H_∞ Type Problem for Sampled-Data Control Systems—A Solution via Minimum Energy Characterization

Yoshikazu Hayakawa, Shinji Hara, and Yutaka Yamamoto

Abstract—**This paper aims at deriving a solution for H_∞ type problem for sampled-data control systems. The solution is given in terms of an equivalent discrete-time H_∞ problem. The reduction procedure is viewed and characterized from the viewpoint of minimum energy principle and J-unitary transformations.**

I. Introduction

The recent studies of sampled-data systems place strong emphasis on the treatment of built-in intersample behavior, especially the H_∞ control problem for sampled-data control system which has been studied extensively ([7], [4], [17], [11], [8], [16], [14], [12], [1], just to name a few). Except in [16], [14], where a direct solution in terms of Riccati equations has been obtained, most approaches reduce the original problem to a norm-equivalent discrete-time H_∞ control problem.

The present note also follows this line, but intends to give a yet different solution via an intuitive minimum energy principle. The

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Y. Hayakawa is with the Department of Electronic-Mechanical Engineering, Faculty of Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-01, Japan.

S. Hara is with the Department of System Science, Tokyo Institute of Technology, 4259 Nagatsuda, Midoriku, Yokohama-shi, Kanagawa 227, Japan.

Y. Yamamoto is with the Division of Applied Systems Science, Faculty of Engineering, Kyoto University, Kyoto 606-01, Japan.

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problem is stated as follows: We are given a generalized plant

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{1.1}$$

and characterize stabilizing feedback digital controllers $u = \mathcal{H}K\mathcal{S}y$ that satisfy $\|T_{zw}\|_\infty < \gamma$ for a prespecified γ, where $\mathcal{S}$ and $\mathcal{H}$ are sample and hold operations, and T_{zw} denotes the transfer function from w to z when closing the loop with K.

In this note we first observe that the solution to the special case $P_{11} = 0$ is directly obtainable by characterizing the disturbance inputs with minimum energy. The general case can be reduced to this case by the J-unitary transformation introduced by Bamieh and Pearson [1]. We then perform yet another J-unitary transformation to make the reduced "A" matrix to be the same as the originally sampled transition matrix e^{Ah}. This has the advantage that stabilizability is readily seen to be preserved by the whole procedure.

II. Problem Statement

Consider the sampled-data feedback system given by Fig. 1. Let

$$\dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) \tag{2.1}$$

$$z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) \tag{2.2}$$

$$y(t) = C_2 x(t) \tag{2.3}$$

be a realization of $P(s)$, where we assume that P is stabilizable and detectable, $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{m_1}$, $u(t) \in \mathbf{R}^{m_2}$, $z(t) \in \mathbf{R}^{p_1}$, $y(t) \in \mathbf{R}^{p_2}$, and $D_{21} = 0$. Introduce the lifted variables

$$x_d[k] := x(kh). \tag{2.4}$$

$$w_c(k, \theta) := w(t),\ \theta = t - kh,\ kh \le t < (k+1)h,\ k = 1, 2, \cdots \tag{2.5}$$

$$z_c(k, \theta) := z(t),\ \theta = t - kh,\ kh \le t < (k+1)h,\ k = 1, 2, \cdots. \tag{2.6}$$

It is now a standard fact [17], [19], [1] that (2.1)–(2.3) are represented by the time-invariant discrete-time equations as

$$\Sigma_0 : \begin{bmatrix} x_d[k+1] \\ z_c(k, \theta) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d & \mathbf{B}_1 & B_{2d} \\ \mathbf{C}_1 & \mathbf{D}_{11} & \mathbf{D}_{12} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ w_c(k, \cdot) \\ u_d[k] \end{bmatrix} \tag{2.7}$$

where the operators A_d, $\mathbf{B}_1$, B_{2d}, etc., are defined by

$$A_d := e^{Ah} : \mathbf{R}^n \to \mathbf{R}^n \tag{2.8}$$

$$\mathbf{B}_1 w := \int_0^h e^{A(h-\sigma)} B_1 w(\sigma) d\sigma : L_2^{m_1}[0, h] \to \mathbf{R}^n \tag{2.9}$$

$$B_{2d} := \int_0^h e^{A(h-\sigma)} B_2 H(\sigma)\, d\sigma : \mathbf{R}^{m_2} \to \mathbf{R}^n \tag{2.10}$$

$$\mathbf{C}_1 := C_1 e^{A\theta} : \mathbf{R}^n \to L_2^{p_1}[0, h] \tag{2.11}$$

$$C_{2d} := C_2 : \mathbf{R}^n \to \mathbf{R}^{p_2} \tag{2.12}$$

$$\mathbf{D}_{11} w(\cdot) := D_{11} w(\cdot) + \int_0^\theta C_1 e^{A(\theta-\sigma)} B_1 w(\sigma)\, d\sigma : L_2^{m_1}[0, h] \to L_2^{p_1}[0, h] \tag{2.13}$$

$$\mathbf{D}_{12} := D_{12} H(\theta) + \int_0^\theta C_1 e^{A(\theta-\sigma)} B_2 H(\sigma) d\sigma : \mathbf{R}^{m_2} \to L_2^{p_1}[0, h]. \tag{2.14}$$

Fix a digital controller $K(z)$. Let T_{zw} be the transfer function from w to z when closing the loop with K, and $\|T_{zw}\|_\infty$ its H_∞ norm in

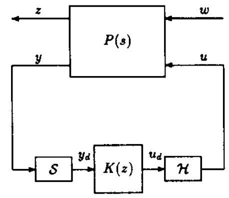

Fig. 1. Sampled-data feedback system.

the sense of discrete-time system as described above. Let

$$J(\Sigma_0, K) := \|T_{zw}\|_\infty = \sup_{w \in L_2[0, \infty)} \frac{\|z\|_{L_2}}{\|w\|_{L_2}}. \tag{2.15}$$

The objective is to derive a finite-dimensional system Σ' so that $J(\Sigma_0, K) < \gamma$ if and only if $J(\Sigma', K) < \gamma$. When it is obvious which system Σ is under consideration, we write $J(K)$ in place of $J(\Sigma, K)$.

III. Minimum Energy Characterization

In this section, we consider the special case where the direct transmission P_{11} from w to z is identically zero. Although highly specialized, this case is of interest on its own for the following two reasons: 1) as an important special case, the robust stability problem can be studied in this setting (see Section IV), and 2) based on the result of [1], the general case can be, in a sense, reduced to this special case (see Section V).

The main result is stated as follows.

Theorem 3.1: Suppose $P_{11} = 0$ in Σ_0, and let * denote the adjoint. Then the following three induced-norm optimization problems are equivalent.

- $J(\Sigma_0, K) < \gamma$

$$\Sigma_0 : \begin{bmatrix} x_d[k+1] \\ z_c(k, \theta) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d & \mathbf{B}_1 & B_{2d} \\ \mathbf{C}_1 & 0 & \mathbf{D}_{12} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ w_c(k, \cdot) \\ u_d[k] \end{bmatrix} \tag{3.1}$$

- $J(\Sigma_2, K) < \gamma$

$$\Sigma_2 : \begin{bmatrix} x_d[k+1] \\ z_c(k, \theta) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d & \tilde{B}_{1d} & B_{2d} \\ \mathbf{C}_1 & 0 & \mathbf{D}_{12} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \tilde{w}_d[k] \\ u_d[k] \end{bmatrix} \tag{3.2}$$

where $\tilde{B}_{1d}$ is a constant matrix defined as

$$\tilde{B}_{1d} := (\mathbf{B}_1 \mathbf{B}_1^*)^{1/2}. \tag{3.3}$$

- $J(\Sigma_4, K) < \gamma$

$$\Sigma_4 : \begin{bmatrix} x_d[k+1] \\ \hat{z}_d[k] \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d & \tilde{B}_{1d} & B_{2d} \\ \hat{C}_{1d} & 0 & \hat{D}_{12d} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \tilde{w}_d[k] \\ u_d[k] \end{bmatrix} \tag{3.4}$$

where $\hat{C}_{1d}$ and $\hat{D}_{12d}$ are constant matrices defined as

$$[\hat{C}_{1d} \quad \hat{D}_{12d}] := \left\{ \begin{pmatrix} \mathbf{C}_1^* \\ \mathbf{D}_{12}^* \end{pmatrix} (\mathbf{C}_1 \mathbf{D}_{12}) \right\}^{1/2}. \tag{3.5}$$

The reduction from Σ_0 to $\Sigma_4$in the above theorem consists of two steps.

a) Norm Preserving Discretization on Disturbance Input: This step replaces the input term $\mathbf{B}_1 w_c(k, \cdot)$ by a discrete-time term $\tilde{B}_{1d}\tilde{w}_d[k]$, so that

$$J(\Sigma_0, K) < \gamma \Leftrightarrow J(\Sigma_2, K) < \gamma.$$

b) Norm Preserving Discretization on Controlled Output: This step introduces a discrete time output $\hat{z}_d[k]$, so that

$$J(\Sigma_2, K) < \gamma \Leftrightarrow J(\Sigma_4, K) < \gamma.$$

Step a) is carried out by a particularly simple procedure based on the following idea [10].

First we observe that $\mathbf{D}_{11} = 0$, so that z_c is affected by w_c only through feedback and hence by state x_d and control input u_d. Therefore, if the same $u_d[k]$ is applied at time step k, then the controlled output z_c is determined by $x_d[k]$. This means that the worst disturbance input w_c that gives rise to the L_2-induced norm $J(\Sigma_0, K)$ may be characterized as the one having the minimum L_2-norm among those yielding the same state $x_d[k]$. This simple minimum energy principle is the key to the reduction process here; indeed, such an element can be characterized easily by linear quadratic (LQ) technique.

Step b) is very simple, just to define the discrete-time output $\hat{z}_d[k]$ satisfying

$$\hat{z}_d[k]^T \hat{z}_d[k] = \int_0^h z_c(k, \sigma)^T z_c(k, \sigma)\, d\sigma. \tag{3.6}$$

Remark 3.2: Let G be a linear time-invariant, stable, finite-dimensional, continuous-time system. Then it is easy to see the following: 1) $G\mathcal{H}$ is a mapping from l_2 into L_2, and 2) when G is strictly proper, $\mathcal{S}G$ is a mapping from L_2 into l_2. Under these observation, Chen and Francis [4] derived formulas to calculate l_2/L_2 induced norm of $G\mathcal{H}$ and L_2/l_2 induced norm of $\mathcal{S}G$ via operator theory. Theorem 3.1 gives us quite the same results as Chen and Francis [4] straightforwardly in terms of state-space formulas. In fact, $G\mathcal{H}$ can be reduced to a finite dimensional discrete-time system by the norm preserving discretization on z_c as shown in Step b), and $\mathcal{S}G$ can be done through applying Step a).

Now we prove Theorem 3.1 to show Steps a) and b) in detail.

Step a): $J(\Sigma_0, K) < \gamma \Leftrightarrow J(\Sigma_2, K) < \gamma$.

Consider the original system Σ_0 and introduce the following equivalence relation $\mathcal{R}$ in the disturbance input space $L_2^{m_1}[0, \infty)$.

Definition 3.3: Let w_c and w_c' be in $L_2^{m_1}[0, \infty)$. Define $w_c \mathcal{R} w_c'$ if $x_d[k] = x_d'[k]$ for all k and u_d, where x_d and x_d' are the state variables driven by w_c and w_c', respectively.

It is trivial that the relation $w_c \mathcal{R} w_c'$ holds if and only if

$$\mathbf{B}_1 w_c(k, \cdot) = \mathbf{B}_1 w_c'(k, \cdot) \text{ for all } k. \tag{3.7}$$

Now given any w_c, we want to find a $w_c^\bullet$ with minimum L_2 norm in the same equivalence class. This is motivated by the following (straightforward but quite important) minimum energy principle.

Lemma 3.4 (Minimum Energy Principle): Let $L_2^{m_1}[0, \infty)/\mathcal{R}$ be the quotient space modulo the equivalence class defined above, and let

$$\mathcal{M} := \{w_c^\bullet \in L_2^{m_1}[0, \infty) \colon w_c^\bullet \mathcal{R} w_c \Rightarrow \|w_c^\bullet\| \le \|w_c\|\}$$

i.e., $\mathcal{M}$ is the set of representatives with minimum energy. Then

$$J(\Sigma_0, K) = \sup_{w_c^\bullet \in \mathcal{M}} \frac{\|z_c\|_{L_2}}{\|w_c^\bullet\|_{L_2}}. \tag{3.8}$$

Proof: Recall that

$$J(\Sigma_0, K) = \sup_{w_c \in L_2^{m_1}[0, \infty)} \frac{\|z_c\|_{L_2}}{\|w_c\|_{L_2}}. \tag{3.9}$$

Clearly, the right-hand side of (3.8) is less than or equal to that of (3.9). But if w_c and $w_c^\bullet$ belong to the same equivalence class, then the numerators of these two are the same by the very definition of our equivalence relation $\mathcal{R}$, so that the right-hand side of (3.9) is less than or equal to that of (3.8). Hence they are identical. □

This lemma asserts that to characterize the L_2-induced norm, we can confine our attention to the minimum energy elements $w^\bullet$. The next lemma gives a characterization for such $w^\bullet$ by using LQ technique [2].

Lemma 3.5: For any $w_c \in L_2^{m_1}[0, \infty)$, $w_c^\bullet$ that gives the minimum norm in the same equivalence class is given by

$$w_c^\bullet(k, \theta) = \mathbf{B}_1^* W_0^\dagger \mathbf{B}_1 w_c(k, \cdot) \text{ for all } k \tag{3.10}$$

where $W_0^\dagger$ is the pseudo-inverse of

$$W_0 := \mathbf{B}_1 \mathbf{B}_1^*. \tag{3.11}$$

Remark 3.6: From (3.11), it is easy to see that W_0 is nonsingular if and only if (A, B_1) is controllable. Since W_0 is a symmetric and positive semidefinite matrix, there exists an orthogonal matrix $Q \in \mathbf{R}^n$ such that

$$QW_0Q^T = \operatorname{diag}[w_1, \cdots, w_{m_{d1}}, 0, \cdots, 0]$$

where $w_i > 0$ for $i = 1, \cdots, m_{d1}$ and $m_{d1} := \operatorname{rank} W_0$, i.e., m_{d1} is equal to the dimension of controllable subspace of (A, B_1). Therefore, $W_0^\dagger$ can be given by

$$W_0^\dagger = Q^T \operatorname{diag}\left[\frac{1}{w_1}, \cdots, \frac{1}{w_{m_{d1}}}, 0, \cdots, 0\right] Q.$$

Hereafter we will denote $(W_0^{\dagger\frac{1}{2}})$ by $W_0^{\dagger\frac{1}{2}}$.

Define

$$\mathbf{B}_0 := W_0^{\dagger\frac{1}{2}} \mathbf{B}_1, \qquad \tilde{w}_d[k] := \mathbf{B}_0 w_c(k, \cdot). \tag{3.12}$$

Note that $\tilde{w}_d[k]$ is a finite-dimensional vector in $\mathbf{R}^{m_{d1}}$, but not in $\mathbf{R}^n$. In fact

$$\begin{aligned} Q\tilde{w}_d[k] &= Q\mathbf{B}_0 w_c(k, \cdot) \\ &= \operatorname{diag}\left[\frac{1}{\sqrt{w_1}}, \cdots, \frac{1}{\sqrt{w_{m_{d1}}}}, 0, \cdots, 0\right] Q\mathbf{B}_1 w_c(k, \cdot) \end{aligned}$$

means that $Q_2 \tilde{w}_d[k] = 0$ with $Q_2 \in \mathbf{R}^{(n-m_{d1})\times n}$ being the lower submatrix of Q. Then we have

$$w_c^\bullet(k, \theta) = \mathbf{B}_0^* \mathbf{B}_0 w_c(k, \cdot) = \mathbf{B}_0^* \tilde{w}_d[k] = \mathbf{B}_1^* W_0^{\dagger\frac{1}{2}} \tilde{w}_d[k]$$

$$\begin{aligned} \|w_c^\bullet(k, \cdot)\|_{L_2[0, h]}^2 &= \tilde{w}_d^T[k] W_0^{\dagger\frac{1}{2}} W_0 W_0^{\dagger\frac{1}{2}} \tilde{w}_d[k] \\ &= (Q\tilde{w}_d[k])^T \operatorname{diag}[I_{m_{d1}}, 0] Q\tilde{w}_d[k] \\ &= \tilde{w}_d^T[k] Q^T Q \tilde{w}_d[k] = \|\tilde{w}_d[k]\|^2 \end{aligned}$$

and therefore

$$\|w_c^\bullet\|_{L_2} = \|\tilde{w}_d\|_{l_2} \tag{3.13}$$

$$\begin{aligned} \mathbf{B}_1 w_c^\bullet(k, \cdot) &= \mathbf{B}_1 \mathbf{B}_1^* W_0^{\dagger\frac{1}{2}} \tilde{w}_d[k] \\ &= W_0^{1/2} \tilde{w}_d[k]. \end{aligned} \tag{3.14}$$

Hence the correspondence

$$L_2^{m_1}[0, \infty) \ni w_c^\bullet \to \tilde{w}_d \in l_2^{m_{d1}}[0, \infty)$$

is norm-preserving, and by (3.14) we can replace the input operator by $W_0^{1/2}$. This clearly completes the proof of the reduction process Step a).

Step b): $J(\Sigma_2, K) < \gamma \Leftrightarrow J(\Sigma_4, K) < \gamma$.

The relation (3.6), i.e., $\|\hat{z}_c(k, \cdot)\|_{L_2[0, h]} = \|\hat{z}_d[k]\|$ directly leads to Step b), where

$$\|\hat{z}_c(k, \cdot)\|^2_{L_2[0, h]} = [x_d^T[k], u_d^T[k]]\begin{bmatrix}\mathbf{C}_1^*\\ \mathbf{D}_{12}^*\end{bmatrix}[\mathbf{C}, \mathbf{D}_{12}]\begin{bmatrix}x_d[k]\\ u_d[k]\end{bmatrix} \tag{3.15}$$

$$\|\hat{z}_d[k]\|^2 = [x_d^T[k], \hat{u}_d^T[k]]\begin{bmatrix}\hat{C}_{1d}^T\\ \hat{D}_{12d}^T\end{bmatrix}[\hat{C}_{1d}, \hat{D}_{12d}]\begin{bmatrix}x_d[k]\\ u_d[k]\end{bmatrix}. \tag{3.16}$$

Theorem 3.1 has been proven completely.

The procedure above relies upon the classification of the intersample input functions. Each equivalence class consists of those that yield the same state x and hence the same output function resulting from x. This latter property is valid because the output z is affected only through x. This is where the hypothesis $P_{11} = 0$ becomes effective. Clearly for a given intersample input w_c, its corresponding equivalence class is $\{w_c + \ker \mathbf{B}_1\}$, i.e., it is affine in w. It is easy to recognize that the input $w_c^\bullet$ with minimum energy gives rise to the worst excitation from the H_∞ control problem viewpoint. As seen above, characterization of such inputs is an easy application of the classical minimum norm problem (e.g., [2]). This clearly gives rise to a realization of the quotient space $L^2[0, h]/\ker \mathbf{B}_1$, and once this space is fixed and the induced system with this quotient space as the space of (intersample) inputs, finding a norm-equivalent finite-dimensional system is fairly straightforward. It should be noted that a reasoning similar to the minimum energy principle described here has been independently applied in [18], the proof of Theorem 3.1, where sampled-data H_∞ control problem on finite horizon is solved using game-theoretic methods.

One may also note that in [1] Bamieh and Pearson have derived a solution for this case. Instead of using the notion of minimum energy principle as discussed above, they made use of the orthogonal decomposition $L^2[0, h] \cong \ker \mathbf{B}_1 \dot{+} (\ker \mathbf{B}_1)^\perp$. Of course, it is well known that the two procedures are mathematically equivalent (see, e.g., [13]), but it seems interesting to note that this orthogonal complement admits the quite concrete realization via the intuitive minimum energy principle as given above, because then the H_∞ solution in this case is nothing but an application of the LQ solution.

IV. Robust Stability

Now, as an application of the case $P_{11} = 0$, we present a robust stability problem. The robust stabilization problem for an additive perturbation of plant in sampled-data control system has been considered in [9] and [5]. Here we discuss the robust stabilization problem for an multiplicative perturbation of plant, where a continuous-time plant $G(s)$ belongs to the class

$$\mathcal{G}(G_0, \delta) := \left\{G(s) = (I + \Delta(s))G_0(s) \middle| \begin{array}{l}\Delta(s)\text{: stable.}\\ \sigma_{\max}(\Delta(j\omega)) \le |\delta(j\omega)|\text{: } \forall\omega \in R\end{array}\right\} \tag{4.1}$$

and $\delta(s)$ is a strictly proper outer function. The robust stabilization problem is to find a discrete-time controller $K(z)$ that stabilizes any $G(s) \in \mathcal{G}(G_0, \delta)$.

The same argument as in [9] leads that the robust stabilization problem is just an H_∞ type problem where the generalized plant $P(s)$ is

$$P(s) = \begin{bmatrix}0 & -G_0(s)\\ \delta(s)I & -G_0(s)\end{bmatrix}. \tag{4.2}$$

Note that $P_{11}(s) = 0$.

Suppose that the state-space realizations of $G_0(s)$ and $\delta(s)I$ are given by

$$G_0(s) = \left[\begin{array}{c|c}A & B\\ \hline C & 0\end{array}\right], \qquad \delta(s)I = \left[\begin{array}{c|c}A_\delta & B_\delta\\ \hline C_\delta & 0\end{array}\right].$$

Then the lifted variables allows us to represent $P(s)$ as

$$\begin{bmatrix}x_d[k+1]\\ z_c(k, \theta)\\ y_d[k]\end{bmatrix} = \begin{bmatrix}\begin{bmatrix}A_d & 0\\ 0 & A_{\delta d}\end{bmatrix} & \begin{bmatrix}0\\ \mathbf{B}_\delta\end{bmatrix} & \begin{bmatrix}-B_\delta\\ 0\end{bmatrix}\\ [\mathbf{C} \;\; 0] & 0 & \mathbf{D}\\ [C_d \;\; C_{\delta d}] & 0 & 0\end{bmatrix}\begin{bmatrix}x_d[k]\\ w_c(k, \cdot)\\ u_d[k]\end{bmatrix}$$

where

$$A_d = e^{Ah}, \; A_{\delta d} = e^{A_\delta h}, \; B_d = \int_0^h e^{A(h-\sigma)}BH(\sigma)\,d\sigma,$$

$$C_d = C, \; C_{\delta d} = C_\delta$$

$$\mathbf{B}_\delta w(\cdot) = \int_0^h e^{A_\delta(h-\sigma)}B_1 w(\sigma)\,d\sigma, \; \mathbf{C} = Ce^{A\theta},$$

$$\mathbf{D} = \int_0^\theta Ce^{A\sigma}BH(\sigma)\,d\sigma.$$

By using Theorem 3.1, we immediately obtain the following corollary.

Corollary 4.1: Define a discrete-time generalized plant P_d as

$$P_d(z) = \begin{bmatrix}0 & -\hat{G}_d(z)\\ \tilde{\delta}_d(z) & -G_d(z)\end{bmatrix} = \left[\begin{array}{cc|cc}A_d & 0 & 0 & -B_d\\ 0 & A_{\delta d} & \tilde{B}_{\delta d} & 0\\ \hline \hat{C}_d & 0 & 0 & \hat{D}_d\\ C_d & C_{\delta d} & 0 & 0\end{array}\right] \tag{4.3}$$

where

$$\tilde{B}_{\delta d} = \{\mathbf{B}_\delta \mathbf{B}_\delta^*\}^{1/2}, \; [\hat{C}_d \;\; \hat{D}_d] = \left\{\begin{bmatrix}\mathbf{C}^*\\ \mathbf{D}^*\end{bmatrix}[\mathbf{C} \;\; \mathbf{D}]\right\}^{1/2}. \tag{4.4}$$

If there exists a solution $K(z)$ with $J(P_d, K) < 1$ to the H_∞ problem for discrete-time system, then the sampled-data system with $\mathcal{G}(G_0, \delta)$ is robustly stabilizable via $K(z)$.

Remark 4.2: $G_d(z)$ and $\hat{G}_d(z)$ in the corollary are determined only by the nominal plant $G_0(s)$, while $\tilde{\delta}_d(z)$ is characterized only by the perturbation bound $\delta(s)$.

The corollary shows the sufficient condition of robust stabilization. If the perturbation class is enlarged to include h-periodic perturbations, the condition would be also necessary in the same way as in the case of additive perturbations [15]. See also [6] for further study of robust stability conditions under linear time-invariant perturbations.

V. General Case

In this section, we derive a solution for the general case, which gives four induced-norm optimization problems equivalent to the original one. We here assume, without loss of generality, that $\gamma = 1$, i.e., we derive norm-equivalent problems for $J(K) < 1$. We also assume the induced norm $\|\mathbf{D}_{11}\| < 1$, taken in the sense of an operator in $L_2[0, h]$. This is a necessary condition for the solvability of the original problem, because $\|\mathbf{D}_{11}\| \le \|T_{zw}\|_\infty$ always holds. Indeed, since $\mathbf{D}_{11}$ reflects the effect due to the behavior that cannot be controlled by a sample-holded feedback, its induced norm can never be reduced in the present framework.

The following theorem derives four systems Σ_1–Σ_4 which satisfy the induced norm bound $J(\Sigma, K) < 1$. Note that the internal stability of (Σ, K) will be considered later, that is, in the following theorem, we are interested only in the induced norm bound under input-output stability (without internal stability). See Theorem 5.2 and Remark 5.3.

Theorem 5.1: The following five induced-norm optimization problems are equivalent.

- $J(\Sigma_0, K) < 1$, where Σ_0 is defined by (2.7).
- $J(\Sigma_1, K) < 1$

$$\Sigma_1 : \begin{bmatrix} x_d[k+1] \\ \tilde{z}_c(k, \theta) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} \tilde{A}_d & \tilde{\mathbf{B}}_1 & \tilde{B}_{2d} \\ \tilde{\mathbf{C}}_1 & 0 & \tilde{\mathbf{D}}_{12} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \tilde{w}_c(k, \cdot) \\ u_d[k] \end{bmatrix} \tag{5.1}$$

where

$$\tilde{A}_d = A_d + \mathbf{B}_1 \mathbf{D}_{11}^* (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1} \mathbf{C}_1 \tag{5.2}$$

$$\tilde{\mathbf{B}}_1 = \mathbf{B}_1 (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1/2} \tag{5.3}$$

$$\tilde{B}_{2d} = B_{2d} + \mathbf{B}_1 \mathbf{D}_{11}^* (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1} \mathbf{D}_{12} \tag{5.4}$$

$$\tilde{\mathbf{C}}_1 = (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} \mathbf{C}_1 \tag{5.5}$$

$$\tilde{\mathbf{D}}_{12} = (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} \mathbf{D}_{12}. \tag{5.6}$$

- $J(\Sigma_2, K) < 1$

$$\Sigma_2 : \begin{bmatrix} x_d[k+1] \\ \tilde{z}_c(k, \theta) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} \tilde{A}_d & \tilde{B}_{1d} & \tilde{B}_{2d} \\ \tilde{\mathbf{C}}_1 & 0 & \tilde{\mathbf{D}}_{12} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \tilde{w}_d[k] \\ u_d[k] \end{bmatrix} \tag{5.7}$$

where

$$\begin{aligned} \tilde{B}_{1d} &= (\tilde{\mathbf{B}}_1 \tilde{\mathbf{B}}_1^*)^{1/2} \\ &= [\mathbf{B}_1 (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1} \mathbf{B}_1^*]^{1/2}. \end{aligned} \tag{5.8}$$

- $J(\Sigma_3, K) < 1$

$$\Sigma_3 : \begin{bmatrix} x_d[k+1] \\ \hat{z}_c(k, \theta) \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d & \hat{B}_{1d} & B_{2d} \\ \hat{\mathbf{C}}_1 & \hat{\mathbf{D}}_{11} & \hat{\mathbf{D}}_{12} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \hat{w}_d[k] \\ u_d[k] \end{bmatrix} \tag{5.9}$$

where

$$\hat{B}_{1d} = \tilde{B}_{1d} (\tilde{B}_{1d}^{\dagger} \mathbf{B}_1 \mathbf{B}_1^* \tilde{B}_{1d}^{\dagger})^{1/2} \tag{5.10}$$

$$\hat{\mathbf{C}}_1 = (I - \hat{\mathbf{D}}_{11} \hat{\mathbf{D}}_{11}^*)^{1/2} \tilde{\mathbf{C}}_1 \tag{5.11}$$

$$\hat{\mathbf{D}}_{11} = -(I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} \mathbf{D}_{11} \mathbf{B}_1^* \{\mathbf{B}_1 (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1} \mathbf{B}_1^*\}^{\dagger \frac{1}{2}} \tag{5.12}$$

$$\hat{\mathbf{D}}_{12} = (I - \hat{\mathbf{D}}_{11} \hat{\mathbf{D}}_{11}^*)^{1/2} \tilde{\mathbf{D}}_{12}. \tag{5.13}$$

- $J(\Sigma_4, K) < 1$

$$\Sigma_4 : \begin{bmatrix} x_d[k+1] \\ \hat{z}_d[k] \\ y_d[k] \end{bmatrix} = \begin{bmatrix} A_d & \hat{B}_{1d} & B_{2d} \\ \hat{C}_{1d} & \hat{D}_{11d} & \hat{D}_{12d} \\ C_{2d} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_d[k] \\ \hat{w}_d[k] \\ u_d[k] \end{bmatrix} \tag{5.14}$$

where

$$[\hat{D}_{1d} \quad \hat{D}_{11d} \quad \hat{D}_{12d}] = \left\{ \begin{pmatrix} \hat{\mathbf{C}}_1^* \\ \hat{\mathbf{D}}_{11}^* \\ \hat{\mathbf{D}}_{12}^* \end{pmatrix} \begin{pmatrix} \hat{\mathbf{C}}_1 & \hat{\mathbf{D}}_{11} & \hat{\mathbf{D}}_{12} \end{pmatrix} \right\}^{1/2}. \tag{5.15}$$

Proof: The proof is divided into the following four steps:

Step 1) $J(\Sigma_0, K) < 1 \Leftrightarrow J(\Sigma_1, K) < 1$
Step 2) $J(\Sigma_1, K) < 1 \Leftrightarrow J(\Sigma_2, K) < 1$
Step 3) $J(\Sigma_2, K) < 1 \Leftrightarrow J(\Sigma_3, K) < 1$
Step 4) $J(\Sigma_3, K) < 1 \Leftrightarrow J(\Sigma_4, K) < 1$

The outline of the proof in each step is as follows.

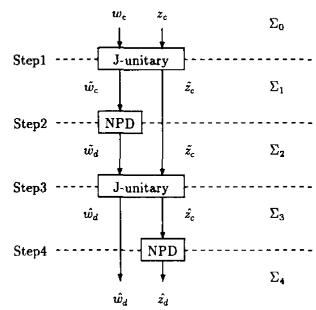

Fig. 2. Schematic diagram of signals.

Step 1): This step is essentially the same as [1]. We introduce a J-unitary transformation

$$\begin{bmatrix} z_c(k, \cdot) \\ w_c(k, \cdot) \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \begin{bmatrix} \tilde{z}_c(k, \cdot) \\ \tilde{w}_c(k, \cdot) \end{bmatrix} \tag{5.16}$$

to get an equivalent system with $\tilde{\mathbf{D}}_{11} = 0$, where

$$\begin{aligned} \mathbf{L} &= \begin{bmatrix} \mathbf{L}_{11} & \mathbf{L}_{12} \\ \mathbf{L}_{21} & \mathbf{L}_{22} \end{bmatrix} \\ &= \begin{bmatrix} (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} & (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} \mathbf{D}_{11} \\ \mathbf{D}_{11}^* (I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} & (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1/2} \end{bmatrix}. \end{aligned} \tag{5.17}$$

Note that $\mathbf{L}$ defined above is well defined under the assumption of $\|\mathbf{D}_{11}\| < 1$, which is a necessary condition for the solvability of the original problem.

Step 2): The influence of the continuous-time disturbance input $\tilde{w}_c(k, \cdot)$ can be replaced by an equivalent discrete-time disturbance input $\tilde{w}_d[k]$ by an approach of minimum energy characterization discussed in Section III, i.e., $\tilde{\mathbf{B}}_1 \tilde{w}_c(k, \cdot)$ can be replaced by $(\tilde{\mathbf{B}}_1 \tilde{\mathbf{B}}_1^*)^{1/2} \tilde{w}_d[k]$.

Step 3): One of the drawback in the system obtained in Step 2) is that it has different $(\tilde{A}_d, \tilde{B}_{2d})$ from the original plant. We introduce another J-unitary transformation

$$\begin{bmatrix} \tilde{z}_c(k, \cdot) \\ \tilde{w}_d[k] \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{L}}_{11} & \tilde{\mathbf{L}}_{12} \\ \tilde{\mathbf{L}}_{21} & \tilde{L}_{22} \end{bmatrix} \begin{bmatrix} \hat{z}_c(k, \cdot) \\ \hat{w}_d[k] \end{bmatrix} \tag{5.18}$$

to get an equivalent system with A_d and B_{2d} in the state transition and the input matrices, respectively, where

$$\tilde{\mathbf{L}} = \begin{bmatrix} \tilde{\mathbf{L}}_{11} & \tilde{\mathbf{L}}_{12} \\ \tilde{\mathbf{L}}_{21} & \tilde{L}_{22} \end{bmatrix} = \begin{bmatrix} (I - \tilde{\mathbf{T}}\tilde{\mathbf{T}}^*)^{-1/2} & (I - \tilde{\mathbf{T}}\tilde{\mathbf{T}}^*)^{-1/2} \tilde{\mathbf{T}} \\ \tilde{\mathbf{T}}^* (I - \tilde{\mathbf{T}}\tilde{\mathbf{T}}^*)^{-1/2} & (I - \tilde{\mathbf{T}}^* \tilde{\mathbf{T}})^{-1/2} \end{bmatrix} \tag{5.19}$$

and

$$\tilde{\mathbf{T}} = -(I - \mathbf{D}_{11}\mathbf{D}_{11}^*)^{-1/2} \mathbf{D}_{11} \mathbf{B}_1^* [\mathbf{B}_1 (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1} \mathbf{B}_1^*]^{\dagger \frac{1}{2}}. \tag{5.20}$$

Note that $\tilde{\mathbf{L}}$ is well defined, since we readily see that $\|\tilde{\mathbf{T}}\| < 1$ holds under the assumption of $\|\mathbf{D}_{11}\| < 1$.

Step 4): This discretization step with respect to the controlled variable z is essentially same as Step b) in Theorem 3.1. □

Summarizing the above, Steps 1) and 3) consist of J-unitary transformations, while Steps 2) and 4) give norm preserving discretizations (NPD). This is illustrated in Fig. 2. At this point, we note that our reduction procedures have the following features:

- All these reductions involve transformations in one intersample period only. Hence dynamics does not enter into the formulas. Moreover, x_d, u_d, and y_d are not changed in each step. Only w and z are subject to changes as

$$(w_c, z_c) \to (\tilde{w}_c, \tilde{z}_c) \to (\tilde{w}_d, \tilde{z}_c) \to (\hat{w}_d, \hat{z}_c) \to (\hat{w}_d, \hat{z}_d).$$

- Since the variables u and y are absent in the reduction process, modifications involving u or y can be handled without any change. For example, the design of controllers with computational delays can be done with the same reduced discrete-time system, because this change involve u only, which is irrelevant to the whole reduction processes.

To complete the derivation of an equivalent discrete-time H_∞ control problem, it remains to see the internal stability of (Σ, K). If A_d and B_{2d} matrices in the equivalent discrete-time problem are different from the original ones of Σ_0, the equivalence of internal stability has to be proven as in [1]. In our final form Σ_4, however, the A_d and B_{2d} matrices are converted back to the original ones of Σ_0 (and hence similar to those of [11], [8]), so that the stabilizability and detectability of (A_d, B_{2d}, C_{2d}) are easily seen to be preserved. Thus we can augment Theorem 5.1 with the internally stabilizing property as follows.

Theorem 5.2: The following statements are equivalent:

1) K internally stabilizes Σ_0 and $J(\Sigma_0, K) < 1$, where Σ_0 is defined by (2.7).
2) K internally stabilizes Σ_4 and $J(\Sigma_4, K) < 1$, where Σ_4 is defined by (5.14).

Remark 5.3: In the above theorem, it is claimed only that the internal stability of (Σ_4, K) is equivalent to one of the original system (Σ_0, K), however, the internal stability of (Σ, K) holds for all the systems Σ_1–Σ_4. In fact, the same technique as in [1] can be used to prove the equivalence of internal stability for Σ_0, Σ_1, and Σ_2. In addition, the equivalence for Σ_0, Σ_3, and Σ_4 is trivial from that they have the same A_d, B_{2d}, and C_{2d} matrices.

As mentioned before, our equivalent discrete-time system Σ_4 has advantage of that the A_d, B_{2d}, C_{2d} matrices are converted back to the original ones of Σ_0. This fact holds even if problem $J(\Sigma_0, K) < \gamma$ is considered instead of $J(\Sigma_0, K) < 1$, i.e., the A_d, B_{2d}, C_{2d} matrices are independent of γ. Therefore the stabilizability and detectability of (A_d, B_{2d}, C_{2d}) are also independent of γ. On the other hand, you need to check the stabilizability of (A_d, B_{2d}) every γ if you use an equivalent discrete-time system with (A_d, B_{2d}) being dependent on γ, e.g., [1].

In addition, whenever the sampling period h is nonpathological, it is well known that (A_d, B_{2d}, C_{2d}) of Σ_4 is stabilizable and detectable if the continuous-time generalized plant P in (2.1)–(2.3) is stabilizable and detectable. Therefore we do not have to check whether Σ_4 is stabilizable and detectable.

VI. Concluding Remarks

We have given an equivalent discrete-time system for the given H_∞ type problem of a sampled-data systems. The reduction process is independent of the (discrete-time) dynamics and the choice of a feedback gain $K(z)$ from y to u. Hence some modifications relevant to the design of K only do not require any change in the final equivalent system; we need only solve the final problem with a different design specification (e.g., controller subject to computational delays). We note that this is also due to the fact that the reduction process classifies the set of disturbance inputs w according to the state $x(kh)$.

Appendix

We now give a state-space form of the equivalent discrete-time system Σ_4 shown in the Section V. For simplicity, we assume here $D_{11} = 0$ and $H(t) = I$ (zero-order hold).

Theorem A.1: Let

$$\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ 0 & I_{m_2} \end{bmatrix} := \exp\left\{ \begin{pmatrix} A & B_2 \\ 0 & 0 \end{pmatrix} h \right\} \tag{A.1}$$

$$\begin{bmatrix} \Psi_{11} & 0 \\ \Psi_{21} & \Psi_{22} \end{bmatrix} := \exp\left\{ \begin{pmatrix} -A^T & 0 \\ B_1 B_1^T & A \end{pmatrix} h \right\} \tag{A.2}$$

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 \\ 0 & 0 & I_{m_2} & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & I_{m_2} \end{bmatrix} := \exp\left\{ \begin{pmatrix} -A^T & -C_1^T C_1 & -C_1^T D_{12} & 0 \\ B_1 B_1^T & A & B_2 & 0 \\ 0 & 0 & 0 & 0 \\ B_2^T & D_{12}^T C_1 & D_{12}^T D_{12} & 0 \end{pmatrix} h \right\} \tag{A.3}$$

and define

$$W_0 := \mathbf{B}_1 \mathbf{B}_1^* = \Psi_{21} \Psi_{11}^{-1} \tag{A.4}$$

$$W := \mathbf{B}_1 (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1} \mathbf{B}_1^* = \Gamma_{21} \Gamma_{11}^{-1} \tag{A.5}$$

$$V_{cc} := \mathbf{C}_1^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{C}_1 = -\Gamma_{11}^{-1} \Gamma_{12} \tag{A.6}$$

$$V_{cd} := \mathbf{C}_1^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{12} = -\Gamma_{11}^{-1} \Gamma_{13} \tag{A.7}$$

$$V_{dd} := \mathbf{D}_{12}^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{12} = \Gamma_{43} - \Gamma_{41} \Gamma_{11}^{-1} \Gamma_{13} \tag{A.8}$$

$$M_1 := \mathbf{C}_1^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{11} \mathbf{B}_1^* = \Gamma_{11}^{-1} - \Phi_{11}^T \tag{A.9}$$

$$M_2 := \mathbf{D}_{12}^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{11} \mathbf{B}_1^* = \Gamma_{41} \Gamma_{11}^{-1} - \Phi_{12}^T \tag{A.10}$$

$$N := W^{\dagger \frac{1}{2}} W_0 W^{\dagger \frac{1}{2}} \tag{A.11}$$

$$M := W^{\dagger \frac{1}{2}} W W^{\dagger \frac{1}{2}}. \tag{A.12}$$

Then a state-space realization of Σ_4 expressed as (5.14) is given by

$$A_d = \Phi_{11}, \quad B_{2d} = \Phi_{12}, \quad C_{2d} = C_2 \tag{A.13}$$

$$\hat{B}_{1d} = W^{\frac{1}{2}} N^{\frac{1}{2}} \tag{A.14}$$

$$[\hat{C}_{1d}, \hat{D}_{11d}, \hat{D}_{12d}] = \left\{ \begin{bmatrix} V_{cc} & 0 & V_{cd} \\ 0 & M & 0 \\ V_{cd}^T & 0 & V_{dd} \end{bmatrix} - \begin{bmatrix} M_1 W^{\dagger \frac{1}{2}} \\ -N^{\frac{1}{2}} \\ M_2 W^{\dagger \frac{1}{2}} \end{bmatrix} \begin{bmatrix} M_1 W^{\dagger \frac{1}{2}} \\ -N^{\frac{1}{2}} \\ M_2 W^{\dagger \frac{1}{2}} \end{bmatrix}^T \right\}^{1/2}. \tag{A.15}$$

Proof: It is obvious that A_d, B_{2d}, C_{2d} and $\hat{B}_{1d}$ are as given above.

The following equations are straightforward to be derived

$$\begin{aligned} \mathbf{D}_{11}^* \mathbf{D}_{11} &= W^{\dagger \frac{1}{2}} \mathbf{B}_1 \mathbf{D}_{11}^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{11} \mathbf{B}_1^* W^{\dagger \frac{1}{2}} \\ &= W^{\dagger \frac{1}{2}} \mathbf{B}_1 \{ (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1} - I \} \mathbf{B}_1^* W^{\dagger \frac{1}{2}} \\ &= W^{\dagger \frac{1}{2}} \mathbf{B}_1 (I - \mathbf{D}_{11}^* \mathbf{D}_{11})^{-1} \mathbf{B}_1^* W^{\dagger \frac{1}{2}} - W^{\dagger \frac{1}{2}} \mathbf{B}_1 \mathbf{B}_1^* W^{\dagger \frac{1}{2}} \end{aligned} \tag{A.16}$$

$$\begin{aligned} \begin{bmatrix} \hat{\mathbf{C}}_1^* \\ \hat{\mathbf{D}}_{12}^* \end{bmatrix} [\hat{\mathbf{C}}_1 \ \hat{\mathbf{D}}_{12}] &= \begin{bmatrix} \mathbf{C}_1^* \\ \mathbf{D}_{12}^* \end{bmatrix} (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} [\mathbf{C}_1 \ \mathbf{D}_{12}] \\ &\quad - \begin{bmatrix} \mathbf{C}_1^* \\ \mathbf{D}_{12}^* \end{bmatrix} (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{11} \mathbf{B}_1^* W^{-1} \mathbf{B}_1 \mathbf{D}_{11}^* \\ &\quad \cdot (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} [\mathbf{C}_1 \ \mathbf{D}_{12}] \end{aligned} \tag{A.17}$$

$$\hat{\mathbf{C}}_1^* \hat{\mathbf{D}}_{11} = \mathbf{C}_1^* (I - \mathbf{D}_{11} \mathbf{D}_{11}^*)^{-1} \mathbf{D}_{11} \mathbf{B}_1^* W^{\dagger \frac{1}{2}} N^{\frac{1}{2}} \tag{A.18}$$

$$\hat{\mathbf{D}}_{11}^{*}\hat{\mathbf{C}}_{1} = (\hat{\mathbf{C}}_{1}^{*}\hat{\mathbf{D}}_{11})^{T} \tag{A.19}$$

$$\hat{\mathbf{D}}_{11}^{*}\hat{\mathbf{D}}_{12} = N^{\frac{1}{2}}W^{\dagger\frac{1}{2}}\mathbf{B}_{1}\mathbf{D}_{11}^{*}(I - \mathbf{D}_{11}\mathbf{D}_{11}^{*})^{-1}\mathbf{D}_{12} \tag{A.20}$$

$$\hat{\mathbf{D}}_{12}^{*}\hat{\mathbf{D}}_{11} = (\hat{\mathbf{D}}_{11}^{*}\hat{\mathbf{D}}_{12})^{T} \tag{A.21}$$

These equations lead to (A.15). The state-space computations of W_0, W, V_{cc}, V_{cd}, V_{dd}, M_1, M_2, N, and M based on three exponentials (A.1), (A.2) and (A.3) can be verified by the same technique as in [1]. □

Remark A.2:

1) We need three exponentiations of sizes $n + m_2$, $2n$, and $2(n + m_2)$, where n and m_2 are the dimensions of the state $x(t)$ and the control input $u(t)$, respectively.
2) If we consider a problem $J(K) < \gamma$ instead of $J(K) < 1$, only $\mathbf{C}_1$ and $\mathbf{D}_{12}$ should be replaced by $\mathbf{C}_1/\gamma$ and $\mathbf{D}_{12}/\gamma$ in the above formulas. Hence, recalculation is required only for Γ in the γ-iteration for the optimization, since Φ and Ψ are independent of γ.

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