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$H_\infty$ Type Problem for Sampled-Data Control Systems—A Solution via Minimum Energy Characterization

Yoshikazu Hayakawa, Shinji Hara, and Yutaka Yamamoto

Abstract—This paper aims at deriving a solution for $H_\infty$ type problem for sampled-data control systems. The solution is given in terms of an equivalent discrete-time $H_\infty$ problem. The reduction procedure is viewed and characterized from the viewpoint of minimum energy principle and $J$-unitary transformations.

I. INTRODUCTION

The recent studies of sampled-data systems place strong emphasis on the treatment of built-in intersample behavior, especially the $H_\infty$ control problem for sampled-data control system which has been studied extensively ([7], [4], [17], [11], [8], [16], [14], [12], [1], just to name a few). Except in [16], [14], where a direct solution in terms of Riccati equations has been obtained, most approaches reduce the original problem to a norm-equivalent discrete-time $H_\infty$ control problem.

The present note also follows this line, but intends to give a yet different solution via an intuitive minimum energy principle. The

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Y. Hayakawa is with the Department of Electrical-Mechanical Engineering, Faculty of Engineering, Nagoya University, Furo-cho, Chikusa-ku, Nagoya 464-01, Japan.
S. Hara is with the Department of System Science, Tokyo Institute of Technology, 4259 Nagatsuda, Midoriga, Yokohama-ku, Kanagawa 227, Japan.
Y. Yamamoto is with the Division of Applied Systems Science, Faculty of Engineering, Kyoto University, Kyoto 606-01, Japan.

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problem is stated as follows: We are given a generalized plant
\[
\begin{pmatrix}
  z \\
  y
\end{pmatrix} =
\begin{bmatrix}
  P_{11} & P_{12} \\
  P_{21} & P_{22}
\end{bmatrix}
\begin{pmatrix}
  w \\
  u
\end{pmatrix}
\]
(1.1)
and characterize stabilizing feedback digital controllers \( u = \mathcal{H}(K) \hat{S}_p \)
that satisfy \( \|T_{ux}\|_{\infty} < \gamma \) for a prespecified \( \gamma \), where \( S \) and \( \hat{S}_p \) are sample and hold operations, and \( T_{ux} \) denotes the transfer function from \( w \) to \( z \) when closing the loop with \( K \).

In this note we first observe that the solution to the special case \( P_{11} = 0 \) is directly obtainable by characterizing the disturbance inputs with minimum energy. The general case can be reduced to this case by the \( J \)-unitary transformation introduced by Baniieh and Pearson [1]. We then perform yet another \( J \)-unitary transformation to make the reduced "4" matrix be the same as the originally sampled transition matrix \( e^{AB} \). This has the advantage that stabilizability is readily seen to be preserved by the whole procedure.

II. PROBLEM STATEMENT

Consider the sampled-data feedback system given by Fig. 1. Let
\[
\begin{align*}
  x(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\
  z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\
  y(t) &= C_2 z(t)
\end{align*}
\]
(2.1) (2.2) (2.3)
be a realization of \( P(s) \), where we assume that \( P \) is stabilizable and detectable, \( x(t) \in \mathbb{R}^n, w(t) \in \mathbb{R}^{m_1}, u(t) \in \mathbb{R}^{m_2}, z(t) \in \mathbb{R}^{p_1}, y(t) \in \mathbb{R}^{p_2}, \) and \( D_{21} = 0 \). Introduce the lifted variables
\[
x_{d}[k] = x(\bar{t}k),
\]
(2.4)
where \( \bar{t} = t - kh \), \( k \in \mathbb{Z} \). It is now a standard fact [17], [19], [1] that (2.1)-(2.3) are represented by the time-invariant discrete-time equations as
\[
\begin{align*}
  x_{d}[k+1] &= \begin{bmatrix}
  A_d & B_1 & B_{2d} \\
  C_1 & D_{11} & D_{12} \\
  C_{2d} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_{d}[k] \\
  u_{e}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
\end{align*}
\]
(2.7)
where the operators \( A_d, B_1, B_{2d}, \) etc., are defined by
\[
A_d := e^{Ah} - R^n
\]
\[
B_{1d} := \int_0^1 e^{A(\sigma - \tau)} B_1 w(\sigma)d\sigma : L^{m_1}_2[0, h] \rightarrow R^n
\]
\[
B_{2d} := \int_0^1 e^{A(\sigma - \tau)} B_2 z(\sigma)d\sigma : L^{m_2}_2[0, h] \rightarrow R^n
\]
\[
C_{1d} := C_1 e^{Ah} : L^{p_1}_2[0, h] \rightarrow R^{p_1}
\]
\[
C_{2d} := C_2 : R^{p_2} \rightarrow R^{p_2}
\]
\[
D_{11} w(\cdot) := D_{11} w(\cdot) + \int_0^1 C_1 e^{A(\sigma - \tau)} B_1 w(\sigma)d\sigma : L^{m_1}_2[0, h] \rightarrow L^2[0, h]
\]
\[
D_{12} := D_{12} H(\bar{t}) + \int_0^1 C_1 e^{A(\sigma - \tau)} B_2 z(\sigma)d\sigma : R^{m_2} \rightarrow L^2[0, h].
\]
(2.12) (2.13) (2.14)

Fix a digital controller \( K(z) \). Let \( T_{ux} \) be the transfer function from \( w \) to \( z \) when closing the loop with \( K \), and \( \|T_{ux}\|_{\infty} \) its \( H_{\infty} \) norm in the sense of discrete-time system as described above. Let
\[
J(\Sigma_0, K) := \|T_{ux}\|_{\infty} = \sup_{w \in L^{m_1}_2[0, \infty]} \|z_{d}[\cdot] \|_{L^2[0, \infty]}
\]
(2.15)
The objective is to derive a finite-dimensional system \( \Sigma' \) so that \( J(\Sigma_0, K') < \gamma \) if and only if \( J(\Sigma', K') < \gamma \). When it is obvious which system \( \Sigma \) is under consideration, we write \( J(K) \) in place of \( J(\Sigma, K) \).

III. MINIMUM ENERGY CHARACTERIZATION

In this section, we consider the special case where the direct transmission \( P_{11} \) from \( w \) to \( z \) is identically zero. Although highly specialized, this case is of interest on its own for the following two reasons: 1) as an important special case, the robust stability problem can be studied in this setting (see Section IV), and 2) based on the result of [1], the general case can be, in a sense, reduced to this special case (see Section V).

The main result is stated as follows.

Theorem 3.1: Suppose \( P_{11} = 0 \) in \( \Sigma_0 \) and let \( * \) denote the adjoint. Then the following three induced-norm optimization problems are equivalent:

- \( J(\Sigma_0, K) < \gamma \)

\[
\begin{bmatrix}
  x_{d}[k+1] \\
  z_{d}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
= \begin{bmatrix}
  A_d & B_1 & B_{2d} \\
  C_1 & D_{11} & D_{12} \\
  C_{2d} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_{d}[k] \\
  u_{e}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
\]
(3.1)

- \( J(\Sigma_2, K) < \gamma \)

\[
\begin{bmatrix}
  x_{d}[k+1] \\
  z_{d}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
= \begin{bmatrix}
  A_d & B_{1d} & B_{2d} \\
  C_1 & D_{11} & D_{12} \\
  C_{2d} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_{d}[k] \\
  u_{e}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
\]
(3.2)

where \( B_{1d} \) is a constant matrix defined as
\[
\hat{B}_{1d} = \left( B_1 \right)_t^{1/2}.
\]
(3.3)

- \( J(\Sigma_4, K) < \gamma \)

\[
\begin{bmatrix}
  x_{d}[k+1] \\
  z_{d}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
= \begin{bmatrix}
  A_d & \hat{B}_{1d} & \hat{B}_{2d} \\
  C_1 & \hat{D}_{11} & \hat{D}_{12} \\
  C_{2d} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_{d}[k] \\
  u_{e}(k, \cdot) \\
  u_{d}[k]
\end{bmatrix}
\]
(3.4)

where \( \hat{C}_{1d} \) and \( \hat{D}_{12d} \) are constant matrices defined as
\[
\begin{bmatrix}
  \hat{C}_{1d} \\
  \hat{D}_{12d}
\end{bmatrix} = \left( \begin{bmatrix}
  C_{1d} \\
  \left( D_{12d} \right)_t
\end{bmatrix} \right)^{1/2}.
\]
(3.5)

The reduction from \( \Sigma_0 \) to \( \Sigma_4 \) in the above theorem consists of two steps.

a) Norm Preserving Discretization on Disturbance Input: This step replaces the input term \( B_1 w(k, \cdot) \) by a discrete-time term \( \hat{B}_{1d} u_{d}[k] \), so that
\[
J(\Sigma_0, K) < \gamma \Leftrightarrow J(\Sigma_2, K) < \gamma.
\]

b) Norm Preserving Discretization on Controlled Output: This step introduces a discrete time output \( z_{d}[k] \), so that
\[
J(\Sigma_2, K) < \gamma \Leftrightarrow J(\Sigma_4, K) < \gamma.
\]
Step a) is carried out by a particularly simple procedure based on the following idea [10].

First we observe that \( \mathbf{D}_{a} = 0 \), so that \( \dot{z}_e \) is affected by \( w_e \) only through feedback and hence by state \( x_e \) and control input \( u_e \).

Therefore, if the same \( u_e[k] \) is applied at step time \( k \), then the controlled output \( \dot{z}_e \) is determined by \( x_e[k] \). This means that the worst disturbance input \( w_e \) that gives rise to the \( L_2 \)-induced norm \( J(\Sigma_0, K) \) may be characterized as the one having the minimum \( L_2 \)-norm among those yielding the same state \( x_e[k] \). This simple minimum energy principle is the key to the reduction process here; indeed, such an element can be characterized easily by linear quadratic (LQ) technique.

Step b) is very simple, just to define the discrete-time output \( \dot{z}_e[k] \) satisfying

\[
\dot{z}_e[k] = \int_0^k z_e(k, \sigma) \dot{z}_e[k, \sigma] \, d\sigma.
\]

(3.6)

**Remark 3.2:** Let \( G \) be a linear time-invariant, stable, finite-dimensional, continuous-time system. Then it is easy to see the following: 1) \( G \mathcal{H} \) is a mapping from \( L_2 \) into \( L_2 \), and 2) when \( G \) is strictly proper, \( \mathcal{H} \) is a mapping from \( L_2 \) into \( L_2 \). Under these observation, Chen and Francis [4] derived formulas to calculate \( L_2 \)-induced norm of \( G \mathcal{H} \) and \( L_2 \)-induced norm of \( G \)-via operator theory. Theorem 3.1 gives us quite the same results as Chen and Francis [4] straightforwardly in terms of state-space formulas. In fact, \( G \mathcal{H} \) can be reduced to a finite-dimensional discrete-time system by the norm-preserving discretization on \( z_\cdot \) as shown in Step b), and \( \mathcal{H} \) can be done by applying Step a).

Now we prove Theorem 3.1 to show Steps a) and b) in detail.

**Step a):** \( J(\Sigma_0, K) < \gamma \quad \Rightarrow \quad J(\Sigma_2, K) < \gamma \).

Consider the original system \( \Sigma_0 \) and introduce the following equivalence relation \( \mathcal{R} \) in the disturbance input space \( L_2^m \{0, \infty\} \).

**Definition 3.3:** Let \( w_e \) and \( w'_e \) be in \( L_2^m \{0, \infty\} \). Define \( w_e, \mathcal{R}, w'_e \), if \( \dot{z}_e[k] = \dot{w}_e[k] \) for all \( k \) and \( u_e \), where \( x_e \) and \( x'_e \) are the state variables driven by \( w_e \) and \( w'_e \), respectively.

It is trivial that the relation \( w_e, \mathcal{R}, w'_e \) holds if and only if

\[
\mathbf{B}_1 w_e(k, \cdot) = \mathbf{B}_1 w'_e(k, \cdot) \quad \forall k.
\]

(3.7)

Now given any \( w_e \), we want to find a \( u^*_e \) with minimum \( L_2 \) norm in the same equivalence class. This is motivated by the following (straightforward but quite important) minimum energy principle.

**Lemma 3.4 (Minimum Energy Principle):** Let \( L^m_2 \{0, \infty\} / \mathcal{R} \) be the quotient space modulo the equivalence class defined above, and let

\[
\mathcal{M} = \{ u^*_e \in L^m_2 \{0, \infty\} : u^*_e \mathcal{R} w_e \Rightarrow \| u^*_e \|^2 \leq \| u_e \|^2 \}
\]

i.e., \( \mathcal{M} \) is the set of representatives with minimum energy.

Then

\[
J(\Sigma_0, K) = \sup_{u_e \in \mathcal{M}} \| \dot{z}_e[k] \|_{L_2},
\]

(3.8)

**Proof:** Recall that

\[
J(\Sigma_0, K) = \sup_{u_e \in L^m_2 \{0, \infty\}} \| \dot{z}_e[k] \|_{L_2},
\]

(3.9)

Clearly, the right-hand side of (3.8) is less than or equal to that of (3.9). But if \( u_e \) and \( u^*_e \) belong to the same equivalence class, then the numerators of these two are the same by the very definition of our equivalence relation \( \mathcal{R} \), so that the right-hand side of (3.9) is less than or equal to that of (3.8). Hence they are identical.

This lemma asserts that to characterize the \( L_2 \)-induced norm, we can confine our attention to the minimum energy elements \( u^*_e \).

The next lemma gives a characterization for such \( u^*_e \) by using LQ technique [2].

**Lemma 3.5:** For any \( w_e \in L^m_2 \{0, \infty\} \), \( u^*_e \) that gives the minimum norm in the same equivalence class is given by

\[
u^*_e(k, \theta) = \mathbf{B}^*_1 \mathbf{W}^{1/2}_0 \mathbf{B}_1 \dot{w}_e(k, \cdot)
\]

for all \( k \)

(3.10)

where \( \mathbf{W}^{1/2}_0 \) is the pseudo-inverse of \( \mathbf{W}_0 : = \mathbf{B}_1 \mathbf{B}^*_1 \).

**Remark 3.6:** From (3.11), it is easy to see that \( \mathbf{W}_0 \) is nonsingular if and only if \( (A, B_1) \) is controllable. Since \( \mathbf{W}_0 \) is a symmetric and positive semidefinite matrix, there exists an orthogonal matrix \( Q \in \mathbb{R}^n \) such that

\[
\mathbf{QW}_0 \mathbf{Q}^T = \diag[w_1, \ldots, w_{m_{d1}}, 0, \ldots, 0]
\]

where \( w_i > 0 \) for \( i = 1, \ldots, m_{d1} \) and \( m_{d1} = \text{rank} \mathbf{W}_0 \), i.e., \( m_{d1} \) is equal to the dimension of controllable subspace of \( (A, B_1) \).

Therefore, \( \mathbf{W}^{1/2}_0 \) can be given by

\[
\mathbf{W}^{1/2}_0 = \mathbf{Q}^T \diag \left[ \frac{1}{w_1}, \ldots, \frac{1}{w_{m_{d1}}}, 0, \ldots, 0 \right] \mathbf{Q}.
\]

(3.11)

Hereafter we will denote \( \mathbf{W}^{1/2}_0 \) by \( \mathbf{W}^{1/2}_0 \).

Define

\[
\mathbf{B}_0 := \mathbf{W}^{1/2}_0 \mathbf{B}_1, \quad \mathbf{w}_d[k] := \mathbf{B}_0 \dot{w}_e(k, \cdot).
\]

(3.12)

Note that \( \mathbf{w}_d[k] \) is a finite-dimensional vector in \( \mathbb{R}^{n-m_{d1}} \), but not in \( \mathbb{R}^n \). In fact

\[
\mathbf{Q} \mathbf{w}_d[k] = \mathbf{Q} \mathbf{B}_0 \dot{w}_e(k, \cdot)
\]

\[
= \diag \left[ \frac{1}{w_1}, \ldots, \frac{1}{w_{m_{d1}}}, 0, \ldots, 0 \right] \mathbf{Q} \mathbf{B}_1 \dot{w}_e(k, \cdot)
\]

means that \( \mathbf{Q} \mathbf{w}_d[k] = 0 \) with \( \mathbf{Q} \in \mathbb{R}^{n-m_{d1}} \times n \) being the lower submatrix of \( \mathbf{Q} \). Then we have

\[
\| \dot{\mathbf{w}}_d[k] \|_{L_2} = \| \mathbf{B}_0 \dot{w}_e[k] \|_{L_2} = \| \mathbf{B}_1 \mathbf{W}^{1/2}_0 \mathbf{w}_d[k] \|
\]

\[
= \left( \mathbf{Q} \mathbf{w}_d[k] \right)^T \mathbf{Q} \mathbf{w}_d[k]
\]

\[
= \mathbf{w}_d[k]^T \mathbf{Q} \mathbf{w}_d[k] = \| \mathbf{w}_d[k] \|^2
\]

and therefore

\[
\| u^*_e \|_{L_2} = \| \dot{\mathbf{w}}_d[k] \|_{L_2}
\]

(3.13)

\[
\mathbf{B}_1 u^*_e(k, \cdot) = \mathbf{B}_1 \mathbf{B}^*_1 \mathbf{W}^{1/2}_0 \mathbf{w}_d[k]
\]

(3.14)

Hence the correspondence

\[
L^m_2 \{0, \infty\} \ni \dot{w}_e \mapsto \mathbf{w}_d \in L^{m_{d1}}_2 \{0, \infty\}
\]

is norm-preserving, and by (3.14) we can replace the input operator by \( \mathbf{W}^{1/2}_0 \). This clearly completes the proof of the reduction process Step a).
Step b): $J(\Sigma_2, K) < \gamma \Leftrightarrow J(\Sigma_4, K) < \gamma$.
The relation (3.6), i.e., $\|\tilde{z}_2(k, \cdot)\|_{L_2[0, h]} = \|\tilde{z}_4[k]\|$ directly leads to Step b, where
\begin{equation}
\|\tilde{z}_2[k]\|^2 = \|x_2'[k], u_2[k]\|^2 \begin{bmatrix}
C_1 D_{12} & 0 \\
0 & C_5 D_{12}
\end{bmatrix} \begin{bmatrix}
x_2[k] \\
u_2[k]
\end{bmatrix}.
\end{equation}

(3.15)

Then the lifted variables allows us to represent $P(s)$ as
\begin{equation}
\begin{bmatrix}
x_{2d}[k] + 1 \\
\tilde{z}_2(k, \theta)
\end{bmatrix} = \begin{bmatrix}
A_d & 0 \\
B_d & 0
\end{bmatrix} \begin{bmatrix}
x_{2d}[k] \\
u_2[k]
\end{bmatrix} + \begin{bmatrix}
0 & -B_d \\
C_d & 0
\end{bmatrix} \begin{bmatrix}
\dot{z}_2(k, \cdot)
\end{bmatrix}.
\end{equation}

(3.16)

Suppose that the state-space realizations of $G_0(s)$ and $\delta(s)I$ are given by
\begin{equation}
G_0(s) = \begin{bmatrix}
A_0 & B_0 \\
C_0 & 0
\end{bmatrix}, \quad \delta(s)I = \begin{bmatrix}
A_\delta & B_\delta \\
C_\delta & 0
\end{bmatrix}.
\end{equation}

Then the lifted variables allows us to represent $P(s)$ as
\begin{equation}
\begin{bmatrix}
x_{2d}[k] + 1 \\
\tilde{z}_2(k, \theta)
\end{bmatrix} = \begin{bmatrix}
A_d & 0 \\
B_d & 0
\end{bmatrix} \begin{bmatrix}
x_{2d}[k] \\
u_2[k]
\end{bmatrix} + \begin{bmatrix}
0 & -B_d \\
C_d & 0
\end{bmatrix} \begin{bmatrix}
\dot{z}_2(k, \cdot)
\end{bmatrix}.
\end{equation}

(3.16)

Theorem 3.1 has been proven completely.

The procedure above relies upon the classification of the intersample input functions. Each equivalence class consists of those that yield the same state $x$ and hence the same output function resulting from $x$. This latter property is valid because the output $z$ is affected only through $x$. This is where the hypothesis $P_{31} = 0$ becomes effective. Clearly for a given intersample input $w_c$, its corresponding equivalence class is $\{w_c + \ker B_1\}$, i.e., it is affine in $w_c$. It is easy to recognize that the input $w_c$ with minimum energy gives rise to the worst excitation from the $H_\infty$ control problem viewpoint. As seen above, characterization of such inputs is an easy application of the classical minimum norm problem (e.g., [2]). This clearly gives rise to a realization of the quotient space $L_2^0(0, h) = \ker B_1$, and once this space is fixed and the induced system with this quotient space as the space of (intersample) inputs, finding a norm-equivalent finite-dimensional system is fairly straightforward. It should be noted that a reasoning similar to the minimum energy principle discussed above has been independently reviewed in [18], the proof of Theorem 3.1, where sampled-data $H_\infty$ control problem on finite horizon is solved using game-theoretic methods.

One may also note that in [1] Bamieh and Pearson have derived a solution for this case. Instead of using the notion of minimum energy principle as discussed above, they made use of the orthogonal decomposition $L_2^0(0, h) \cong \ker B_1 \oplus (\ker B_1)^\perp$. Of course, it is well known that the two procedures are mathematically equivalent (see, e.g., [13]), but it seems interesting to note that this orthogonal complement admits quite concrete realization via the intuitive minimum energy principle given above, because then the $H_\infty$ solution in this case is nothing but an application of the LQ solution.

IV. ROBUST STABILITY

Now, as an application of the case $P_{31} = 0$, we present a robust stability problem. The robust stabilization problem for an additive perturbation of plant in sample-data control system has been considered in [9] and [5]. Here we discuss the robust stabilization problem for a multiplicative perturbation of plant, where a continuous-time plant $G(s)$ belongs to the class
\begin{equation}
\dot{G}(G_0, \delta) = \left\{ G(s) = (I + \Delta(s))G_0(s) \right\}
\end{equation}

and $\delta(s)$ is a strictly proper outer function. The robust stabilization problem is to find a discrete-time controller $K(z)$ that stabilizes any $G(s) \in \dot{G}(G_0, \delta)$.

The same argument as in [9] leads to that the robust stabilization problem is just an $H_\infty$ type problem where the generalized plant $P(s)$ is
\begin{equation}
P(s) = \begin{bmatrix}
0 & -G_0(s) \\
\delta(s)I & -G_0(s)
\end{bmatrix}.
\end{equation}

(4.1)

Note that $P_{31}(s) = 0$.

Suppose that the state-space realizations of $G_0(s)$ and $\delta(s)I$ are given by
\begin{equation}
G_0(s) = \begin{bmatrix}
A_0 & B_0 \\
C_0 & 0
\end{bmatrix}, \quad \delta(s)I = \begin{bmatrix}
A_\delta & B_\delta \\
C_\delta & 0
\end{bmatrix}.
\end{equation}

Then the lifted variables allows us to represent $P(s)$ as
\begin{equation}
\begin{bmatrix}
x_{2d}[k] + 1 \\
\tilde{z}_2(k, \theta)
\end{bmatrix} = \begin{bmatrix}
A_d & 0 \\
B_d & 0
\end{bmatrix} \begin{bmatrix}
x_{2d}[k] \\
u_2[k]
\end{bmatrix} + \begin{bmatrix}
0 & -B_d \\
C_d & 0
\end{bmatrix} \begin{bmatrix}
\dot{z}_2(k, \cdot)
\end{bmatrix}.
\end{equation}

(4.1)

By using Theorem 3.1, we immediately obtain the following corollary.

Corollary 4.1: Define a discrete-time generalized plant $P_d$ as
\begin{equation}
P_d(z) = \begin{bmatrix}
0 & -G_0(z) \\
\dot{z}_2(z) & -G_0(z)
\end{bmatrix} = \begin{bmatrix}
A_d & 0 \\
0 & A_\delta
\end{bmatrix} \begin{bmatrix}
0 & -B_d \\
C_d & 0
\end{bmatrix} \begin{bmatrix}
0 & 0 \\
\delta(z)I & -G_0(z)
\end{bmatrix}.
\end{equation}

(4.1)

where
\begin{equation}
\dot{B}_d = \begin{bmatrix} B_2 \delta \end{bmatrix}^{1/2}, \quad \dot{C}_d \dot{D}_d = \begin{bmatrix} C D^* \end{bmatrix}^{1/2}.
\end{equation}

(4.3)

If there exists a solution $K(z)$ with $J(P_d, K) < 1$ to the $H_\infty$ problem for discrete-time system, then the sampled-data system with $\dot{G}(G_0, \delta)$ is robustly stabilizable via $K(z)$.

Remark 4.2: $G_d(z)$ and $\dot{G}_0(z)$ in the corollary are determined only by the nominal plant $G_0(s)$, while $\dot{z}_2(z)$ is characterized only by the perturbation bound $\delta(s)$.

The corollary shows the sufficient condition of robust stabilization. If the perturbation class is enlarged to include $h$-periodic perturbations, the condition would be also necessary in the same way as in the case of additive perturbations [15]. See also [6] for further study of robust stability conditions under linear time-invariant perturbations.

V. GENERAL CASE

In this section, we derive a solution for the general case, which gives four induced-norm optimization problems equivalent to the original one. We here assume, without loss of generality, that $\gamma = 1$, i.e., we derive norm-equivalent problems for $J(K) < 1$. We also assume the induced norm $\|D_{11}\| \leq 1$, taken in the sense of an operator in $L_2(0, h)$. This is a necessary condition for the solvability of the original problem, because $\|D_{11}\| \leq \|T_{W_1}\|_\infty$ always holds. Indeed, since $D_{11}$ reflects the effect due to the behavior that cannot be controlled by a sample-held feedback, its induced norm cannot be reduced in the present framework.

The following theorem derives four systems $\Sigma_1 - \Sigma_4$ which satisfy the induced norm bound $J(\Sigma, K) < 1$. Note that the internal stability of $(\Sigma, K)$ will be considered later, that is, in the following theorems, we are interested only in the induced norm bound under input-output stability (without internal stability). See Theorem 5.2 and Remark 5.3.
Theorem 5.1: The following five induced-norm optimization problems are equivalent.

- \( J(\Sigma_0, K) < 1 \), where \( \Sigma_0 \) is defined by (2.7).
- \( J(\Sigma_1, K) < 1 \)

\[
\Sigma_1: \begin{bmatrix}
  x_d[k+1] \\
  z_c(k, \theta) \\
  y_d[k]
\end{bmatrix} =
\begin{bmatrix}
  \hat{A}_d & \hat{B}_d & \hat{B}_{2d} \\
  \hat{C}_d & 0 & \hat{D}_{2d}
\end{bmatrix}
\begin{bmatrix}
  x_d[k] \\
  u_c(k, \cdot) \\
  u_d[k]
\end{bmatrix}
\]  
\( (5.1) \)

where
\[
\hat{A}_d = A_d + B_1 D_1^*(I - D_1^* D_1)^{-1} C_1 
\]  
\( (5.2) \)
\[
\hat{B}_{2d} = B_{2d} + B_1 D_1^*(I - D_1^* D_1)^{-1/2} C_1 
\]  
\( (5.3) \)
\[
\hat{C}_1 = (I - D_1^* D_1)^{-1/2} C_1 
\]  
\( (5.4) \)
\[
\hat{D}_{12} = (I - D_1^* D_1)^{-1/2} D_{12}. 
\]  
\( (5.6) \)

- \( J(\Sigma_2, K) < 1 \)

\[
\Sigma_2: \begin{bmatrix}
  x_d[k+1] \\
  z_c(k, \theta) \\
  y_d[k]
\end{bmatrix} =
\begin{bmatrix}
  \hat{A}_d & \hat{B}_d & \hat{B}_{2d} \\
  \hat{C}_1 & 0 & \hat{D}_{12}
\end{bmatrix}
\begin{bmatrix}
  x_d[k] \\
  u_c(k, \cdot) \\
  u_d[k]
\end{bmatrix}
\]  
\( (5.7) \)

where
\[
\hat{B}_{1d} = (B_1 B_1^*)^{1/2} 
\]  
\( (5.8) \)
\[
\hat{C}_1 = (I - D_1^* D_1)^{-1/2} C_1 
\]  
\( (5.9) \)
\[
\hat{D}_{11} = -(I - D_1^* D_1)^{-1/2} D_1 B_1^*(I - D_1^* D_1)^{-1} B_1^* \frac{1}{2} 
\]  
\( (5.12) \)
\[
\hat{D}_{12} = (I - D_1^* D_1)^{-1/2} D_{12}. 
\]  
\( (5.13) \)

- \( J(\Sigma_4, K) < 1 \)

\[
\Sigma_4: \begin{bmatrix}
  x_d[k+1] \\
  z_d[k] \\
  y_d[k]
\end{bmatrix} =
\begin{bmatrix}
  \hat{A}_d & \hat{B}_d & \hat{B}_{1d} \\
  \hat{C}_d & 0 & \hat{D}_{12}
\end{bmatrix}
\begin{bmatrix}
  x_d[k] \\
  u_c(k, \cdot) \\
  u_d[k]
\end{bmatrix}
\]  
\( (5.14) \)

where
\[
[\hat{B}_{1d}, \hat{D}_{11}, \hat{D}_{12}] = \left( \begin{bmatrix}
  \hat{C}_1^* \\
  \hat{D}_{11} \\
  \hat{D}_{12}
\end{bmatrix} \right)^{1/2} \begin{bmatrix}
  \hat{C}_1 \\
  \hat{D}_{11} \\
  \hat{D}_{12}
\end{bmatrix} 
\]  
\( (5.15) \)

Proof: The proof is divided into the following four steps:

1. Step 1: \( J(\Sigma_0, K) < 1 \) \( \Leftrightarrow \) \( J(\Sigma_1, K) < 1 \)
2. Step 2: \( J(\Sigma_2, K) < 1 \)
3. Step 3: \( J(\Sigma_2, K) < 1 \) \( \Leftrightarrow \) \( J(\Sigma_3, K) < 1 \)
4. Step 4: \( J(\Sigma_4, K) < 1 \)

The outline of the proof in each step is as follows.

- All these reductions involve transformations in one intersample period only. Hence dynamics does not enter into the formulas.
- Moreover, \( x_d, u_c, \) and \( u_d \) are not changed in each step. Only \( w \) and \( z \) are subject to changes as
  - \( (w_c, z_c) \rightarrow (\hat{w}_c, \hat{z}_c) \rightarrow (\hat{u}_c, \hat{z}_c) \rightarrow (\hat{u}_d, \hat{z}_d) \)}
Since the variables $u$ and $y$ are absent in the reduction process, modifications involving $u$ or $y$ can be handled without any change. For example, the design of controllers with computational delays can be done with the same reduced discrete-time system, because this change involves $u$ only, which is irrelevant to the whole reduction processes.

To complete the derivation of an equivalent discrete-time $H_\infty$ control problem, it remains to see the internal stability of $(\Sigma, K)$. If $A_d$ and $B_{2d}$ matrices in the equivalent discrete-time problem are different from the original ones of $\Sigma_0$, the equivalence of internal stability has to be proven as in [1]. In our final form $\Sigma_4$, however, the $A_d$ and $B_{2d}$ matrices are converted back to the original ones of $\Sigma_0$ (and hence similar to those of [11], [8]), so that the stabilizability and detectability of $(A_{2d}, B_{2d}, C_{2d})$ are easily seen to be preserved. Thus we can augment Theorem 5.1 with the internally stabilizing property as follows.

Theorem 5.2: The following statements are equivalent:
1) $K$ internally stabilizes $\Sigma_0$ and $J(\Sigma_0, K) < 1$, where $\Sigma_0$ is defined by (2.7).
2) $K$ internally stabilizes $\Sigma_4$ and $J(\Sigma_4, K) < 1$, where $\Sigma_4$ is defined by (5.14).

Remark 5.3: In the above theorem, it is claimed only that the internal stability of $(\Sigma_4, K)$ is equivalent to one of the original system $(\Sigma_0, K)$, however, the internal stability of $(\Sigma, K)$ holds for all the systems $\Sigma_0, \Sigma_1$, and $\Sigma_2$. In fact, the same technique as in [1] can be used to prove the equivalence of internal stability for $\Sigma_0, \Sigma_1$, and $\Sigma_2$. In addition, the equivalence for $\Sigma_0, \Sigma_3$, and $\Sigma_4$ is trivial from that they have the same $A_d$, $B_{2d}$, and $C_{2d}$ matrices.

As mentioned before, our equivalent discrete-time system $\Sigma_4$ has advantage of that the $A_d, B_{2d}, C_{2d}$ matrices are converted back to the original ones of $\Sigma_0$. This fact holds even if problem $J(\Sigma_0, K) < 1$ is considered instead of $J(\Sigma, K) < 1$, i.e., the $A_d, B_{2d}, C_{2d}$ matrices are independent of $\gamma$. Therefore the stabilizability and detectability of $(A_{2d}, B_{2d}, C_{2d})$ are also independent of $(A_d, B_d, C_d)$ every $\gamma$; if you use an equivalent discrete-time system with $(A_d, B_d, C_d)$ being dependent on $\gamma$, e.g., [1].

In addition, whenever the sampling period $h$ is nonpathological, it is well known that $(A_d, B_{2d}, C_{2d})$ of $\Sigma_4$ is stabilizable and detectable if the continuous-time generalized plant $P$ in (2.1)-(2.3) is stabilizable and detectable. Therefore we do not have to check whether $\Sigma_4$ is stabilizable and detectable.

VI. CONCLUDING REMARKS

We have given an equivalent discrete-time system for the given $H_\infty$ type problem of a sampled-data systems. The reduction process is independent of the (discrete-time) dynamics and the choice of a feedback gain $K(z)$ from $y$ to $u$. Hence some modifications relevant to the design of $K$ only do not require any change in the equivalent system; we need only solve the final problem with a different design specification (e.g., controller subject to computational delays). We note that this is also due to the fact that the reduction process classifies the set of disturbance inputs $w$ according to the state $x(kh)$.

APPENDIX

We now give a state-space form of the equivalent discrete-time system $\Sigma_4$ shown in the Section V. For simplicity, we assume here $D_{11} = 0$ and $H(t) = I$ (zero-order hold).

Theorem A.1: Let
\[
\begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix} := \exp \left\{ \begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} h \right\}
\]

(A.2)

\[
\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} \end{bmatrix} := \exp \left\{ \begin{bmatrix} -A^T & -C_1^T C_1 & -C_1^T D_{21} \\ B_1 B_1^T & A & C_1 \\ 0 & 0 & 0 \end{bmatrix} h \right\}
\]

(A.3)

and define
\[
W_1 := B_2 B_1^+ = \Psi_{21} \Psi_{11}^{-1}
\]

(A.4)

\[
W := B_1 (I - D_{11} D_{11}^{-1}) B_1^+ = \Gamma_{21} \Gamma_{11}^{-1}
\]

(A.5)

\[
V_{\Sigma} := (I - D_{11} D_{11}^{-1})^{-1} C_1 = -\Gamma_{11}^{-1} \Gamma_{12}
\]

(A.6)

\[
V_{\Sigma} := (I - D_{11} D_{11}^{-1})^{-1} D_{11} = -\Gamma_{11}^{-1} \Gamma_{13}
\]

(A.7)

\[
V_{\Sigma} := D_{11} (I - D_{11} D_{11}^{-1})^{-1} D_{11} B_1^+ = \Gamma_{11}^{-1} - \Phi_{11}
\]

(A.8)

\[
M := (I - D_{11} D_{11}^{-1})^{-1} D_{11} B_1^+ = \Gamma_{11}^{-1} - \Phi_{11}
\]

(A.9)

\[
N := D_{11} (I - D_{11} D_{11}^{-1})^{-1} D_{11} - \Gamma_{11}^{-1} - \Phi_{11}
\]

(A.10)

\[
M := D_{11} (I - D_{11} D_{11}^{-1})^{-1} D_{11} B_1^+ = \Gamma_{11}^{-1} - \Phi_{11}
\]

(A.11)

\[
M := D_{11} (I - D_{11} D_{11}^{-1})^{-1} D_{11} B_1^+ = \Gamma_{11}^{-1} - \Phi_{11}
\]

(A.12)

Then a state-space realization of $\Sigma_4$ expressed as (5.14) is given by
\[
A_d = \Phi_{11}, \quad B_{2d} = \Phi_{12}, \quad C_{2d} = \Phi_{11}
\]

(A.13)

\[
\hat{B}_{1,d} = W_1^{1/2} \hat{W}_1^{1/2}
\]

(A.14)

\[
[C_d, \hat{D}_{1,d}, \hat{D}_{2,d}, \hat{D}_{3,d}]
\]

\[
\begin{bmatrix} V_{ee} & 0 & V_{ed} \\ 0 & M & 0 \\ V_{de} & V_{dd} & 0 \end{bmatrix} - \begin{bmatrix} M_1 W_1^{1/2} & -M_2 W_1^{1/2} \\ -N_1 W_1^{1/2} & M_2 W_1^{1/2} \end{bmatrix} \right\}^{1/2}
\]

(A.15)

\[
\begin{bmatrix} \hat{C}_1^T \\ \hat{D}_{12}^T \end{bmatrix} = \begin{bmatrix} \hat{C}_1^T \\ \hat{D}_{12}^T \end{bmatrix} (I - D_{11} D_{11}^{-1})^{-1} [C_1^T, D_{12}^T]
\]

(A.16)

\[
\begin{bmatrix} \hat{C}_1^T \\ \hat{D}_{12}^T \end{bmatrix} = \begin{bmatrix} \hat{C}_1^T \\ \hat{D}_{12}^T \end{bmatrix} (I - D_{11} D_{11}^{-1})^{-1} [C_1, D_{12}]
\]

(A.17)

\[
\begin{bmatrix} \hat{C}_1^T \\ \hat{D}_{12}^T \end{bmatrix} = \begin{bmatrix} \hat{C}_1^T \\ \hat{D}_{12}^T \end{bmatrix} (I - D_{11} D_{11}^{-1})^{-1} [C_1, D_{12}]
\]

(A.18)
\[ D_1^* C_1 = (C_1^* D_{11})^T \quad (A.19) \]
\[ D_1^* D_{12} = N^{-\frac{1}{2}} W^{\frac{1}{2}} B_1 D_{11}^* (I - D_{11} D_{11}^*)^{-1} D_{12} \quad (A.20) \]
\[ D_{12}^* D_{11} = (D_{11}^* D_{12})^T \quad (A.21) \]

These equations lead to (A.15). The state-space computations of \( W_0 \), \( W, V_{ee}, V_{dd}, M_1, M_2, N \), and \( M \) based on three exponentials (A.1), (A.2) and (A.3) can be verified by the same technique as in [1].

**Remark A.2:**
1. We need three exponentials of sizes \( n + m_1, 2n, \) and \( 2(n + m_2) \), where \( n \) and \( m_2 \) are the dimensions of the state \( x(t) \) and the control input \( u(t) \), respectively.
2. If we consider a problem \( J(K) < \gamma \) instead of \( J(K) < 1 \), only \( C_1 \) and \( D_{12} \) should be replaced by \( C_1/\gamma \) and \( D_{12}/\gamma \) in the above formulas. Hence, recalculation is required only if \( \Gamma \) in the \( \gamma \)-iteration for the optimization, since \( \Phi \) and \( \Psi \) are independent of \( \gamma \).

**REFERENCES**


