

(4.2)

We set \( \lambda_0^* = 10^{-4} \) and \( \eta_0^* = 0.1 \). Then,

\[
\lambda_0^* / \mu_0 = D_0 \psi / \eta_0 \sim 10^{-3}.
\]

(4.3)

Since the relaxation rate of a sound with wave-number \( k \) is \( \eta k^2 / \rho \), the time scale of \( \psi \) becomes longer than that of the elastic field by three orders of magnitude. In real solid alloys, these two time scales are much more distinctly separated, probably except for hydrogen-metal systems where the protons diffuse quickly.

In homogeneous one-phase states we have \( e_2 = e_3 = 0 \) and \( e_1 = -\alpha \psi / K \). Here, well known is a parameter \( \eta = |\partial a / \partial \psi| / a \) representing the strength of the composition dependence of the lattice constant \( a \) in a mixture. In our case we have \( \eta \approx \alpha / 2 K = 0.067 \) and the spinodal temperature \( T_s \) in Eq. (2.16) becomes 2.31.

B. Slips and composition changes

Edge dislocations appear in the form of slips or dipole pairs, because a single isolated dislocation requires a very large elastic energy. Slips are thus fundamental units of plastic deformations. In Fig. 2 we show the displacement and the composition around typical slips in a one-phase steady state with length \( 10 \sqrt{2} a \) in the upper plate and \( 10 a \) in the lower plate. Here we initially prepared a slip given by the linear elasticity theory at the critical composition \( \psi(0) = 0 \) and let \( u \) and \( \psi \) relax until the steady state was achieved. The temperature was kept at \( T/T_0 = 2.5 \) and no phase separation occurred. As in the previous simulations, we can see Cot-
and Cottrell atmospheres around the dislocation cores. The maximum and minimum of \( \psi \) at the lattice points close to the dislocation cores are of order \( \pm 0.6 \). Cottrell’s result is obtained as follows: Let \( T \) be much higher than \( T_0 \) and \( \alpha^2/L_0 \) and the gradient term be neglected; then, the condition \( \partial F/\partial \psi = \text{const.} \) yields \( c_A/(1-c_A) = \text{const.} \exp(-U/k_B T) \), where \( U = U_0 \alpha \epsilon_1 \). In our case the maximum of \( |U|/k_B T \) at the lattice points is of order 1 and the accumulation is not very strong.

As a next step, starting with the configuration in Fig. 2, we lowered the temperature to \( T/T_0 = 2 \) to induce spinodal decomposition. Subsequently the Cottrell atmospheres grew into domains and the dislocation cores stayed at the interface regions. The domain size attained finally was of order 50a. Figure 3 illustrates the displacement and the composition in the final steady state, where the maximum and minimum of \( \psi \) are about \( \pm 0.9 \). Léonard and Desai \(^{16} \) obtained similar composition profiles in spinodal decomposition, where the elastic field of dislocations (given by the linear elasticity theory) was fixed in space and time.

Mathematically, slips in steady states satisfy \( \partial F/\partial \epsilon = 0 \) and \( \partial F/\partial \psi = \text{const.} \). Without externally applied strains, they are metastable owing to the Peierls potential energy arising from the discreteness of the lattice structure.\(^{12} \) Although not discussed in this paper, slips become unstable against expansion or shrinkage with increasing applied strain.

### C. Dislocation formation around a hard domain

Figure 4 shows a single large hard (A-rich) domain at the center in the coherent condition at shallow quenching \( T/T_0 = 2 \) after a long equilibration time. Here \( \psi \) is about 0.7 inside the domain and about −0.7 outside. Its shape slightly deviates from sphericity owing to the weak cubic anisotropy assumed in this paper. We next performed a second deeper quenching to \( T/T_0 = 1 \). Subsequent diffusional adjustment of the composition proceeded very slowly, but a discontinuity of the order parameter \( \Delta \psi \) about 1.8 was established relatively rapidly across the interface.\(^{3} \) As a result, at a time about 1000 after the second quenching, the maximum of \( |c_2| \) reached 1/4, the value at the stability limit, in the interface region (see the sentences below Eq. (2.7)). We then observed formation of dislocations and generation of sound waves emitted from the dislocations. The upper panel of Fig. 5 shows the coherent elastic displacement \( u_{\text{coh}} \) just before the dislocation formation, while the lower panel shows the subsequent additional incoherent change \( \partial u = u - u_{\text{coh}} \) after a time interval of 1000. The free energy \( F \) in the state in the lower panel is smaller than that in the upper panel by 152.9 in units of \( \mu_{\text{MB}} \mathbf{a} \). More details are as follows: (i) two pairs of dislocation dipoles (four dislocations) appeared simultaneously in a narrow region; (ii) two of them glided preferentially into the softer region forming two slips perpendicular to each other; and (iii) slips collided in many cases and stopped far from the droplet, resulting in a nearly steady elastic deformation. Thus a half of the dislocation cores stayed at the interface and the others were distributed around the domain. These three processes took only a short time of order 100.
FIG. 9. Domain structure obtained at a shall quench \( T/T_0 = 2 \) in the coherent condition.

D. Dislocation formation in a soft network

Next we examine dislocation formation when hard rectangular domains are densely distributed and wrapped by a percolated soft network. As in Fig. 9, we prepared such a steady domain structure at \( T/T_0 = 2 \) in the coherent condition. As in the previous simulations, 23,24 the hard domains (in gray) are elastically isotropic, while the soft network (in white) is mostly uniaxially stretched. That is, in the soft stripes between the two adjacent hard domains, we obtain \( e_2 \approx 0.2 \) in the horizontal stripes and \( e_2 \approx -0.2 \) in the vertical stripes. We then quenched \( T \) to \( T/T_0 = 1 \) to induce the composition readjustment. Figure 10 displays the resultant time evolution of the total free energy \( F = \int drf \) and the snapshots of \( e \) in Eq. (4.4) at the points A, ..., and E. It demonstrates that \( F \) mainly decreases due to the composition change but sometimes due to appearance and gliding of slips in the soft stripes. Note that the overall composition adjustment occurs slowly on the time scale of \( R^2/D_0 = 10^5 - 10^6 \) where \( R \) is the domain size. In Fig. 11 we show the displacement \( u \) within the square window in B, C, and D, respectively, while in Fig. 12 the bird views of the free energy density \( f \) the square window are given at \( t = 0 \) and 4475 after the second quench. Figure 12 clearly illustrates appearance of the peaks representing the dislocation cores.

E. Uniaxial stretching in two-phase states

Finally we apply a constant uniaxial deformation to initially coherent states with \( \langle \phi \rangle = 0 \) to induce plastic flow. That is, we set \( u_x = u_y = 0 \) at the bottom \( (y = 0) \) and \( u_x = -u_y = eL_0/2 \) at the top \( (y = L_0) \). The applied strain rate was fixed at \( \dot{e} = 10^{-4} \), so \( e = \dot{e} t \) with \( t \) being the time after application of the deformation. In Fig. 13 we plot the average normal stress \( N_1 \) versus the applied strain \( e \) for \( T/T_0 = 3, 2.4, \) and 2 (upper plate), where

FIG. 10. Relaxation of the total free energy \( F \) in units of \( \mu_{20}d_0 = 20k_BT_0 \) after a two-step quench from \( T/T_0 = 2 \) to 1 with the initial configuration in Fig. 9. It mostly relaxes due to the gradual composition adjustment, but it sometimes relaxed due to dislocation formation as enlarged in the inset. Snapshots of \( e \) at the points A~E are given in the lower plates.
\[ N_1 = \langle \sigma_{xx} - \sigma_{yy} \rangle = \frac{1}{p} (\mu_2 \sin(2\pi \phi_2)), \]  \hspace{1cm} (4.5)

where \( \langle \cdots \rangle \) denotes taking the spatial average. The snapshots of \( e \) in Eq. (4.4) are also given at the points a, b, and c (lower plates). For \( T/T_0 = 3 \) the system is in a homogeneous one-phase state and random numbers with variance 0.01 were assigned to \( \psi \) at the lattice points at \( t = 0 \). In the initial state at \( T/T_0 = 2.4 \) the maximum and minimum of \( \psi \) and \( \varepsilon_2 \) are \( \pm 0.32 \) and \( \pm 0.05 \), respectively. At \( T/T_0 = 2 \) these numbers are mag-

FIG. 11. Elastic displacement \( u \) in the marked regions B, C, and D.

FIG. 12. Bird views of the free energy density \( f \) in units of \( k_B T_0 \nu_0^{-1} \) at \( t = 0 \) and 4475 after the two-step quench in Fig. 10.

FIG. 13. Stress-strain curves after application of uniaxial stretching \( \varepsilon = \dot{\varepsilon} t \) with \( \dot{\varepsilon} = 10^{-4} \) for \( T/T_0 = 3, 2.4, \) and 2. There is no dislocation at \( t = 0 \). Snapshots of \( e \) in Eq. (4.4) at points a, b, and c are given below, which represent slip patterns in plastic flow.
of plasticity have been gained. We mention them and give some remarks.

(1) Performing a two-step quench, we have numerically examined dislocation formation around the interface regions, which occur spontaneously in deeply quenched phase separation. Experimentally,\textsuperscript{10} dislocation formation has been observed around growing $\gamma'$ (Al$_3$Sc) precipitates at low volume fractions when the radii exceeded a threshold about 20 nm. Such spontaneous dislocation formation with domain growth has not yet been studied theoretically.

(2) We have found that dislocations glide preferentially into the softer regions with smaller shear moduli and tend to be trapped in the interface regions in agreement with a number of observations.\textsuperscript{7} Theoretically, the composition dependence of the elastic moduli (elastic inhomogeneity) is a crucial ingredient to explain the experiments.

(3) We have applied uniaxial strain to create multiple slips in two-phase alloys which were initially in the coherent condition. The dislocation formation starts in the mostly stretched middle points of the soft stripes. A stress-strain curve in Fig. 13 at deep quenching is very different from the curves in one-phase states. In real two-phase alloys, a similar monotonic increase of the stress without overshoot has been observed, but a considerable amount of defects should pre-exist in such experiments particularly in work-hardened samples.\textsuperscript{7,13,20}

This work is a theoretical step to understand complex phenomena of incoherency in solids. Finally, we mention two future problems which could be studied numerically in our scheme.

(1) The composition has been taken as a single-order parameter. We should investigate dislocation formation in more general phase separation processes involving an order-disorder phase transition\textsuperscript{2,4,16} and in diffusionless (Martensitic) structural phase transitions.\textsuperscript{2,4}

(2) Dislocations move under applied strain. The motion is complicated when they are coupled with an order parameter and when the time scale of the order parameter is slow.\textsuperscript{16,29}

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APPENDIX

Here we assume weak elastic anisotropy and weak elastic inhomogeneity in the coherent condition in two dimensions, supposing shallow quenching. Then we may eliminate the elastic field in terms of $\psi$ using the mechanical equilibrium condition $\nabla \cdot \tau = 0$ in the linear elasticity. We consider the space integral of the last two terms in the free energy density in Eq. (2.3): $\Delta F = \int d\mathbf{r} \left[ \alpha e \phi + f_g \right]$. We assume that $|\mu_{33}|$ and $|\mu_{31}|$ are much smaller than $L_0 = K + \mu_{20}$ and that $\xi_a$...
The third-order terms are known to give rise to pinning of domain growth (and some frustration effects when \( g_2 \) and \( g_3 \) have different signs).\textsuperscript{23,24}

In our simulations we set \( \xi_c = 2(1/1.1-1) \approx -0.18 \) and \( \tau_{\text{cub}} \approx 0.0043 \mu_{20} \), so the domains tend to become square or rectangular with interfaces parallel to the \( x \) or \( y \) axis. Furthermore, we set \( g_2 = g_3/4 \approx 0.0035 \mu_{20} \). For \( \mu_{21} \approx \mu_{31} \) the typical domains in pinned two-phase states \( R_E \) is given by\textsuperscript{23}

\[
R_E \sim \gamma[\mu_{21}(\Delta c)^3],
\]

where \( \gamma \) is the surface tension and \( \Delta c \) is the composition difference between the two phases. Thus \( R_E \) decreases as the quenching becomes deeper.

\begin{align*}
\Delta F &= \int d\mathbf{r} \left[ -\frac{\alpha^2}{2L_0^2} \psi^2 + \frac{1}{2} \tau_{\text{cub}} |\nabla \psi|^2 \right] + \int d\mathbf{r} \left[ g_2 \psi |\nabla \psi|^2 \right] \\
&= 2(\mu_{20}/\mu_{30} - 1) \text{ is small. Then } \Delta F \text{ may be rewritten as}\textsuperscript{23,24}
\end{align*}

\[
\Delta F = \int d\mathbf{r} \left[ -\frac{\alpha^2}{2L_0^2} \psi^2 + \frac{1}{2} \tau_{\text{cub}} |\nabla \psi|^2 \right] + \int d\mathbf{r} \left[ g_2 \psi |\nabla \psi|^2 \right],
\]

where \( \psi \) is obtained from the Laplace equation

\[
\nabla^2 \psi = \psi - \langle \psi \rangle,
\]

with \( \langle \psi \rangle \) being the average order parameter. In the first line of Eq. (A.1) the bilinear terms are written with

\[
\tau_{\text{cub}} = -\left( 2\alpha^2/L_0^2 \right) \mu_{20} \xi_c.
\]

The term proportional to \( \tau_{\text{cub}} \) gives rise to anisotropic domains.\textsuperscript{22} The second line consists of the third-order terms with

\[ g_2 = \mu_{21} \alpha^2/L_0^2, \quad g_3 = \mu_{31} \alpha^2/L_0^2. \]