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Kyoto University
Photonic band structure calculations of two-dimensional Archimedean tiling patterns

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We present a study of photonic band structures of two-dimensional Archimedean tiling patterns. The tilings we have investigated are \((4, 8^2), (6^3), (4, 6, 12)\), and \((3^2, 4, 3, 4)\), which have been discovered computationally and experimentally in self-assembled microphase separation of \(ABC\) star block terpolymer systems. Using plane-wave method, we have calculated eigenvalue equations for various combinations of dielectric contrast on the complex patterns. We demonstrate the existence of complete photonic band gaps in the \((4, 6, 12)\) structure. Furthermore, we find that complete photonic bands readily open in the \((3^2, 4, 3, 4)\) structures in the same way as in dodecagonal quasicrystals. Complex tilings open up a way to construct photonic crystals.

I. INTRODUCTION

Archimedean tilings depicted by Kepler in Harmonices Mundi II (1619) are regular patterns of polygonal tessellation in plane by using regular polygons.\(^1\) According to Kepler, they are congruent because only one type of vertices is permitted in each tiling. It is known that only 11 kinds of Archimedean tilings illustrated in Fig. 1 can fill the whole plane without gaps. Here, a set of integers \((n_1, n_2, n_3, \ldots)\) denotes a tiling of a vertex type in the way that \(n_1\)-gon, \(n_2\)-gon, and \(n_3\)-gon, \ldots, meet consecutively on each vertex. The symbol \((3^2, 4, 3, 4)\), for instance, represents a tiling in which two equilateral triangles, a square, an equilateral triangle, and a square gather edge-to-edge around a vertex.

Recently, several Archimedean tiling structures denoted by \((6^3)\), \((4, 8^2)\), \((4, 6, 12)\), and \((3^2, 4, 3, 4)\) were discovered in the self-organized structures of synthetic polymer systems.\(^2\)–\(^9\) Interestingly, the lattice constants of the structures can reach the wavelength of visible light. Hence, they may open the possibility of constructing novel photonic band gap (PBG) devices, such as waveguides or dielectric mirrors, where the propagation of electromagnetic waves or the spontaneous emission of light is forbidden.\(^10\)–\(^19\) In this paper, we present photonic band structures of the four Archimedean tiling patterns accessible in the self-assembling structures.

To generate such mesoscopic patterns, much attention has been paid to block copolymer systems.\(^20\)–\(^24\) Intramolecular segregation of block copolymers consisting of different polymers covalently linked together gives rise to microphase separations producing periodic morphologies: lamellar, co-continuous, hexagonally cylindrical, and bcc spherical structures are well known. Concerning optical properties, triply periodic cocontinuous structures such as double-diamond and gyroid structures have been extensively investigated by several researchers.\(^25\)–\(^30\)

The focus of the present paper is two-dimensionally complex patterns produced by \(ABC\) star block copolymers (terpolymers) consisting of chemically distinct three polymers linked at one junction. Their melts can form two-dimensional tiling patterns, precisely, polygonal cylindrical phases whose sections are the Archimedean tilings. If the interactions between \(ABC\) polymer components are equally strong, only \((6^3)\), \((4, 8^2)\), and \((4, 6, 12)\) belonging to the single junction class\(^31\) (SJC) can be obtained as direct patterns,\(^5\) where each polygon in the Archimedean tiling directly corresponds to each polymeric microdomain. It is firstly because only three polygons corresponding to \(ABC\) microdomains should meet on a vertex, and secondly because only even polygons should appear, which fact is called even polygon theorem.\(^2,32\) Very recently, an indirect skeleton tiling \((3^2, 4, 3, 4)\) has been obtained.\(^6\) The tiling is more complex than the SJC; however, the skeleton structure is the \((3^2, 4, 3, 4)\) Archimedean tiling. Therefore, we consider three direct tilings, \((6^3)\), \((4, 8^2)\), and \((4, 6, 12)\), and one skeleton tiling, \((3^2, 4, 3, 4)\), in the present paper.

FIG. 1. Archimedean tilings: A set of integers \((n_1, n_2, n_3, \ldots)\) denotes a tiling of a vertex type in the way that \(n_1\)-gon, \(n_2\)-gon, and \(n_3\)-gon, \ldots, meet consecutively on each vertex. Superscripts are employed to abbreviate when possible. There exist only 11 types of tiling by regular polygons, where all vertices are of the same type.
II. MONTE CARLO SIMULATION OF ABC STAR TERPOLYMERS

In this section, we present a phase diagram and an example of the \((3^2, 4, 3, 4)\) structure obtained in our MC simulations. It is worthwhile to mention that dodecagonal quasicrystals are thought to be promising candidates for PBG structures because of their high degree of rotational symmetry. It was suggested that a complete PBG opens with low dielectric contrast in a dodecagonal structure. Although the two-dimensional (2D) space group is \(p4gm\), the \((3^2, 4, 3, 4)\) tiling produces 12-fold-like Fourier peaks, and it is an approximant of the dodecagonal quasicrystal. Therefore, the band structure of \((3^2, 4, 3, 4)\) is worth considering. In this direction, a systematic study has been reported.

The organization of paper is as follows. In Sec. II, we provide an Archimedean tiling phase diagram of \(ABC\) star terpolymers obtained by MC simulations. We explain the MC simulation method and present some results. In Sec. III, we elucidate how to construct tiling patterns with the arbitrary component ratio of \(ABC\) species by a geometric operation in terms of minimal fundamental triangles. The method of photonic band calculations for both \(E\) and \(H\) polarizations is provided in Sec. IV. We have used plane-wave expansion method.\(^{13–18}\) In Sec. V, the main results of the paper are presented. Sec. VI is devoted to discussion and summary.

III. GEOMETRY OF TILINGS

A. Single junction class: \((6^3), (4.8^2)\), and \((4.6.12)\)

Evidently, in real microphase separations, the area ratios of \(ABC\) components are not the same as those of the Archimedean tilings that are by definition composed of regular...
and perpendiculars to the edges of the triangle, then exact Archimedean tilings (a) $6^3$, (b) $4.6.12$, and (c) $4.8^2$ composed of regular polygons are obtained.

To obtain a tiling corresponding to an experiment, put a junction on an appropriate position inside a fundamental triangle and draw three straight or curved lines to three edges such that the lines are perpendicular to the edges at the edges of the triangle (mirror planes). Our SJC photonic crystals we consider here are depicted in Fig. 5. The ratios of components are $A:B:C=1:1:1$ for the $6^3$ structure, $A:B:C=2:1:2$ for the $4.8^2$ structure, and $A:B:C=1:1:2$ for the $4.6.12$ structure, which are typical ratios in numerical experiments. We assume that domain boundaries are straight lines in our calculations. Dielectric contrasts will be described later.

Occasionally, it has been stated that observed tiling types are designated different from the SJC. In view of symmetry, they can be unified into the classification scheme of the SJC. Let us consider limiting cases that the junction is on an edge or a vertex of a minimal fundamental triangle. For example, we can construct (b) a $3.4.6.4$ tiling from (e), which corresponds to the structure that has been obtained by Sioula et al.\textsuperscript{9} In this case, repulsion between triangular and hexagonal regions are stronger than others. There is a number of cases, which is summarized in Table I and Figs. 4(d)–4(n). Finally, we can employ curved domain boundaries instead of straight lines. For example, in terms of the minimal triangle of $4.8^2$, we are able to construct a pattern (n) called St. Andrew’s cross by Yamauchi et al.\textsuperscript{8}

![Minimal fundamental triangles](https://example.com/minimal-triangles.png)

**FIG. 4.** (Color online) Minimal fundamental triangles of (a) $6^3$, (b) $4.6.12$, and (c) $4.8^2$. (d)–(l) Archimedean tilings, which belong to the single junction class. Limiting cases (d)–(l) and the resulting tiling (g) (3.6.3.6), (h) (3.4.6.4), and (i) (4$^3$), respectively. Other limiting cases (j) (3$^3$), (k) (3.6.3.6), (l) (4$^4$), and (m) (3.12$^2$). (n) St. Andrew’s cross having curved polygons from the minimal triangle of $4.8^2$.

![Area ratios of photonic crystals](https://example.com/area-ratios.png)

**FIG. 5.** Area ratios of photonic crystals: (a) $A:B:C=1:1:1$ in the $6^3$ structure, (b) $A:B:C=2:1:2$ in the $4.8^2$ structure, and (c) $A:B:C=1:1:2$ in the $4.6.12$ structure. These ratios are used in photonic band calculations.

| TABLE I. Minimal fundamental triangle type denoted by one of the single junction class (SJC) and their limiting cases [Figs. 4(d)–4(f) and 4(j)–4(m)] and resulting Archimedean tilings. |
|-----------------|-----------------|-----------------|
| SJC             |       | Tiling |
| (d)             | $6^3$ | (3.6.3.6) | (g) |
| (e)             | 4.6.12 | (3.4.6.4) | (h) |
| (f)             | 4.8$^2$ | (4$^3$) | (i) |
| (j)             | 6$^3$ | (3$^3$) |   |
| (k)             | 4.6.12 | (3.6.3.6) |   |
| (l)             | 4.8$^2$ | (4$^4$) | Checkerboard |
| (m)             | 4.6.12 | (3.12$^2$) |   |
| (n)             | 4.8$^2$ | (4.8$^2$) | St. Andrew’s cross |
denoted by

\[ \text{pose the problem. Notice the unit cell of the} \]

A calculated in this paper is

\[ \text{quirements for simplicity:} \]

\[ \text{on triangles. Furthermore, we assume three additional re-} \]

\[ \text{inside them,} \]

\[ \text{three parameters} x, y, \text{and} z \text{are known. Using these three relations, we can determine} \]

\[ \text{A octagon must be} 6:8. \]

\[ \text{The ratios of} A, B, \text{and} C = 9:7:14 \text{or} 9:7:16. \]

\[ \text{The two-} \]

\[ \text{structure is a nontrivial} \]

\[ \text{plane group is} p4gm: \]

\[ \text{The center of a square has} C_4 \text{ symmetry inside them, (2) two types of} B \text{ rectangles are of the same} \]

\[ \text{shape, and (3) all domain boundaries are straight lines. Therefore, we impose the} \]

\[ \text{symmetry on squares and the mirror symmetry on triangles. Furthermore, we assume three additional re-} \]

\[ \text{quirements for simplicity:} (1) \text{Squares have} C_4 \text{ symmetry inside them, (2) two types of} B \text{ rectangles are of the same} \]

\[ \text{shape, and (3) all domain boundaries are straight lines. Consequently, the positions of domains can be assigned by only} \]

\[ \text{three parameters} x, y, \text{and} z \text{ as shown in Fig. 6.} \]

\[ \text{Physically, the density of junctions of molecules on all} \]

\[ \text{vertices seems to be the same.} \]

\[ \text{A:B:C}=9:7:14 \text{ or} 9:7:16. \]

\[ \text{The area ratio we have calculated in this paper is} \]

\[ \text{IV. METHOD OF PHOTONIC BAND CALCULATION} \]

\[ \text{In this section, we outline the calculation method of pho-} \]

\[ \text{tonic band structures for two-dimensional crystalline struc-} \]

\[ \text{tures and show the accuracy of our computation. We employ} \]

\[ \text{the plane-wave expansion method.} \]

\[ \text{A. Plane-wave expansion method} \]

\[ \text{Since the dielectric structure is uniform in the} z \text{ direc-} \]

\[ \text{tion, the relative dielectric constant} \]

\[ \varepsilon(x+a)=\varepsilon(x). \]

\[ \text{The reciprocal lattice vectors} b_1 \text{ and} b_2 \text{ are de-} \]

\[ a_i \cdot b_j = 2 \pi \delta_{ij} , \]

where \( \delta_{ij} \) is the Kronecker delta symbol.

\[ \text{To solve the Maxwell equations, we expand} 1/\varepsilon \text{ by the} \]

\[ 1/\varepsilon = \sum G \kappa(G) e^{iG \cdot x}, \]

where the sum is taken over every reciprocal lattice vector \( G \), which is a linear combination of \( b_1 \) and \( b_2 \):

\[ G = l_1 b_1 + l_2 b_2, \]

where \( l_1 \) and \( l_2 \) are integers. The Fourier coefficient is expressed by

\[ \kappa(G) = 1/ S_0 \int d\mathbf{x} \frac{1}{\varepsilon(x)} e^{-iG \cdot x}, \]

where \( S_0 \) denotes the area of the unit cell.

\[ \text{We assume that the magnetic permeability of the photonic} \]

\[ \text{crystals is equal to that in free space,} \]

\[ \mu_0; B(r,t) = \mu_0 H(r,t). \]

\[ \text{Maxwell equations become} \]

\[ \nabla \times \mathbf{E}(r,t) = -\mu_0 \frac{\partial}{\partial t} \mathbf{H}(r,t), \]

\[ \nabla \times \mathbf{H}(r,t) = \varepsilon_0 \varepsilon(r) \frac{\partial}{\partial t} \mathbf{D}(r,t), \]

where \( \mathbf{E} \) and \( \mathbf{H} \) are the electric and magnetic fields, and \( \mathbf{D} \) and \( \mathbf{B} \) are the displacement and magnetic induction fields, respectively. \( \varepsilon_0 \) is the dielectric constant in free space.

\[ \text{The electromagnetic waves travel in the} x-y \text{ plane; thus,} \]

\[ \mathbf{H} \text{ and} \mathbf{E} \text{ are independent of} z \text{ in the equations. The equations} \]

\[ \text{are then decoupled into two independent sets of equations.} \]

\[ \text{The first set is} \]

\[ \frac{\partial E_z}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t}, \]

\[ \frac{\partial E_z}{\partial x} = \mu_0 \frac{\partial H_z}{\partial t}, \]

\[ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = \varepsilon_0 \varepsilon(x) \frac{\partial E_z}{\partial t}, \]

and the second set is

\[ \frac{\partial H_z}{\partial y} = \varepsilon_0 \varepsilon(x) \frac{\partial E_z}{\partial t}, \]

\[ \frac{\partial H_z}{\partial x} = -\varepsilon_0 \varepsilon(x) \frac{\partial E_z}{\partial t}, \]

\[ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\mu_0 \frac{\partial H_z}{\partial t}. \]

The first set is called \( E \) polarization, where \( \mathbf{E} \) is parallel to the \( z \) axis and \( \mathbf{H} \) is in the \( x-y \) plane. The second set, on the other hand, is called \( H \) polarization, where \( \mathbf{H} \) is parallel to the \( z \) axis and \( \mathbf{E} \) is in the \( x-y \) plane.

\[ \text{We derive first the eigenvalue equation of} E \text{ polarization. From the first set [Eqs. (8)-(10)], we obtain the wave equation in terms of} E_z, \]
where

\[ E_0(x) = u(x)e^{i k_x}, \]

and

\[ u(x + a_l) = u(x) \] \hspace{1cm} \text{(17)}

and \( E_0(x) \) can thus be represented by

\[ E_0(x) = \sum_G \psi_G e^{i (k + G) \cdot x}. \] \hspace{1cm} \text{(18)}

We finally obtain the eigenvalue equation for the expansion coefficients \( \psi_G \) as follows:

\[ \sum_{G'} \kappa(G - G')|k + G|^2 \psi_{G'} = \frac{\omega^2}{c^2} \psi_G. \] \hspace{1cm} \text{(19)}

The coefficient matrix is not Hermitian, and we introduce a new vector defined by

\[ \xi_G = |k + G| \psi_G. \] \hspace{1cm} \text{(20)}

Using this vector, we obtain the eigenvalue equation with a Hermitian matrix:

\[ \sum_{G'} \kappa(G - G')|k + G||k + G'| \xi_{G'} = \frac{\omega^2}{c^2} \xi_G. \] \hspace{1cm} \text{(21)}

In the case of \( H \) polarization, from the second set of equations [Eqs. (11)–(13)], we obtain the wave equation

\[ \left\{ \frac{\partial}{\partial x} \frac{1}{\varepsilon(x)} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\varepsilon(x)} \frac{\partial}{\partial y} \right\} H_z = \frac{\omega^2}{c^2} H_z, \] \hspace{1cm} \text{(22)}

and again we seek the solution of the form \( H_z(x,t) = H_0(x)e^{-i \omega t} \). Then, we obtain

\[ -\left\{ \frac{\partial}{\partial x} \frac{1}{\varepsilon(x)} \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{1}{\varepsilon(x)} \frac{\partial}{\partial y} \right\} H_0(x) = \frac{\omega^2}{c^2} H_0(x), \] \hspace{1cm} \text{(23)}

where \( H_0(x) \) is the eigenfunction of the wave equation.

The Bloch theorem allows us to express \( H_0(x) \) as
The eigenvalue equation with a Hermitian matrix in terms of PBGs where the gaps of both $E$ of $H$ polarization are different. We seek complete photonic band structures of complex patterns is nontrivial and requires a certain care. To check the convergence of eigenvalues and, consequently, to show the accuracy of our results, we first illustrate how Fourier series expansions reproduce the most complicated structure (3\textsuperscript{2}.4.3.4), and second, we show how a band gap converges with changing the number of waves in the expansion in Eq. (3). We select (3\textsuperscript{2}.4.3.4) because its lattice constant is larger than those of SJC patterns in the unit of edge length of polygons.

\begin{equation}
H_0(x) = \sum_G \phi_G e^{i(k+G)\cdot x}.
\end{equation}

The eigenvalue equation with a Hermitian matrix in terms of $\phi_G$, which we should solve numerically, is

\begin{equation}
\sum_{G'} \kappa(G-G')(k+G)\cdot(k+G')\phi_{G'} = \frac{\omega^2}{c^2}\phi_G.
\end{equation}

Since Eqs. (19) and (25) are different, the band structures of $E$ and $H$ polarizations are different. We seek complete PBGs where the gaps of both $E$ and $H$ polarizations overlap.

### B. Accuracy of computation

It is known that the computation of accurate two-dimensional photonic band structures of complex patterns is nontrivial and requires a certain care. To check the convergence of eigenvalues and, consequently, to show the accuracy of our results, we first illustrate how Fourier series expansions reproduce the most complicated structure (3\textsuperscript{2}.4.3.4), and second, we show how a band gap converges with changing the number of waves in the expansion in Eq. (3). We select (3\textsuperscript{2}.4.3.4) because its lattice constant is larger than those of SJC patterns in the unit of edge length of polygons.

Suppose $-l \leq l_i \leq l$, where $l_i$ is $l_1$ or $l_2$ in Eq. (4). The number of plane waves is $(2l+1)^2$. We have checked the Fourier coefficient in Eq. (5) by reproducing $1/e(x)$. The integral Eq. (5) is computed through 256 mesh points in each direction. Figure 7 shows plots of the Fourier series expansions for a (3\textsuperscript{2}.4.3.4) structure with dielectric constants $\varepsilon_A = 13$, $\varepsilon_B = 13$, and $\varepsilon_C = 1$. In Fig. 7(a), there is a substantial amount of large islands. Consequently, it implies that the accuracy of band structures using 529 waves is dubious. On the contrary, the higher expansion by 3721 waves reproduces the complex pattern as shown in Fig. 7(b).

Figure 8 shows the largest band gap of $E$ polarization as a function of $1/l$ for the same structure (Fig. 17). Upper points correspond to the bottom of the air band, and lower points correspond to the top of the dielectric band. From right to left, these points correspond to 529, 1089, 2025, 2601, 3721, and 5625 plane waves, respectively.

Broken lines are least-squares fits to four points with larger $l$. The extrapolation to values with $1/l = 0$ demonstrates that the band gap remains. As far as we have done, the topology of generic band structures (not in detail) for 529 and 3721 waves is the same one for all patterns. To obtain more accurate values as possible, we have chosen 3721 plane waves. Below all results, both figures and values are obtained using 3721 waves.
PHOTONIC BAND STRUCTURE CALCULATIONS OF TWO-...
have a better choice of the air-filling factor. The average of dielectric constant and the fluctuation are

$$\bar{e} = f e_{\text{air}} + (1 - f) e_{\text{die}},$$

$$\left(\Delta e\right)^2 = \frac{f e_{\text{air}}^2 + (1 - f) e_{\text{die}}^2}{\left(f e_{\text{air}} + (1 - f) e_{\text{die}}\right)^2} - 1,$$

where \(f\) is the air-filling factor. For a given choice of dielectric contrasts, the fluctuation has a maximum for a filling factor as follows:

$$f_{\text{max}} = \frac{e_{\text{die}}}{e_{\text{air}} + e_{\text{die}}}.$$

It has been argued that this maximum gives the strongest scattering of electromagnetic waves. In our case of \(e_{\text{air}} = 1\) and \(e_{\text{die}} = 13\), the better choice of the air-filling factor is about 93%. In practice, the higher the air-filling factor is, the larger the PBG is. This argument may be useful when one designs the ratio of components of a block copolymer before exact band calculations are at hand. It should be mentioned that the argument is independent of structures, dimensions, and the degree of ordering, and that it is accordingly by no means accurate.

C. (6^3) structure

We present photonic band structures for a (6^3) tiling with an area ratio of \(A:B:C = 1:1:1\). The plane group is \(p6mm\). Dielectric constants are \(e_A = 13\), \(e_B = 1\), and \(e_C = 1\) in the case of dielectric cylinders (Fig. 9), and \(e_A = 1\), \(e_B = 13\), and \(e_C = 13\) in the case of air cylinders (Fig. 10). Dielectric cylinders have broad PBGs in \(E\) polarization, and narrow ones in \(H\) polarization, but their positions are different. Thus, there is no complete PBG. Air cylinders have PBGs in \(H\) polarization, and very narrow one in \(E\) polarization.

It is known that a triangular lattice composed of air cylinders with larger fractions of air has the complete PBG. However, the air-filling factor of the (6^3) is 33%, which is too small to obtain the complete PBG.

D. (4.8^2) structure

We present photonic band structures for a (4.8^2) tiling with an area ratio of \(A:B:C = 2:1:2\). The plane group is \(p4mm\). There are two ways to construct dielectric cylinders: \(e_A = 13\), \(e_B = 1\), \(e_C = 1\) (Fig. 11) and \(e_A = 1\), \(e_B = 13\), \(e_C = 13\). The latter has large PBGs in \(E\) polarization and a narrow PBG in \(H\) polarization. These band gaps are very close but do not overlap. The latter has PBGs only in \(E\) polarization. Similarly, there are two ways to construct air cylinders: \(e_A = 1\), \(e_B = 13\), \(e_C = 13\) (Fig. 12) and \(e_A = 13\), \(e_B = 1\), \(e_C = 13\). Only in Fig. 12 we see PBGs in \(H\) polarization. In any case, there is no complete PBG in the (4.8^2) tiling. We think that these results are essentially the same as the results of about 40% or 20% cylinders on square lattices.

E. (4.6.12) structure

We present photonic band structures for a (4.6.12) tiling with an area ratio of \(A:B:C = 1:1:2\). The plane group is \(p6mm\). There are three ways to construct dielectric cylinders: \(e_A = 13\), \(e_B = 1\), and \(e_C = 1\) (Fig. 13), \(e_A = 13\), \(e_B = 1\), and \(e_C = 1\), and \(e_A = 13\), \(e_B = 13\), and \(e_C = 13\). All cases have large PBGs in waves of \(E\) polarization, and \(A\) and \(C\) dielectric cylinders have PBGs in waves of \(H\) polarization. More importantly, in the case of \(A\) dielectric cylinders, there are complete PBGs with 75% air-filling factor. The structure looks like that composed of \(C\) air cylinders. Let \(\omega\) be the frequency at the middle of a band gap and \(\Delta\omega\) be the gap frequency width. The scale independent gap-midgap ratios \(\Delta\omega/\omega\) are 9.3 × 10^{-2} and 3.0 × 10^{-2}.

On the other hand, there are three ways to construct air cylinders: \(e_A = 1\), \(e_B = 13\), \(e_C = 13\), \(e_A = 13\), \(e_B = 13\), \(e_C = 13\); and \(e_A = 13\), \(e_B = 13\), \(e_C = 1\) (Fig. 14). All cases have large PBGs in waves of \(H\) polarization. \(B\) and \(C\) air cylinders have \(E\) PBGs as well, but there is no complete PBGs.

The \(C\) air cylinders form the triangular lattice. However, like the case of (6^3), the fraction of the \(C\) air cylinders cannot be far beyond 0.5, and the air cylinders do not show complete PBGs (Fig. 14). Increasing the fraction of \(C\) component, which is necessary for complete PBGs, will make the structure undergo a microphase separation to another structure. Instead of enlarging \(C\) air cylinders, \(A\) dielectric

FIG. 17. (Color online) Photonic band structure for (3^2, 4, 3, 4) with dielectric constants \(e_A = 13\), \(e_B = 1\), and \(e_C = 1\): \(E\) polarization (solid line) and \(H\) polarization (broken line). The gap-midgap ratio \(\Delta\omega/\omega\) is 5.4 × 10^{-2}.

FIG. 18. Electric field of near \(M\)-point \(E\) polarization mode in the dielectric band for (3^2, 4, 3, 4) with dielectric constants \(e_A = 13\), \(e_B = 13\), and \(e_C = 1\). Different shades indicate the plus and minus of the field. The field mainly concentrates into \(A\) regions, leading to the reduction of energy.
cylinders forming the honeycomb lattice is promising as shown in Fig. 13. The fact that the honeycomb lattice has the complete PBG has been known. In this case, since B regions do not have the high dielectric constant, a gap of H polarization around 0.3 seen in Fig. 14 disappears.

We present photonic band structures for a (3\(^2\), 4.3.4) tiling with an area ratio of A:B:C = 9:7:14. The plane group is \(p4gm\), which is a nonsymmorphic group leading to the sticking together of bands. There are three ways to construct dielectric cylinders: \(\varepsilon_A = 13, \varepsilon_B = 1, \varepsilon_C = 1\) (Fig. 15); \(\varepsilon_A = 1, \varepsilon_B = 13, \varepsilon_C = 1\); and \(\varepsilon_A = 1, \varepsilon_B = 1, \varepsilon_C = 13\) (Fig. 16). All cases have large PBGs in waves of E polarization and H polarization. More importantly, in the case of A and C dielectric cylinders, they have complete PBGs. The centers of these PBGs are at almost the same position. The gap-midgap ratios \((\Delta\omega/\omega)\) for A dielectric cylinders are \(2.2 \times 10^{-2}\) and \(3.1 \times 10^{-2}\), and that of C ones is \(5.1 \times 10^{-2}\).

On the other hand, there are three ways to construct air cylinders: \(\varepsilon_A = 1, \varepsilon_B = 13, \varepsilon_C = 13\); \(\varepsilon_A = 13, \varepsilon_B = 1, \varepsilon_C = 13\); and \(\varepsilon_A = 13, \varepsilon_B = 13, \varepsilon_C = 1\) (Fig. 17). All cases have PBGs in waves of H polarization. In Fig. 17, there are again complete PBGs. The center of the PBG in this case is at almost half compared to the case of dielectric cylinders. It is quite interesting that both C dielectric and C air cylinders of the same structure have complete PBGs. The gap-midgap ratio \((\Delta\omega/\omega)\) for C air cylinders is \(5.4 \times 10^{-2}\). When we increase air-filling factor up to 50\% \((A:B:C = 9:7:16)\), the gap-midgap ratio becomes \(6.6 \times 10^{-2}\).

We show the electric fields with E polarization for C air cylinders with respect to the gap \((\omega a/2\pi c < 0.5)\) in Fig. 17.

### Table II. Summary of the photonic band structures. Structures are dielectric cylinders (D) with \(\varepsilon_a = 13\) and \(\varepsilon_b = \varepsilon_c = 1\) or air cylinders (A) with \(\varepsilon_a = 1\) and \(\varepsilon_b = \varepsilon_c = 13\), where \(\alpha, \beta,\) and \(\gamma\) stand for combinations of A, B, and C. In the case of \((3^2, 4.3.4)\), C* implies data for \(A9B7C16\). In the fourth column, E or H means its polarization has the largest PBG. In the second to the last column, when complete PBGs exist, the gap-midgap ratios of the complete PBGs \((\Delta\omega/\omega)\) are listed, where \(\omega\) is the eigenangular frequency at the middle of a gap and \(\Delta\omega\) is the gap width.

<table>
<thead>
<tr>
<th>Tiling</th>
<th>Cylinder</th>
<th>(\alpha)</th>
<th>Polarization</th>
<th>(\Delta\omega/\omega)</th>
<th>Fig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(6(^2))</td>
<td>D</td>
<td>A</td>
<td>E</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>(p6mm)</td>
<td>A</td>
<td>A</td>
<td>H</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>(4.8(^2))</td>
<td>D</td>
<td>A</td>
<td>E</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>(p4mm)</td>
<td>B</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>H</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(4.6.12)</td>
<td>D</td>
<td>A</td>
<td>E</td>
<td>0.093, 0.030</td>
<td>13</td>
</tr>
<tr>
<td>(p6mm)</td>
<td>B</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>H</td>
<td></td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(3(^2), 4.3.4)</td>
<td>D</td>
<td>A</td>
<td>E</td>
<td>0.022, 0.031</td>
<td>15</td>
</tr>
<tr>
<td>(p4gm)</td>
<td>B</td>
<td>E</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>E</td>
<td></td>
<td>0.051</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>A</td>
<td>A</td>
<td>H</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>H</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>H</td>
<td></td>
<td>0.054</td>
<td>17</td>
</tr>
<tr>
<td></td>
<td>C*</td>
<td>H</td>
<td></td>
<td>0.066</td>
<td></td>
</tr>
</tbody>
</table>
An eigenfunction of the dielectric band at a point near the M point a little way off to the Γ point is displayed in Fig. 18. The field concentrates mainly into A regions. The eigenfunction of the same point at the bottom of the air band is shown in Fig. 19. The field is mainly concentrated into B regions; however, it invades C air cylinders more and thus increases the displacement energy.

VI. DISCUSSION AND SUMMARY

We have investigated photonic band structures of two-dimensional Archimedean tiling patterns, (4.82), (62), (4.6.12), and (32.4.3.4). Double circles correspond to the cases of complete photonic band gaps. The least-squares fit is estimated from solid or open data points; the slope is 0.19 or 0.46, respectively.

PBGs (double circles) has nothing to do with the dielectric fluctuations. Of course, when a complete band gap exists for a lattice, the ratio correlates with the fluctuation.

FIG. 20. The largest gap-midgap ratio against dielectric fluctuations: E polarization of dielectric cylinders (solid circle) and H polarization of air cylinders (open circle) are plotted for (62), (4.82), (4.6.12), and (32.4.3.4). Double circles correspond to the cases of complete photonic band gaps. The least-squares fit is estimated from solid or open data points; the slope is 0.19 or 0.46, respectively.

An eigenfunction of the dielectric band at a point near the M point a little way off to the Γ point is displayed in Fig. 18. The field is mainly concentrated into A regions. The eigenfunction of the same point at the bottom of the air band is shown in Fig. 19. The field is mainly concentrated into B regions; however, it invades C air cylinders more and thus increases the displacement energy.

In Table II, we summarize the results. We call attention that the dielectric fluctuations defined by Eq. (27) and the gap-midgap ratio of E polarization of dielectric cylinders and that of H polarization of air cylinders, see Fig. 20. This plot displays the gap-midgap ratio of the largest band gap for E polarization of dielectric cylinders (solid circle) and H polarization of air cylinders (open circle). With increasing the dielectric fluctuations, the ratio for both polarizations increases. However, it appears that the existence of complete photonic band gaps is akin to the dodecagonal quasicrystals: our ongoing study will clarify the difference between the band structure of (32.4.3.4) and that of the higher approximant of dodecagonal quasicrystals.

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5 A. Takano, S. Wada, S. Sato, T. Araki, K. Hirahara, T. Kazama, S.
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Single junction class is the abbreviation of single junction point line class in Ref. 2. See Theorem 4.2 in the paper.


T. Dotera, Philos. Mag. (to be published).
