A Note on Linear Input/Output Maps of Bounded-Type

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Abstract—**This note studies a class of constant, linear, continuous-time input/output maps, which is called input/output maps of bounded-type. An algebraic condition for an input/output map to be of bounded-type is given. Its application to realization theory (especially for delay-differential systems) is considered. An example is given to illustrate how it gives rise to a concrete realization procedure.**

I. Introduction

In this note we study a class of constant, linear, continuous-time input/output maps, which we call input/output maps of *bounded-type*. An input/output map f (or its impulse response A) is said to be *T-bounded* (of *bounded-type*, or simply, *bounded*) if its canonical realization in the sense defined in [7] has the following property: its initial state determination procedure is well posed on the basis of observation of output data on the finite interval $[0,T]$. We prove the following result. An impulse response A is of bounded-type if it is annihilated by a suitable distribution q of compact support, i.e., $q*A|_{[0,\infty)}=0$ ($*$ denotes convolution). In particular, an impulse response which is of the form $A=Q^{-1}*P$ (under certain conditions on Q and P) is of bounded-type. This shows, for example, that input/output maps of retarded delay-differential systems are of bounded-type.

II. Preliminaries

Fix a field k, which is either $\boldsymbol{R}$ or $\boldsymbol{C}$, and consider k-valued functions and systems over k. We confine ourselves to linear constant (stationary) and continuous-time systems. Let Ω and Γ denote the following spaces: $\Omega:=\cup_{n>0}L^2[-n,0]$; $\Gamma:=L^2_{\text{loc}}[0,\infty)$. Our input space is the m-fold product of Ω, namely Ω^m, and the output space is the p-fold product of Γ, i.e., Γ^p; that is, we have m-input and p-output channels [7]. These spaces are equipped with the obvious left shift operators, σ_t and $\tilde{\sigma}_t$, respectively, which are strongly continuous semigroups [7]. We then consider the zero-initial state response associated to an impulse response A: Let A be a $p\times m$ matrix whose entries are measures on $[0,\infty)$ regular at 0. Then the *constant linear input/output map* f_A *associated to* A is defined by $f_A(\omega):=\pi(A*\omega)$, $\omega\in\Omega^m$, where $\pi\psi:=\psi|_{[0,\infty)}$ and $*$ denotes convolution. A is called the *impulse response matrix* of f_A.

In [7] and [8], the (unique) canonical realization of f_A is given as follows. The state space $X:=\overline{\operatorname{Im} f_A}$ (the closure of $\operatorname{Im} f_A$ in Γ^p); the semigroup of the system is simply $\tilde{\sigma}_t$—the restriction of $\tilde{\sigma}_t$ to $\overline{\operatorname{Im} f_A}$; the state-transition is given by $\phi(t,x,u):=\tilde{\sigma}_t x+f(\sigma_t^l u)$, where $u\in(L^2[0,t])^m$ is an input, $x\in\overline{\operatorname{Im} f_A}$ is the initial state, and $(\sigma_t^l u)(s):=u(s+t)$; the output equation is given by the inclusion map j: $\overline{\operatorname{Im} f_A}\to\Gamma^p$: $x\to x$. This system is *canonical* in the sense that it is *quasi-reachable*, i.e., its reachable set is dense in the state space, and it is topologically observable, i.e., its initial state determination is well posed. We say that f_A (or A) is *T-bounded* (of *bounded-type*) if it further satisfies the property that $\overline{\operatorname{Im} f_A}$ is determined by its partial data on $[0,T]$, i.e., π_T: $\overline{\operatorname{Im} f_A}\to\overline{\operatorname{Im} f_A|_{[0,T]}}$ is a topological isomorphism. (This is equivalent to requiring that $\overline{\operatorname{Im} f_A}$ be isomorphic to Hilbert space; in general, it is only a Fréchet space.) Here $\overline{\operatorname{Im} f_A|_{[0,T]}}$ is considered as a subspace of $(L^2[0,T])^p$, of course.

We prepare some language from distribution theory. Let $\mathscr{D}'_+$ denote the set of distributions on $\boldsymbol{R}$ with support bounded on the left. For $q\in\mathscr{D}'_+$, define $l(q)$ to be the infimum of supp q, i.e., $l(q):=\inf\{t;\ t\in\operatorname{supp} q\}$. $\mathscr{E}'(\boldsymbol{R}^-)$ denotes the subspace of $\mathscr{D}'_+$ consisting of those with compact support contained in $(-\infty,0]$. $\mathscr{D}[0,\infty)$ is the space of all infinitely differentiable functions on the real line having compact support contained in $[0,\infty)$; and $\mathscr{D}'[0,\infty)$ is its dual space. We then extend the truncation mapping π as follows. Given a distribution $q\in\mathscr{D}'_+$, define $\pi q\in\mathscr{D}'[0,\infty)$ by $\langle\pi q,\psi\rangle:=\langle q,\psi\rangle$, where the right-hand side denotes the value of q at ψ regarding as an element of $\mathscr{D}(\boldsymbol{R})$. For a distribution q, ord q denotes its (global) *order* [4]; it is of order r (>0) if it acts continuously on C^r-functions but not on C^{r-1}-functions. Measures which are not functions are of order 0. A function ψ is of order $-r$ if r is the largest integer such that $(d/dt)^r\psi$ is a measure.

We adopt the following duality between Ω^1 and Γ^1: $\langle\omega,\gamma\rangle:=\int_0^\infty\omega(-t)\gamma(t)\,dt$. With respect to this duality, we have the following.

Proposition (2.1): $(\Omega^m)'=\Gamma^m$, $(\Gamma^p)'=\Omega^p$, and $(L^2[0,T])'=L^2[-T,0]$. Further, the adjoint of an input/output map f_A: $\Omega^m\to\Gamma^p$ is again an input/output map $f_{A'}$: $\Omega^p\to\Gamma^m$, where A' denotes the transpose of A.

Proof: Direct calculation. □

III. Main Theorem

Main Theorem (3.1): Let f_A be a constant linear input/output map associated to impulse response A. Suppose there exists $q\in\mathscr{E}'(\boldsymbol{R}^-)$ such that

1) $q^{-1}\in\mathscr{D}'_+$ exists (with respect to convolution);
2) $\operatorname{ord} q^{-1}=-\operatorname{ord} q$;
3) $\pi(q*A)=0$.

Then f_A is T-bounded for any T greater than $-l(q)$.

Proof: We first show that there exists a measure $\tilde{q}$ such that i) $\tilde{q}^{-1}$ is also a measure, ii) $\pi(\tilde{q}*A)=0$, and iii) $l(\tilde{q})$ is close to $l(q)$. Let r be the order of q. Take any $\epsilon>0$, and let χ_ϵ be the function given by $\chi_\epsilon(t)=1$ for $t\in[-\epsilon,0]$ and $\chi_\epsilon(t)=0$ for $t\notin[-\epsilon,0]$. The convolutional inverse of χ_ϵ is the first-order differential operator $\delta'*(\delta_{-\epsilon}-\delta)^{-1}$. Let $\tilde{q}:=q*(\chi_\epsilon)^r$ (the power r is taken with respect to convolution). Then $\tilde{q}$ becomes a measure. Since the convolutional inverse of χ_ϵ is a first-order differential operator, convolving it with q^{-1} increases its order by 1 (see [10, p. 162]

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for a similar argument). Hence, $\tilde{q}^{-1}$ also becomes a measure. By taking ϵ sufficiently small, $l(\tilde{q})$ can be made arbitrarily close to $l(q)$. Obviously, $\pi(\tilde{q} * A) = 0$.

Note that $\pi(\tilde{q} * A') = 0$. This implies that $\tilde{q}$ annihilates any element in $\Omega^p/\ker f_{A'}$ with respect to the module action of measures on $\Omega^p/\ker f_{A'}$ considered in [6]. Hence, by [6, Lemma 5.8] (or by [3, Corollary 6.1]), it follows that the map

$$\left(L^2[-T,0]\right)^p / \left(\ker f_{A'} \cap \left(L^2[-T,0]\right)^p\right) \to \Omega^p/\ker f_{A'} \quad (3.2)$$

induced by the natural inclusion j_T: $(L^2[-T,0])^p \to \Omega^p$ is surjective for any $T > -l(q)$. (It is also trivially injective.) Since by Proposition (2.1) we have $(\Gamma^p)' = \Omega^p$, $(f_A)' = f_{A'}$, etc., we obtain $\Omega^p/\ker f_{A'} = (\overline{\operatorname{Im} f_A})'$, $(L^2[-T,0])^p/(\ker f_{A'} \cap (L^2[-T,0])^p) = (\overline{\operatorname{Im} f_A|_{[0,T]}})'$, and the mapping (3.2) is the adjoint of the projection π_T: $\overline{\operatorname{Im} f_A} \to \overline{\operatorname{Im} f_A|_{[0,T]}}$: $\psi \to \psi|_{[0,T]}$. Hence, the adjoint π_T' here is bijective.

We now prove that the above π_T (restricted to $\overline{\operatorname{Im} f_A}$) is bijective. The injectivity follows easily from the Hahn–Banach theorem. Since the image of π_T' is shown to be the whole space, it is trivially weakly closed in $(\overline{\operatorname{Im} f_A})'$. Then by a well-known theorem on surjections of Fréchet spaces [5, Theorem 37.2], π_T is surjective.

Since π_T is continuous, it is a topological isomorphism due to the open mapping theorem. This readily implies $\overline{\operatorname{Im} f_A|_{[0,T]}} = \operatorname{Im} f_A|_{[0,T]}$ and π_T: $\overline{\operatorname{Im} f_A} \to \overline{\operatorname{Im} f_A|_{[0,T]}}$ is a topological isomorphism. Hence, f_A is T-bounded. □

Corollary (3.3): Let f_A be a constant linear input/output map with impulse response A. Suppose that there exist $p \times p$ and $p \times m$ matrices with entries in $\mathscr{E}'(\boldsymbol{R}^-)$ such that

1) $(\det Q)^{-1} \in \mathscr{D}'_+$ exists;

2) $\operatorname{ord}(\det Q)^{-1} = -\operatorname{ord}\det Q$;

3) $A = Q^{-1} * P$.

Then f_A is T-bounded for any T greater than $-l(\det Q)$.

Proof: $(\det Q) * Q^{-1} * P = (\operatorname{adj} Q) * P$. Since each entry of $(\operatorname{adj} Q) * P$ belongs to $\mathscr{E}'(\boldsymbol{R}^-)$, $\pi((\operatorname{adj} Q) * P) = 0$ clearly follows, hence the result. □

Remark (3.4): The condition on the order of q in the above results is automatically satisfied if q is of the following form: $q = (d/dt)^r \delta_a +$ lower order terms. This is the case for many applications, for example, for delay-differential systems.

IV. Applications

Example (4.1) (Delay-Differential Systems): Consider a subring of $\mathscr{E}'(\boldsymbol{R}^-)$: $k[\delta', \delta_{a_1}, \cdots, \delta_{a_r}]$, that is, the subring generated by Dirac distributions and their derivatives $\delta', \delta_{a_1}, \cdots, \delta_{a_r}$ $(a_i < 0)$. It is shown in [2] that any nonzero q in this ring admits an inverse in $\mathscr{D}'_+$. Hence, if $A = Q^{-1} * P$ for some matrices Q and P with entries in this ring, A is of bounded-type in view of Corollary (3.3) and Remark (3.4).

Example (4.2) (Periodic Impulse Responses): Suppose that an impulse response A is a periodic function of period T. It is easy to see $\pi((\delta_{-T} - \delta) * A) = 0$. Since $\delta_{-T} - \delta$ is easily seen to be invertible over $\mathscr{D}'_+$, A is T-bounded.

We shall now see how the present framework is applied to compute the canonical realization of A.

Theorem (4.3): Let f_A be an input/output map with impulse response A. Suppose that A satisfies the conditions of Corollary (3.3) for matrices Q and P. Suppose also that Q and P are left coprime in the following sense: there exist matrices R and S with entries in $\mathscr{E}'(\boldsymbol{R}^-)$ such that $Q * R + P * S = I_P$. Then we have

$$\overline{\operatorname{Im} f_A} = \{\gamma \in \Gamma^p;\ \pi(Q * \gamma) = 0\}.$$

Proof: Omitted. See [9]. □

Let us see how this theorem can be applied to compute canonical realizations.

Example (4.4) (Realization of a Retarded Delay-Differential System): Let $A(t)$ be the impulse response given by

$$A(t) := \begin{cases} 0 & \text{for } 0 \leqslant t < 1, \\ \displaystyle\sum_{i=0}^{n-1} (t-i-1)^i / i! & \text{for } n \leqslant t < n+1. \end{cases}$$

In terms of distributions, we have $A = (\delta'_{-1} - \delta)^{-1} * \delta$, which trivially satisfies the assumptions of Theorem (4.3). Therefore, if γ is smooth, it belongs to $\overline{\operatorname{Im} f_A}$ iff $\gamma'(t+1) - \gamma(t) = 0$ for all $t \geqslant 0$. In other words,

$$\gamma(t) = \gamma(1) + \int_1^t \gamma(\tau - 1)\, d\tau \quad (4.5)$$

for $1 \leqslant t < 2$. Iterating this formula successively, we see that $\gamma|_{[0,1]}$ and $\gamma(1)$ completely determine the values of $\gamma(t)$ for all $t \geqslant 0$. Taking the closure of all such γ's in Γ^1, we see that $\overline{\operatorname{Im} f_A}$ is isomorphic to $L^2[0,1] \times \boldsymbol{R}$. Denote an element of $L^2[0,1] \times \boldsymbol{R}$ by $(z(\theta), x)$ instead of $(\gamma(t), \gamma(1))$. We now want to derive the differential equation description in the following form: $dx/dt = Fx + Gu$, $y = Hx$. For this, we need only to take (F,G,H) as follows [8]: i) $F :=$ the infinitesimal generator of the semigroup $\bar{\sigma}_t$ in $\overline{\operatorname{Im} f_A}$; ii) $G := f_A(\delta) = A$; $H\gamma := \gamma(0)$ for all γ in $\overline{\operatorname{Im} f_A}$ continuous at 0. Since $\bar{\sigma}_t$ is the left shift operator in the present case, its infinitesimal generator is the differential operator d/dt. In order that $(z(\theta), x)$ be differentiable in $L^2[0,1] \times \boldsymbol{R}$, $z(\theta)$ must belong to the first-order Sobolev space $W_2^1[0,1]$ and $z(1) = x$. If the initial state is $(z(\theta), x)$, then the second coordinate at $t = \epsilon$ is given by

$$x_\epsilon = x + \int_0^\epsilon z(\theta)\, d\theta$$

by (4.5). Taking the limit $(x_\epsilon - x)/\epsilon$ as $\epsilon \to 0$, we have $F(z(\theta), x) = ((d/d\theta) z(\theta), z(0))$, and $D(F) = \{(z, x);\ z \in W_2^1[0,1],\ z(1) = x\}$. Also, $G = (0,1)$ in this representation. Finally, $H(z(\theta), x) = z(0)$ is obvious by definition. We have thus obtained the following functional differential equation description for the canonical realization of A:

$$\frac{d}{dt}\begin{pmatrix} z_t \\ x_t \end{pmatrix} = \begin{pmatrix} (\partial/\partial\theta) z_t(\theta) \\ z_t(0) \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t); \qquad y(t) = z_t(0). \quad (4.6)$$

This is nothing but the M_2-space model for such a retarded delay-differential system introduced by [1].

V. Concluding Remarks

The dual result of Theorem (4.3), which characterizes $\ker f_A$ instead of $\overline{\operatorname{Im} f_A}$, for scalar (i.e., $p = m = 1$) input/output maps has been obtained by [3] in a slightly different setting. Our result here is more consistent with our basic approach which places more emphasis on observability than reachability. We also feel that it is more suitable for obtaining differential equation descriptions.

It is of interest to point out that M_2-space models arise naturally as a result of the canonical construnction as shown in Example (4.4).

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