JOURNAL OF MATHEMATICAL PHYSICS VOLUME 43, NUMBER 12 DECEMBER 2002

Magnetic translation groups in an n-dimensional torus and their representations

Shogo Tanimura[a)]
Department of Engineering Physics and Mechanics, Kyoto University, Kyoto 606-8501, Japan

(Received 7 May 2002; accepted 6 August 2002)

A charged particle in a uniform magnetic field in a two-dimensional torus has a discrete noncommutative translation symmetry instead of a continuous commutative translation symmetry. We study topology and symmetry of a particle in a magnetic field in a torus of arbitrary dimensions. The magnetic translation group (MTG) is defined as a group of translations that leave the gauge field invariant. We show that the MTG in an n-dimensional torus is isomorphic to a central extension of a cyclic group $\mathbf{Z}_{\nu_1}\times\cdots\times\mathbf{Z}_{\nu_{2l}}\times T^m$ by $U(1)$ with $2l+m=n$. We construct and classify irreducible unitary representations of the MTG in a three-torus and apply the representation theory to three examples. We briefly describe a representation theory for a general n-torus. The MTG in an n-torus can be regarded as a generalization of the so-called noncommutative torus. © *2002 American Institute of Physics.* [DOI: 10.1063/1.1513208]

I. INTRODUCTION

Many people have been studying dynamics of an electrically charged particle in a magnetic field for various interests. Landau found that the energy spectrum of an electron becomes discrete when a magnetic field is applied, and explained the diamagnetic property of a metal. The quantum Hall effect looked a peculiar phenomenon when it was first discovered but today it is understood as a universal phenomenon observable in a two-dimensional electron system in a magnetic field. Dynamics of charged particles in a magnetic field is still an active research area.

Here we examine a group-theoretical aspect of the quantum system in a magnetic field. In particular we compare symmetry in a torus with symmetry in a Euclidean space. We would like to understand how the symmetry structure of the dynamical system is affected by the topological structure of the underlying space. It is known that the translation symmetry group becomes noncommutative when a uniform magnetic field is introduced into the Euclidean space. Moreover, the translation symmetry group becomes discrete when the underlying space is replaced by a torus. In this article we consider a vector potential

$$A=\sum_{j,k=1}^{n} x_j\omega_{jk}dx_k+\sum_{j=1}^{n} \alpha_j dx_j \tag{1.1}$$

over an n-dimensional torus $T^n=\mathbf{R}^n/\mathbf{Z}^n$. Here ω_{jk} are arbitrary integers and α_j are real numbers. Then the corresponding magnetic field is given by the two-form

$$B=dA=\sum_{j,k=1}^{n} \frac{1}{2}(\omega_{jk}-\omega_{kj})dx_j\wedge dx_k\,. \tag{1.2}$$

We conclude that the magnetic translation group (MTG) in T^n is

[a)]Electronic mail: tanimura@kues.kyoto-u.ac.jp

$$S_A=(\mathbf{R}\times_\omega\Omega^n)/(\mathbf{Z}\times_\omega\mathbf{Z}^n), \tag{1.3}$$

where Ω^n is a subgroup of $\mathbf{R}^n$ defined by $\Omega^n=\{v\in\mathbf{R}^n|(\omega-{}^t\omega)v\in\mathbf{Z}^n\}$ and the group operation in $\mathbf{R}\times_\omega\mathbf{R}^n$ is defined by

$$(x_0,x_1,\ldots,x_n)\cdot(y_0,y_1,\ldots,y_n)=\left(x_0+y_0+\sum_{j,k=1}^{n}x_j\omega_{jk}y_k,x_1+y_1,\ldots,x_n+y_n\right). \tag{1.4}$$

This characterization of the magnetic translation symmetry is one of the main results of this article. The MTG is actually a central extension of a cyclic group

$$\mathbf{Z}_{\mu_1}\times\cdots\times\mathbf{Z}_{\mu_l}\times\mathbf{Z}_{\nu_1}\times\cdots\times\mathbf{Z}_{\nu_l}\times T^m \quad (2l+m=n) \tag{1.5}$$

by $S^1=U(1)$. We build a complete set of irreducible representations of the MTG in T^3. We also describe a method to build irreducible representations of the MTG in T^n.

We would like to briefly review studies by other people on a quantum system in a magnetic field. Brown[1] found that the translation symmetry of an electron in a lattice in a uniform magnetic field is noncommutative and that the quantum system obeys a projective representation of the translation group. At the almost same time[2] and later[3] Zak built a representation theory of the lattice translation group in a magnetic field. Ashby and Miller[4] considered a space–time lattice of a finite size in uniform electric and magnetic fields and proposed an electromagnetic translation group. Avron, Herbst, and Simon have been studying spectral problems of the Schrödinger operators in a magnetic field in a series of papers.[5–8] Particularly, in Ref. 6 they examined a system of particles in a uniform magnetic field and characterized a constant of motion analogous to the total momentum. Dubrovin and Novikov[9,10] studied the spectrum of the Pauli operator in a two-dimensional lattice with a periodic magnetic field and intensively analyzed the gap structure above the ground state. Asch, Over, and Seiler[11] clarified how the inequivalent Hamiltonians on a torus in a magnetic field are induced from a Hamiltonian on the universal covering space of the torus. In a series of studies[12–17] Lulek, Florek, Lipinski, and Walcerz established a systematic method to construct central extensions of a finitely generated Abelian group. Their results are equivalent to the MTGs in a lattice. Kuwabara[18,19] is studying relations between the trajectories of a classical particle and the spectra of its quantized system and has obtained many results. Gruber[20] also examined quantization of a particle on a Riemannian manifold in a magnetic field from a view point of geometric quantization.

As reviewed above, a lot of studies on dynamics and symmetry in a magnetic field have been done. Although MTGs in a finite lattice and in an infinite lattice have been much investigated, the MTG in a torus of arbitrary dimensions is not yet fully investigated. Motivated by a recent study[21,22] on extra dimensions of the space–time, Sakamoto *et al.*[23–25] are developing field theoretical models in which the translation symmetry of an extra circle is spontaneously broken by a nontrivial boundary condition in the extra S^1. Moreover, we are developing models[26,27] in which the rotation symmetry of an extra two-sphere is spontaneously broken by a magnetic monopole in the extra S^2. So we would like to understand how a background gauge field in a compact space influences symmetry structure of a quantum system. Hence we decide to investigate symmetry in a magnetic field in a torus.

This article is organized as follows. In Sec. II we shall examine how symmetry of a quantum system in a magnetic field is changed when the underlying two-dimensional Euclidean space is replaced by a two-dimensional torus. In Sec. III we extend our discussion to an n-dimensional torus. We introduce a noncommutative group structure into $\mathbf{R}^{n+1}$ and use it to construct a magnetic fiber bundle, which is a bundle over T^n with a fiber S^1. In Sec. IV we classify topological structures of the bundles. In Sec. V we define connections, which are generalizations of a vector potential, and classify them. In Sec. VI we define a magnetic translation group as a group of lifted translations that leave the connection invariant. In Sec. VII we build a representation theory of the MTG for T^3 and illustrate the theory by a few examples. In Sec. VIII we describe an outline of the representation theory of the MTG for a general T^n. Section IX is devoted to conclusions and discussions. To reach the main result quickly the reader may read only Secs. III, V, and VI.

II. SYMMETRIES IN A MAGNETIC FIELD

This section is devoted to exercises to get ideas about the problem. The reader may skip this section and restart from Sec. III without missing the main course of the article.

A. Euclidean space

Let us begin our discussion by examining symmetry of quantum mechanics of a particle in the uniform magnetic field in $\mathbf{R}^2$. It is a well-known system and becomes a starting point to explore further nontrivial systems.

A uniform magnetic field $B dx \wedge dy = dA$ is derived from a vector potential $A = Bx\, dy$. The Schrödinger equation is

$$H\psi = \left[-\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial}{\partial y} - iBx\right)^2 \right] \psi(x,y) = E\psi. \tag{2.1}$$

Then the operators

$$\tilde{P}_x := -i\frac{\partial}{\partial x} - By, \quad P_y := -i\frac{\partial}{\partial y} \tag{2.2}$$

commute with H. These generate unitary transformations

$$(U_x(a)\psi)(x,y) = e^{-i\tilde{P}_x a}\psi(x,y) = e^{iBay}\psi(x-a,y), \tag{2.3}$$

$$(U_y(b)\psi)(x,y) = e^{-iP_y b}\psi(x,y) = \psi(x,y-b). \tag{2.4}$$

It is to be noted that $U_x(a)$ is a combination of a translation in the x-direction by the length a and a gauge transformation. It is also to be noted that the translation in the x-direction and the one in the y-direction do not commute but satisfy

$$U_x(a)U_y(b)(U_x(a))^{-1}(U_y(b))^{-1} = e^{iBab}. \tag{2.5}$$

The momentum generates a continuous symmetry and enables us to separate the variables. For example, if we put the eigenvalue of P_y as k, the wave function is factorized as

$$\psi(x,y) = e^{iky}\phi(x). \tag{2.6}$$

Then the Schrödinger equation (2.1) is rewritten as

$$H\psi = e^{iky}\left[-\frac{1}{2}\left(\frac{\partial}{\partial x}\right)^2 + \frac{1}{2}(k - Bx)^2 \right]\phi(x) = e^{iky}E\phi(x) \tag{2.7}$$

and is reduced to the equation of a harmonic oscillator. Hence the energy eigenvalues are given by

$$E = |B|(n + \tfrac{1}{2}) \quad (n = 0,1,2,\ldots) \tag{2.8}$$

and are called the Landau levels. Each eigenvalue is infinitely degenerated with respect to $-\infty < k < \infty$.

B. Torus

Next we turn to a two-dimensional torus. The two-torus T^2 is defined as the quotient space $\mathbf{R}^2/\mathbf{Z}^2$. Namely, the points in $\mathbf{R}^2$,

$$(x,y) \sim (x+1,y) \sim (x,y+1), \tag{2.9}$$

are identified as a single point in T^2. If we impose a pseudoperiodic condition

$$\psi(x+1,y) = e^{iBy}\psi(x,y), \quad \psi(x,y+1) = \psi(x,y), \tag{2.10}$$

on the wave function, the Schrödinger equation (2.1) is well defined over T^2. In other words, on the space of functions satisfying the pseudoperiodic condition, the operator H becomes self-adjoint. To make the two conditions in (2.10) compatible each other we need to have

$$\psi(x+1,y+1) = e^{iB(y+1)}\psi(x,y+1) = e^{iB}e^{iBy}\psi(x,y) = \psi(x+1,y) = e^{iBy}\psi(x,y). \tag{2.11}$$

Hence we should have $e^{iB}=1$. Namely, in the magnetic field strength

$$B = 2\pi\nu, \tag{2.12}$$

ν must be an integer. We call ν the magnetic flux number of the torus.

The operators $\tilde{P}_x$ and P_y in (2.2) commute with H defined in (2.1). However, when they act on a wave function satisfying the pseudoperiodic condition (2.10), they do not give back a function satisfying the pseudoperiodic condition but instead give

$$P_y\psi(x+1,y) = e^{iBy}(P_y+B)\psi(x,y), \tag{2.13}$$

$$\tilde{P}_x\psi(x,y+1) = (\tilde{P}_x - B)\psi(x,y). \tag{2.14}$$

Hence, the actions of these operators are not closed in the space of pseudoperiodic functions. Thus we get a lesson that *the generator of infinitesimal translation does not exist in the torus*. However, it is still possible to construct operators for finite translations. We let the finite translation operators (2.3) and (2.4) act on a pseudoperiodic function (2.10), and examine whether the resultant functions satisfy the pseudoperiodic condition. Using the flux quantization (2.12) we get

$$\begin{aligned}(U_x(a)\psi)(x,y+1) &= e^{iBa(y+1)}\psi(x-a,y+1)\\ &= e^{iBa}e^{iBay}\psi(x-a,y)\\ &= e^{2\pi i\nu a}(U_x(a)\psi)(x,y),\end{aligned} \tag{2.15}$$

$$\begin{aligned}(U_y(b)\psi)(x+1,y) &= \psi(x+1,y-b)\\ &= e^{iB(y-b)}\psi(x,y-b)\\ &= e^{-iBb}e^{iBy}\psi(x,y-b)\\ &= e^{-2\pi i\nu b}e^{iBy}(U_y(b)\psi)(x,y).\end{aligned} \tag{2.16}$$

Therefore, the transformed wave functions, $U_x(a)\psi$ and $U_y(b)\psi$, satisfy the pseudoperiodic condition (2.10) if and only if

$$\nu a, \nu b \in \mathbf{Z}. \tag{2.17}$$

Consequently, the lengths of shifts, a and b, are restricted to integral multiples of $1/\nu$. Moreover, on a pseudoperiodic function the shifts by the unit length act as

$$(U_x(1)\psi)(x,y) = e^{iBy}\psi(x-1,y) = \psi(x,y), \tag{2.18}$$

$$(U_y(1)\psi)(x,y) = \psi(x,y-1) = \psi(x,y). \tag{2.19}$$

Hence $U_x(1)$ and $U_y(1)$ are identity operators. Thus the operators $U_x(1/\nu)$ and $U_y(1/\nu)$ generate a cyclic group $\mathbf{Z}_\nu = \mathbf{Z}/\nu\mathbf{Z}$ of the order ν. However, as seen in (2.5) their commutator produces a nontrivial phase factor. Thus we conclude that the symmetry of the quantum system in the torus magnetic field is described by a projective representation of $\mathbf{Z}_\nu \times \mathbf{Z}_\nu$.

The group of translations of the quantum system in the magnetic field is called a magnetic translation group (abbreviated as MTG). A more precise definition of the MTG will be given in Sec. VI. In the torus the MTG becomes discrete and finite. Its representation is constructed as follows. Let $\{|0\rangle, |1\rangle, \ldots, |\nu-1\rangle\}$ be a basis of the representation space. Then we define the action of the translation operators by

$$U_x(n_x/\nu)|q\rangle = e^{2\pi i n_x q/\nu}|q\rangle, \tag{2.20}$$

$$U_y(n_y/\nu)|q\rangle = |q + n_y (\mathrm{mod}\ \ \nu)\rangle, \tag{2.21}$$

for $n_x, n_y \in \mathbf{Z}$. We can easily verify that they satisfy

$$U_x(n_x/\nu)U_y(n_y/\nu)(U_x(n_x/\nu))^{-1}(U_y(n_y/\nu))^{-1}|q\rangle = e^{i(2\pi\nu)(n_x/\nu)(n_y/\nu)}|q\rangle, \tag{2.22}$$

which is homomorphic to the commutator (2.5). This representation is irreducible and its dimension is ν. Hence each energy eigenvalue (2.8) is degenerated by ν folds.

C. Three-torus

Let us examine the case of a three-dimensional torus briefly to motivate further discussion. With real constants (b_1, b_2, b_3) a vector potential

$$A = b_1 x_2 dx_3 + b_2 x_3 dx_1 + b_3 x_1 dx_2 \tag{2.23}$$

gives rise to a magnetic field

$$B = dA = b_1 dx_2 \wedge dx_3 + b_2 dx_3 \wedge dx_1 + b_3 dx_1 \wedge dx_2. \tag{2.24}$$

The Hamiltonian is then given by

$$H\psi = -\frac{1}{2}\left[\left(\frac{\partial}{\partial x_1} - i b_2 x_3\right)^2 + \left(\frac{\partial}{\partial x_2} - i b_3 x_1\right)^2 + \left(\frac{\partial}{\partial x_3} - i b_1 x_2\right)^2\right]\psi(x_1, x_2, x_3). \tag{2.25}$$

On the three-torus the wave function must satisfy a set of conditions

$$\begin{aligned} \psi(x_1+1, x_2, x_3) &= e^{i b_3 x_2}\psi(x_1, x_2, x_3), \\ \psi(x_1, x_2+1, x_3) &= e^{i b_1 x_3}\psi(x_1, x_2, x_3), \\ \psi(x_1, x_2, x_3+1) &= e^{i b_2 x_1}\psi(x_1, x_2, x_3), \end{aligned} \tag{2.26}$$

which is a generalization of the the pseudoperiodic condition (2.10) of the two-torus.

We would like to find a complete set of translation operators that commute with H and are compatible with the pseudoperiodic condition (2.26). Of course, if the magnetic field is parallel to one of the axes, the system is reduced to the two-torus as has been discussed by Zak.[3] For example, if $(b_1, b_2, b_3) = (0,0,B)$, the Hamiltonian (2.25) and the condition (2.26) are reduced to (2.1) and (2.10), respectively. However, it is a highly nontrivial and not yet fully solved problem to find a complete symmetry group for an inclined magnetic field (b_1, b_2, b_3). Thus we decide to develop a more systematic method to construct the translation symmetry group for a generic magnetic field in the n-torus.

III. MAGNETIC FIBER BUNDLE

We shall extend the previous consideration on the two-dimensional torus to arbitrary dimensions. What we will do in the rest of this article is to construct $U(1)$ principal fiber bundles over an n-dimensional torus T^n, to classify the bundles, to introduce $U(1)$ connections with constant curvatures over T^n, to define the MTG as the stability group of each connection, and to construct the representations of the MTGs. Throughout this article we are identifying S^1 with $U(1)$.

Let us begin with construction of S^1 principal fiber bundles over T^n. For this purpose we introduce a noncommutative group structure into $\mathbf{R}^{n+1}$ as follows. Take an $n\times n$ matrix ω which consists of integers, $\omega_{jk}\in\mathbf{Z}$ $(j,k=1,\ldots,n)$. The matrix ω is not necessarily antisymmetric. Define a product of $(x_0,x_1,\ldots,x_n),(y_0,y_1,\ldots,y_n)\in\mathbf{R}^{n+1}$ by

$$(x_0,x_1,\ldots,x_n)\cdot(y_0,y_1,\ldots,y_n):=\left(x_0+y_0+\sum_{j,k=1}^{n}x_j\omega_{jk}y_k,x_1+y_1,\ldots,x_n+y_n\right). \tag{3.1}$$

In the following we abbreviate the notation of the vectors as $x=(x_1,\ldots,x_n)\in\mathbf{R}^n$. We write the inner product of vectors as $xy=\Sigma_{j=1}^n x_j\, y_j$ and the bilinear form as $x\omega y=\Sigma_{j,k=1}^n x_j\omega_{jk}y_k$. It is easily verified that the set $\mathbf{R}^{n+1}$ becomes a group with this product operation; the associativity is satisfied as

$$\begin{aligned}((x_0,x)\cdot(y_0,y))\cdot(z_0,z)&=(x_0+y_0+x\omega y,x+y)\cdot(z_0,z)\\&=(x_0+y_0+z_0+x\omega y+(x+y)\omega z,(x+y)+z)\\&=(x_0+y_0+z_0+x\omega y+x\omega z+y\omega z,x+y+z)\\&=(x_0+y_0+z_0+x\omega(y+z)+y\omega z,x+(y+z))\\&=(x_0,x)\cdot((y_0,y)\cdot(z_0,z)),\end{aligned} \tag{3.2}$$

the unit element is given by $(0,0)\in\mathbf{R}\times\mathbf{R}^n$, and the inverse element of $(x_0,x)\in\mathbf{R}\times\mathbf{R}^n$ is given by

$$(x_0,x)^{-1}=(-x_0+x\omega x,-x). \tag{3.3}$$

The set $\mathbf{R}^{n+1}$ equipped with this group structure is denoted by $\mathbf{R}\times_\omega\mathbf{R}^n$. A commutator is calculated as

$$\begin{aligned}(x_0,x)\cdot(y_0,y)\cdot(x_0,x)^{-1}\cdot(y_0,y)^{-1}&=(x_0+y_0+x\omega y,x+y)\cdot(-x_0+x\omega x,-x)\cdot(-y_0+y\omega y,-y)\\&=(y_0+x\omega y+x\omega x-(x+y)\omega x,y)\cdot(-y_0+y\omega y,-y)\\&=(x\omega y+x\omega x-(x+y)\omega x+y\omega y-y\omega y,0)=(x\omega y-y\omega x,0),\end{aligned} \tag{3.4}$$

and therefore $\mathbf{R}\times_\omega\mathbf{R}^n$ is Abelian if and only if ω is a symmetric matrix. The natural projection map $\mathbf{R}\times_\omega\mathbf{R}^n\to\mathbf{R}^n$ becomes a group homomorphism. As its kernel $\mathbf{R}\times_\omega\{0\}$ is contained in the center of $\mathbf{R}\times_\omega\mathbf{R}^n$, the group $\mathbf{R}\times_\omega\mathbf{R}^n$ is a central extension of $\mathbf{R}^n$ by $\mathbf{R}$.

The subset $\mathbf{Z}\times_\omega\mathbf{Z}^n=\{(m_0,m_1,\ldots,m_n)|m_0,m_j\in\mathbf{Z}\}$ is also a subgroup of $\mathbf{R}\times_\omega\mathbf{R}^n$ but it is not isomorphic to the standard Abelian group $\mathbf{Z}^{n+1}$. The subgroup $\mathbf{Z}\times_\omega\mathbf{Z}^n$ acts freely on $\mathbf{R}\times_\omega\mathbf{R}^n$ from the left via the group operation. Hence the space of orbits

$$P_\omega^{n+1}:=(\mathbf{Z}\times_\omega\mathbf{Z}^n)\backslash(\mathbf{R}\times_\omega\mathbf{R}^n) \tag{3.5}$$

becomes a smooth manifold.

The group operation also induces action of the group $\mathbf{R}\times_\omega\mathbf{R}^n$ on the space P_ω^{n+1} from the right. The subgroups $\mathbf{Z}\times_\omega\{0\}\subset\mathbf{R}\times_\omega\{0\}$ are contained in the center of $\mathbf{R}\times_\omega\mathbf{R}^n$ and hence their actions from the right are equivalent to those from the left. The subgroups $\mathbf{R}\times_\omega\{0\}$ and $\mathbf{Z}$

$\times_{\omega}\{0\}$ are isomorphic to $\mathbf{R}$ and $\mathbf{Z}$, respectively. Thus $\mathbf{R}$ acts on P_{ω}^{n+1} but its subgroup $\mathbf{Z}$ acts trivially on P_{ω}^{n+1} since $\mathbf{Z}$ is contained in the dividing group $\mathbf{Z}\times_{\omega}\mathbf{Z}^n$ of the quotient space (3.5). Therefore the action of $\mathbf{R}$ is reduced to the effective action of $S^1=\mathbf{R}/\mathbf{Z}$ on P_{ω}^{n+1}. The space of orbit P_{ω}^{n+1}/S^1 is diffeomorphic to a torus T^n. Consequently we obtain a principal fiber bundle with the canonical projection map $\pi_{\omega}:P_{\omega}^{n+1}\to T^n$ with a structure group S^1. We call this fiber bundle a *magnetic fiber bundle twisted by the matrix* ω. The procedure to construct the magnetic fiber bundle is summarized by the following commutative diagram:

$$\begin{array}{ccccc}
\mathbf{Z}\times_{\omega}\{0\} & \rightarrow & \mathbf{Z}\times_{\omega}\mathbf{Z}^n & \rightarrow & \mathbf{Z}^n \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{R}\times_{\omega}\{0\} & \rightarrow & \mathbf{R}\times_{\omega}\mathbf{R}^n & \rightarrow & \mathbf{R}^n \\
\downarrow & & \downarrow & & \downarrow \\
S^1 & \rightarrow & P_{\omega}^{n+1} & \underset{\pi_{\omega}}{\rightarrow} & T^n
\end{array}. \tag{3.6}$$

A function $f:P_{\omega}^{n+1}\to\mathbf{C}$ is identified with a function $f:\mathbf{R}\times_{\omega}\mathbf{R}^n\to\mathbf{C}$ that is invariant under action of $\mathbf{Z}\times_{\omega}\mathbf{Z}^n$ from the left as

$$f(m_0+x_0+m\omega x,m+x)=f(x_0,x),\quad (m_0,m)\in\mathbf{Z}\times_{\omega}\mathbf{Z}^n. \tag{3.7}$$

Moreover, when the function $f:P_{\omega}^{n+1}\to\mathbf{C}$ satisfies

$$f(x_0+t,x)=e^{-2\pi it}f(x_0,x),\quad t\in\mathbf{R}, \tag{3.8}$$

it is called an equivariant function on P_{ω}^{n+1}. Hence the equivariant function f has the property

$$f(x_0,x+m)=e^{2\pi im\omega x}f(x_0,x),\quad m\in\mathbf{Z}^n. \tag{3.9}$$

This is a generalization of the pseudoperiodic condition (2.10),

$$\psi(x+1,y)=e^{2\pi i\nu y}\psi(x,y),\quad \psi(x,y+1)=\psi(x,y). \tag{3.10}$$

In fact, if we take the matrix

$$\omega=\begin{pmatrix} 0 & \nu \\ 0 & 0 \end{pmatrix}, \tag{3.11}$$

the general condition (3.9) of T^n is reduced to the specific one (3.10) of T^2.

IV. EQUIVALENT MAGNETIC BUNDLES

In the above construction each magnetic fiber bundle is specified by an integral matrix ω. However, it can happen that different matrices ω and ω' give rise to equivalent fiber bundles. In this section we prove that ω and ω' induce equivalent fiber bundles if and only if the difference $\omega'-\omega$ is a symmetric integral matrix. Therefore, we may choose a representative matrix ω such that $\omega_{jk}=0$ for $j\geqslant k$. Namely, the upper triangle matrix

$$\omega=\begin{pmatrix}
0 & \omega_{12} & \omega_{13} & \cdots & \omega_{1,n-1} & \omega_{1n} \\
0 & 0 & \omega_{23} & \cdots & \omega_{2,n-1} & \omega_{2n} \\
0 & 0 & 0 & \cdots & \omega_{3,n-1} & \omega_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \omega_{n-1,n} \\
0 & 0 & 0 & \cdots & 0 & 0
\end{pmatrix} \tag{4.1}$$

with integers ω_{jk} can be taken as a standard form of the matrix ω. The reader will not miss the main result of the article even if he skips this section and restarts from Sec. V.

Here we introduce three kinds of isomorphisms that convert a bundle specified by a matrix ω to a bundle specified by another matrix ω'.

Let us introduce the first kind of bundle isomorphism. When a symmetric matrix σ of integral elements, $\sigma_{jk}=\sigma_{kj}\in\mathbf{Z}$, satisfies

$$\sum_{j,k=1}^{n} m_j\sigma_{jk}m_k\in 2\mathbf{Z} \tag{4.2}$$

for any $m=(m_1,\ldots,m_n)\in\mathbf{Z}^n$, we call σ an even symmetric matrix. This requirement for σ is equivalent to demanding that the off-diagonal elements σ_{jk} are integers and that the diagonal elements σ_{jj} are even integers. Here we will show that two magnetic bundles P_{ω}^{n+1} and $P_{\omega+\sigma}^{n+1}$ are isomorphic each other for any even symmetric matrix σ. For this purpose let us define a map $\phi_\sigma:\mathbf{R}\times_\omega\mathbf{R}^n\to\mathbf{R}\times_{\omega+\sigma}\mathbf{R}^n$ by

$$\phi_\sigma(x_0,x):=(x_0+\tfrac{1}{2}x\sigma x,x). \tag{4.3}$$

Existence of the inverse map is obvious; it is given by $\phi_\sigma^{-1}(x_0,x)=(x_0-\frac{1}{2}x\sigma x,x)$. It is easily verified that the map ϕ_σ is a group isomorphism as

$$\begin{aligned}\phi_\sigma((x_0,x)\cdot_\omega(y_0,y))&=\phi_\sigma(x_0+y_0+x\omega y,x+y)\\&=(x_0+y_0+x\omega y+\tfrac{1}{2}(x+y)\sigma(x+y),x+y)\\&=(x_0+y_0+\tfrac{1}{2}x\sigma x+\tfrac{1}{2}y\sigma y+x(\omega+\sigma)y,x+y)\\&=(x_0+\tfrac{1}{2}x\sigma x,x)\cdot_{\omega+\sigma}(y_0+\tfrac{1}{2}y\sigma y,y)\\&=\phi_\sigma(x_0,x)\cdot_{\omega+\sigma}\phi_\sigma(y_0,y),\end{aligned} \tag{4.4}$$

where we have distinguished the product operation of $\mathbf{R}\times_{\omega+\sigma}\mathbf{R}^n$ from that of $\mathbf{R}\times_\omega\mathbf{R}^n$. The map ϕ_σ sends the integer subgroup $\mathbf{Z}\times_\omega\mathbf{Z}^n$ to $\mathbf{Z}\times_{\omega+\sigma}\mathbf{Z}^n$, since σ is even as required in (4.2). Therefore, ϕ_σ induces a diffeomorphism

$$(\phi_\sigma)_*:(\mathbf{Z}\times_\omega\mathbf{Z}^n)\backslash(\mathbf{R}\times_\omega\mathbf{R}^n)\to(\mathbf{Z}\times_{\omega+\sigma}\mathbf{Z}^n)\backslash(\mathbf{R}\times_{\omega+\sigma}\mathbf{R}^n). \tag{4.5}$$

Moreover, since ϕ_σ is the identity map when it is restricted on $\mathbf{R}\times_\omega\{0\}$,

$$\phi_\sigma((t,0)\cdot_\omega(x_0,x))=\phi_\sigma(t,0)\cdot_{\omega+\sigma}\phi_\sigma(x_0,x)=(t,0)\cdot_{\omega+\sigma}\phi_\sigma(x_0,x), \tag{4.6}$$

thus $(\phi_\sigma)_*$ is equivariant with respect to the action of S^1. It is also clear that $\pi_\omega=\pi_{\omega+\sigma}\circ(\phi_\sigma)_*$. Thus we conclude that the map $(\phi_\sigma)_*$ is an isomorphism between the principal fiber bundles P_ω^{n+1} and $P_{\omega+\sigma}^{n+1}$.

Next we shall introduce the second kind of bundle isomorphism. We identify a diagonal matrix $\Delta=\mathrm{diag}(\Delta_1,\Delta_2,\ldots,\Delta_n)$ with a vector $\Delta=(\Delta_1,\Delta_2,\ldots,\Delta_n)\in\mathbf{Z}^n$. Then we define a map $\phi_\Delta:\mathbf{R}\times_\omega\mathbf{R}^n\to\mathbf{R}\times_{\omega+\Delta}\mathbf{R}^n$ by

$$\phi_\Delta(x_0,x):=\left(x_0+\frac{1}{2}x\Delta x+\frac{1}{2}\Delta x,x\right)=\left(x_0+\frac{1}{2}\sum_{j=1}^{n}(x_j\Delta_jx_j+\Delta_jx_j),x\right). \tag{4.7}$$

It is also easily verified that ϕ_Δ is a group isomorphism as

$$\begin{aligned}\phi_\Delta((x_0,x)\cdot_\omega(y_0,y)) &= \phi_\Delta(x_0+y_0+x\omega y,x+y)\\ &= (x_0+y_0+x\omega y+\tfrac{1}{2}(x+y)\Delta(x+y)+\tfrac{1}{2}\Delta(x+y),x+y)\\ &= (x_0+y_0+\tfrac{1}{2}x\Delta x+\tfrac{1}{2}\Delta x+\tfrac{1}{2}y\Delta y+\tfrac{1}{2}\Delta y+x(\omega+\Delta)y,x+y)\\ &= (x_0+\tfrac{1}{2}x\Delta x+\tfrac{1}{2}\Delta x,x)\cdot_{\omega+\Delta}(y_0+\tfrac{1}{2}y\Delta y+\tfrac{1}{2}\Delta y,y)\\ &= \phi_\Delta(x_0,x)\cdot_{\omega+\Delta}\phi_\Delta(y_0,y). \end{aligned} \tag{4.8}$$

Note that when x_j is an integer, $x_j^2+x_j=x_j(x_j+1)$ is always an even integer and hence $\frac{1}{2}\Delta_j(x_j^2+x_j)$ is an integer. Therefore the map ϕ_Δ sends the integer subgroup $\mathbf{Z}\times_\omega\mathbf{Z}^n$ to $\mathbf{Z}\times_{\omega+\Delta}\mathbf{Z}^n$. Moreover, ϕ_Δ sends $\mathbf{R}\times_\omega\{0\}$ to $\mathbf{R}\times_{\omega+\Delta}\{0\}$ identically. Thus the induced map $(\phi_\Delta)_*$ becomes an isomorphism between the principal fiber bundles P_ω^{n+1} and $P_{\omega+\Delta}^{n+1}$.

There is the third kind of bundle isomorphism, which will be used when we classify connections later. For each $\varepsilon=(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)\in\mathbf{Z}^n$ we define a map $\phi_\varepsilon:\mathbf{R}\times_\omega\mathbf{R}^n\to\mathbf{R}\times_\omega\mathbf{R}^n$ by

$$\phi_\varepsilon(x_0,x):=(x_0+\varepsilon x,x)=\left(x_0+\sum_{j=1}^n\varepsilon_j x_j,x\right). \tag{4.9}$$

It is also easily verified that ϕ_ε is a group isomorphism as

$$\begin{aligned}\phi_\varepsilon((x_0,x)\cdot_\omega(y_0,y)) &= \phi_\varepsilon(x_0+y_0+x\omega y,x+y)\\ &= (x_0+y_0+x\omega y+\varepsilon(x+y),x+y)\\ &= (x_0+\varepsilon x+y_0+\varepsilon y+x\omega y,x+y)\\ &= (x_0+\varepsilon x,x)\cdot_\omega(y_0+\varepsilon y,y)=\phi_\varepsilon(x_0,x)\cdot_\omega\phi_\varepsilon(y_0,y). \end{aligned} \tag{4.10}$$

The map ϕ_ε sends the integer subgroup $\mathbf{Z}\times_\omega\mathbf{Z}^n$ to $\mathbf{Z}\times_\omega\mathbf{Z}^n$. Moreover, ϕ_ε sends $\mathbf{R}\times_\omega\{0\}$ to $\mathbf{R}\times_\omega\{0\}$ identically. Thus the group isomorphism ϕ_ε induces an automorphism $(\phi_\varepsilon)_*$ of the principal fiber bundle P_ω^{n+1}.

As a summary, we write down a combined isomorphism of the three kinds of maps

$$(\phi_\varepsilon\circ\phi_\Delta\circ\phi_\sigma)(x_0,x):=(x_0+\tfrac{1}{2}x(\sigma+\Delta)x+\tfrac{1}{2}\Delta x+\varepsilon x,x). \tag{4.11}$$

By adding an integral diagonal matrix Δ to an even symmetric matrix σ, we can make any integral symmetric matrix $\sigma'=\sigma+\Delta$. Therefore, by combining the first and second kinds of isomorphisms, ϕ_σ and ϕ_Δ, we can establish an isomorphism between P_ω^{n+1} and $P_{\omega+\sigma'}^{n+1}$ for any integral symmetric matrix σ'. In other words, the set of magnetic fiber bundles has a one-to-one correspondence with $\mathrm{Mat}(n,\mathbf{Z})/\mathrm{Sym}(n,\mathbf{Z})$, where the quotient is taken in the sense of additive groups.

V. CONNECTION

In this section we define the vector potentials that yield uniform magnetic fields in an n-dimensional torus. We use the words, a vector potential, a gauge field, and a connection, to describe the same notion. Magnetic field strength and curvature are an identical notion.

Let us define a differential one-form A on $\mathbf{R}\times_\omega\mathbf{R}^n$ by

$$A:=-dx_0+\sum_{j,k=1}^n x_j\omega_{jk}dx_k+\sum_{j=1}^n\alpha_j dx_j=-dx_0+x\omega dx+\alpha dx \tag{5.1}$$

with a real vector $\alpha\in\mathbf{R}^n$. These parameters $\alpha=(\alpha_1,\ldots,\alpha_n)$ characterize the Aharonov–Bohm effect. The action of $(m_0,m)\in\mathbf{Z}\times_\omega\mathbf{Z}^n$ from the left of $\mathbf{R}\times_\omega\mathbf{R}^n$ defines a map $\varphi:(x_0,x)\mapsto(m_0+x_0+m\omega x,m+x)$. Note that the one-form A is invariant under the transformation by φ as

$$\varphi^* A = -(dx_0 + m\omega dx) + (m+x)\omega dx + \alpha dx = A. \tag{5.2}$$

Thus A can be regarded as a one-form on $P_\omega^{n+1} = (\mathbf{Z}\times_\omega \mathbf{Z}^n)\backslash(\mathbf{R}\times_\omega \mathbf{R}^n)$. It is also obvious that A is invariant under a transformation $(x_0, x) \mapsto (x_0 + t, x)$ for any $t \in \mathbf{R}$. Moreover, A satisfies

$$\left\langle \frac{\partial}{\partial x_0}, A \right\rangle = -1 \tag{5.3}$$

by the definition. In the above equation, $\langle \cdot , \cdot \rangle$ denotes the pairing of a vector and a one-form. Thus A satisfies the axiom of a connection form of the principal bundle $\pi_\omega : P_\omega^{n+1} \to T^n$.

We can classify the connections using isomorphism maps introduced in the last section. The connection $A_{\omega,\alpha}$ defined by (5.1) is parametrized by an integral matrix $\omega \in \mathrm{Mat}(n,\mathbf{Z})$ and a real vector $\alpha \in \mathbf{R}^n$. For any even symmetric matrix $\sigma \in \mathrm{EvenSym}(n,\mathbf{Z})$ and integral vectors $\Delta, \varepsilon \in \mathbf{Z}^n$, the combined isomorphism (4.11) induces a transformation

$$\begin{aligned}
(\phi_\varepsilon \circ \phi_\Delta \circ \phi_\sigma)^* & A_{\omega+\sigma+\Delta, \alpha + 1/2\,\Delta + \varepsilon} \\
&= -d(x_0 + \tfrac{1}{2} x(\sigma+\Delta)x + \tfrac{1}{2}\Delta x + \varepsilon x) + x(\omega+\sigma+\Delta)dx + (\alpha + \tfrac{1}{2}\Delta + \varepsilon)dx \\
&= -dx_0 - x(\sigma+\Delta)dx - \tfrac{1}{2}\Delta dx - \varepsilon dx + x(\omega+\sigma+\Delta)dx + (\alpha + \tfrac{1}{2}\Delta + \varepsilon)dx \\
&= -dx_0 + x\omega dx + \alpha dx \\
&= A_{\omega,\alpha}
\end{aligned} \tag{5.4}$$

via pullback. Thus the connections are classified by the equivalence relation

$$(\omega,\alpha) \sim (\omega+\sigma+\Delta, \alpha + \tfrac{1}{2}\Delta + \varepsilon), \quad \sigma \in \mathrm{EvenSym}(n,\mathbf{Z}); \Delta, \varepsilon \in \mathbf{Z}^n \tag{5.5}$$

among $(\omega,\alpha) \in \mathrm{Mat}(n,\mathbf{Z}) \times \mathbf{R}^n$.

Next we define a covariant derivative of the equivariant function f by

$$Df := df - 2\pi i A f. \tag{5.6}$$

Of course, on the right-hand side, $i = \sqrt{-1}$. The curvature form F is defined by

$$F := dA = \sum_{j,k=1}^{n} \omega_{jk} dx_j \wedge dx_k = \sum_{j,k=1}^{n} \frac{1}{2}(\omega_{jk} - \omega_{kj}) dx_j \wedge dx_k, \tag{5.7}$$

which gives a constant magnetic field. Hence the first Chern class is uniquely specified by the integral antisymmetrized matrix $(\omega - {}^t\omega)$. It is known[28] that an S^1-fiber bundle has a one-to-one correspondence with the first Chern class. Therefore, by choosing $\omega \in \mathrm{Mat}(n,\mathbf{Z})$ appropriately, we can construct any principal fiber bundles over T^n with the fiber S^1.

VI. MAGNETIC TRANSLATION GROUP

Now we shall examine translation symmetry of the vector potential A of the uniform magnetic field. In this section we shall give a precise definition of the MTG in T^n and express the MTG in a more concrete form. We will prove that the MTG is

$$S_A = (\mathbf{R}\times_\omega \Omega^n)/(\mathbf{Z}\times_\omega \mathbf{Z}^n), \tag{6.1}$$

where Ω^n is a subgroup of $\mathbf{R}^n$ defined by $\Omega^n = \{ v \in \mathbf{R}^n \,|\, (\omega - {}^t\omega) v \in \mathbf{Z}^n \}$ and the group operation is taken in the sense of (3.1). This is one of the main results of this article.

We begin by defining the MTG. A vector $v \in \mathbf{R}^n$ generates a translation of T^n by

$$\tau_v : T^n \to T^n, \quad x \mapsto x + v. \tag{6.2}$$

When a map $\tilde{\tau}_v : P^{n+1}_\omega \to P^{n+1}_\omega$ satisfies the commutative diagram

$$\begin{array}{ccc} & S^1 & \\ \swarrow & & \searrow \\ P^{n+1}_\omega & \xrightarrow{\tilde{\tau}} & P^{n+1}_\omega \\ \pi_\omega \downarrow & & \downarrow \pi_\omega \\ T^n & \xrightarrow{\tau} & T^n \end{array} \tag{6.3}$$

the map $\tilde{\tau}_v$ is called a lift of the translation τ_v. The lifted translations that leave the connection A invariant form a group

$$S_A := \{ \tilde{\tau}_v : P^{n+1}_\omega \to P^{n+1}_\omega | v \in \mathbf{R}^n, \pi_\omega \circ \tilde{\tau}_v = \tau_v \circ \pi_\omega ; \ \ \tilde{\tau}^*_v A = A \}. \tag{6.4}$$

We call it the stability group of A, or the magnetic translation group (MTG).

Let us write down the lifted translation in a more explicit form. We use $(x_0, x) \in \mathbf{R}^{n+1}$ as a coordinate of P^{n+1}_ω. Since $\pi_\omega \circ \tilde{\tau}_v = \tau_v \circ \pi_\omega$, the lift $\tilde{\tau}_v$ of (6.2) must have the form

$$\tilde{\tau}_v : (x_0, x) \mapsto (x_0 + \theta(x_0, x, v), x + v). \tag{6.5}$$

To make $\tilde{\tau}_v$ commutative with the action of $e^{2\pi i w_0} \in S^1$ the function θ must satisfy

$$x_0 + w_0 + \theta(x_0 + w_0, x, v) = x_0 + \theta(x_0, x, v) + w_0, \tag{6.6}$$

namely, θ must satisfy

$$\theta(x_0 + w_0, x, v) = \theta(x_0, x, v) \tag{6.7}$$

for any $w_0 \in \mathbf{R}$. Therefore, the function θ is independent of x_0. To become a map of P^{n+1}_ω, the map $\tilde{\tau}_v$ must send an orbit of the left-action of $\mathbf{Z} \times_\omega \mathbf{Z}^n$ to an orbit of the same group. In other words, for any $(m_0, m) \in \mathbf{Z} \times_\omega \mathbf{Z}^n$ there must exist an element $(m_0', m') \in \mathbf{Z} \times_\omega \mathbf{Z}^n$ that satisfies

$$\tilde{\tau}_v((m_0, m) \cdot (x_0, x)) = (m_0', m') \cdot \tilde{\tau}_v(x_0, x). \tag{6.8}$$

The above equation is rewritten as

$$(m_0 + x_0 + m\omega x + \theta(m + x, v), m + x + v) = (m_0' + x_0 + \theta(x, v) + m'\omega(x + v), m' + x + v),$$

which is equivalent to a set of equations

$$m = m', \tag{6.9}$$

$$m_0 + m\omega x + \theta(m + x, v) = m_0' + \theta(x, v) + m'\omega(x + v). \tag{6.10}$$

The last equation implies that

$$\theta(m + x, v) - \theta(x, v) - m\omega v = m_0' - m_0 \in \mathbf{Z} \tag{6.11}$$

for any $m \in \mathbf{Z}^n$. In reverse order, any function $\theta(x, v)$ satisfying the condition (6.11) defines a lifted translation $\tilde{\tau}_v$ by (6.5). The lifted translation $\tilde{\tau}_v$ is actually a combination of a spatial shift by v with a gauge transformation by θ. Hence, we finish characterizing the lifted translations.

Let the lifted translation $\tilde{\tau}_v$ act on the connection form A of (5.1) via pull-back. Then it gives

$$\tilde{\tau}_v^* A = -(dx_0 + d\theta) + (x+v)\omega d(x+v) + \alpha d(x+v) = A - d\theta + v\omega dx. \tag{6.12}$$

Hence, to leave the connection invariant as $\tilde{\tau}_v^* A = A$, the function θ must satisfy a differential equation $d\theta = v\omega dx$. Thus we have

$$\theta(x,v) = v\omega x + v_0 \tag{6.13}$$

with a constant $v_0 \in \mathbf{R}$. To make $\tilde{\tau}_v$ a map of P_ω^{n+1}, the function θ must satisfy the condition (6.11), which requires that

$$\theta(m+x,v) - \theta(x,v) - m\omega v = v\omega m - m\omega v = -m(\omega - {}^t\omega)v \in \mathbf{Z} \tag{6.14}$$

for any $m \in \mathbf{Z}^n$. Therefore, the vector $v \in \mathbf{R}^n$ is required to satisfy

$$(\omega - {}^t\omega)v \in \mathbf{Z}^n. \tag{6.15}$$

We call the vector v satisfying (6.15) a magnetic shift. A set of the magnetic shifts is denoted by

$$\Omega^n := \{ v \in \mathbf{R}^n | (\omega - {}^t\omega)v \in \mathbf{Z}^n \}. \tag{6.16}$$

The set Ω^n becomes an additive subgroup of $\mathbf{R}^n$. When the antisymmetrized matrix $(\omega - {}^t\omega)$ is nondegenerated, Ω^n is discrete. The lifted translation $\tilde{\tau}_v$ defined by (6.5) with (6.13) becomes

$$\tilde{\tau}_v : (x_0, x) \mapsto (x_0 + \theta(x_0, x, v), x+v) = (x_0 + v\omega x + v_0, x+v) = (v_0, v)\cdot(x_0, x), \tag{6.17}$$

and therefore the action of $\tilde{\tau}_v$ is identified with the action of $(v_0, v) \in \mathbf{R} \times_\omega \Omega^n$ on P_ω^{n+1} from the left. However, the subgroup $\mathbf{Z} \times_\omega \mathbf{Z}^n \subset \mathbf{R} \times_\omega \Omega^n$ acts on P_ω^{n+1} trivially. Thus the stability group S_A of the connection A is identified as

$$S_A = (\mathbf{R} \times_\omega \Omega^n)/(\mathbf{Z} \times_\omega \mathbf{Z}^n). \tag{6.18}$$

This is one of the main results of this article. Note that S_A is a central extension of a compact Abelian group $\Omega^n/\mathbf{Z}^n$ by $S^1 = \mathbf{R}/\mathbf{Z}$.

Actually, there is another way to characterize the group S_A. The group $\mathbf{R} \times_\omega \Omega^n$ is a normalizer of $N = \mathbf{Z} \times_\omega \mathbf{Z}^n$ in $G = \mathbf{R} \times_\omega \mathbf{R}^n$. In other words, the subgroup H defined by

$$H := \{ h \in G | \forall n \in N, hnh^{-1} \in N \} \tag{6.19}$$

coincides with $\mathbf{R} \times_\omega \Omega^n$. The above statement is easily proved as follows. A straightforward calculation yields

$$(x_0, x)\cdot(m_0, m)\cdot(x_0, x)^{-1} = (m_0 + x(\omega - {}^t\omega)m, m). \tag{6.20}$$

Therefore the necessary and sufficient condition for $(x_0, x) \in \mathbf{R} \times_\omega \mathbf{R}^n$ to bring the above element into $N = \mathbf{Z} \times_\omega \mathbf{Z}^n$ is that $(\omega - {}^t\omega)x \in \mathbf{Z}^n$, or that $x \in \Omega^n$. Thus $N = \mathbf{Z} \times_\omega \mathbf{Z}^n$ is a normal subgroup of $H = \mathbf{R} \times_\omega \Omega^n$, and hence the quotient group $S_A = H/N$ is well defined.

VII. REPRESENTATIONS OF THE MTG IN A THREE-TORUS

A unitary representation theory of the MTG is significant for spectral analyses of the Laplace operator and the Dirac operator in a background gauge field. In this section we examine a three-dimensional torus and construct a complete set of representations. This is another main result of this article. In the next section we will discuss an outline of the representation theory of the MTG for arbitrary dimensions.

A. Method

The MTG was identified as $S_A=(\mathbf{R}\times_\omega\Omega^n)/(\mathbf{Z}\times_\omega\mathbf{Z}^n)$ at (6.18). We would like to express the MTG in terms of generators and relations. Here we concentrate on the three-dimensional torus. Let us take the matrix

$$\omega=\begin{pmatrix} 0 & b_3 & -b_2 \\ 0 & 0 & b_1 \\ 0 & 0 & 0 \end{pmatrix} \tag{7.1}$$

with positive integers b_1, b_2, and b_3. Then antisymmetrization of ω yields

$$\omega-{}^t\omega=\begin{pmatrix} 0 & b_3 & -b_2 \\ -b_3 & 0 & b_1 \\ b_2 & -b_1 & 0 \end{pmatrix}. \tag{7.2}$$

The characteristic equation of $(\omega-{}^t\omega)$ is

$$\det(\lambda-(\omega-{}^t\omega))=\lambda(\lambda^2+b_1^2+b_2^2+b_3^2). \tag{7.3}$$

Hence, its eigenvalues are

$$\lambda=0,\pm iB \tag{7.4}$$

with $B:=\sqrt{b_1^2+b_2^2+b_3^2}$. We assume that $B\neq 0$. The eigenspace for $\lambda=0$ is spanned by

$$b=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{7.5}$$

The action of $(\omega-{}^t\omega)$ on a vector $v\in\mathbf{R}^3$ is equivalent to the vector product $(\omega-{}^t\omega)v=v\times b$. The magnetic shift group (6.16) now becomes

$$\Omega^3=\{v\in\mathbf{R}^3|(\omega-{}^t\omega)v\in\mathbf{Z}^3\}. \tag{7.6}$$

The linear subspace $\mathbf{R}b$ spanned by b of (7.5) is a subgroup of Ω^3.Let us define a generator e_0 by

$$D_0:=\mathrm{GCD}\{b_1,b_2,b_3\}, \tag{7.7}$$

$$e_0:=\frac{1}{D_0}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}. \tag{7.8}$$

Here the GCD is an abbreviation of the greatest common divisor while the LCM is an abbreviation of the least common multiple. It is obvious that e_0 is in $\mathbf{Z}^3$ and that $(\omega-{}^t\omega)e_0=0$. The vector e_0 is a minimal integral vector in the sense that there is no real number s such that $0<s<1$ and $se_0\in\mathbf{Z}^3$. There exist other vectors e_1, e_2, $\in\mathbf{Q}^3$ that generate Ω^3 as

$$\Omega^3=\mathbf{R}e_0\oplus\mathbf{Z}e_1\oplus\mathbf{Z}e_2. \tag{7.9}$$

Here $\mathbf{Q}$ is the whole set of rational numbers. From (7.6) these generators e_1 and e_2 must satisfy

$$(\omega-{}^t\omega)e_1,(\omega-{}^t\omega)e_2\in\mathbf{Z}^3. \tag{7.10}$$

These vectors $\{e_1,e_2\}$ are minimal magnetic shifts in the sense that there is no real number s such that

$$0<s<1, \quad s(\omega-{}^t\omega)e_i\in\mathbf{Z}^3, \tag{7.11}$$

for each $i=1,2$. Moreover, there are positive integers ν_1 and ν_2 such that

$$\nu_1 e_1,\nu_2 e_2\in\mathbf{Z}^3. \tag{7.12}$$

We demand that the integers $\{\nu_1,\nu_2\}$ are the smallest cycles in the sense that there is no integer m such that

$$0<m<\nu_i, \quad m e_i\in\mathbf{Z}^3, \tag{7.13}$$

for each $i=1,2$. Consequently, the decomposition (7.9) yields

and

$$\Omega^3/\mathbf{Z}^3=(\mathbf{R}/\mathbf{Z})\oplus(\mathbf{Z}/\nu_1\mathbf{Z})\oplus(\mathbf{Z}/\nu_2\mathbf{Z}). \tag{7.14}$$

Thus an arbitrary element g of $S_A=(\mathbf{R}\times_\omega\Omega^3)/(\mathbf{Z}\times_\omega\mathbf{Z}^3)$ is parametrized as

$$g=(s,te_0+n_1e_1+n_2e_2), \quad s,t\in\mathbf{R}/\mathbf{Z};n_1\in\mathbf{Z}/\nu_1\mathbf{Z};n_2\in\mathbf{Z}/\nu_2\mathbf{Z}. \tag{7.15}$$

Let us examine the commutator (3.4). It is clear that the element $(s,0)$ commutes with any element. Since $(\omega-{}^t\omega)e_0=0$, the element $(0,te_0)$ also commutes with any element. On the other hand, $(0,e_1)$ and $(0,e_2)$ produce a nonvanishing commutator

$$(0,e_1)\cdot(0,e_2)\cdot(0,e_1)^{-1}\cdot(0,e_2)^{-1}=(\gamma,0) \tag{7.16}$$

with

$$\gamma:=e_1(\omega-{}^t\omega)e_2=e_1\cdot(e_2\times b). \tag{7.17}$$

From (7.10) and (7.12) we can see that

$$\nu_1\gamma,\nu_2\gamma\in\mathbf{Z}. \tag{7.18}$$

Hence γ is a rational number. Let d be the greatest common divisor of ν_1 and ν_2. If we put $\nu_1=dp_1$ and $\nu_2=dp_2$, then p_1 and p_2 are mutually prime. The above equation (7.18) implies that $d\gamma$ is an integer. So we have

$$d:=\mathrm{GCD}\{\nu_1,\nu_2\}, \quad \ell:=d\gamma\in\mathbf{Z}. \tag{7.19}$$

Before constructing the representation of the MTG, we need to know how the generators generate an arbitrary element of the MTG. From the multiplication rule of the group $\mathbf{R}\times_\omega\mathbf{R}^n$ we deduce that for $x,y\in\mathbf{R}^n$

$$\begin{aligned}(\tfrac{1}{2}x\omega x,x)\cdot(\tfrac{1}{2}y\omega y,y)&=(\tfrac{1}{2}x\omega x+\tfrac{1}{2}y\omega y+x\omega y,x+y)\\&=(\tfrac{1}{2}x\omega y-\tfrac{1}{2}y\omega x+\tfrac{1}{2}(x+y)\omega(x+y),x+y)\\&=(\tfrac{1}{2}x(\omega-{}^t\omega)y,0)\cdot(\tfrac{1}{2}(x+y)\omega(x+y),x+y)\end{aligned} \tag{7.20}$$

and

$$(\tfrac{1}{2}x\omega x,x)^{-1}=(\tfrac{1}{2}x\omega x,-x). \tag{7.21}$$

Iteration of (7.20) yields

$$(\tfrac{1}{2}x\omega x,x)^n=(\tfrac{1}{2}n^2x\omega x,nx),\quad n\in\mathbf{Z}. \tag{7.22}$$

Furthermore, (7.20) implies

$$(\tfrac{1}{2}s^2x\omega x,sx)\cdot(\tfrac{1}{2}t^2x\omega x,tx)=(\tfrac{1}{2}(s+t)^2x\omega x,(s+t)x),\quad s,t\in\mathbf{R}. \tag{7.23}$$

By a tedious calculation we can show

$$(s,x+y+z)=(s-X,0)\cdot(\tfrac{1}{2}x\omega x,x)\cdot(\tfrac{1}{2}y\omega y,y)\cdot(\tfrac{1}{2}z\omega z,z) \tag{7.24}$$

with

$$X=\tfrac{1}{2}(x+y+z)\omega(x+y+z)+\tfrac{1}{2}x(\omega-{}^t\omega)y+\tfrac{1}{2}x(\omega-{}^t\omega)z+\tfrac{1}{2}y(\omega-{}^t\omega)z. \tag{7.25}$$

Thus an arbitrary element of the MTG is expressed as

$$\begin{aligned} g&=(s,te_0+n_1e_1+n_2e_2)\\ &=(s-\tfrac{1}{2}(te_0+n_1e_1+n_2e_2)\omega(te_0+n_1e_1+n_2e_2)-\tfrac{1}{2}\gamma n_1n_2,0)\\ &\quad\cdot(\tfrac{1}{2}t^2e_0\omega e_0,te_0)\cdot(\tfrac{1}{2}e_1\omega e_1,e_1)^{n_1}\cdot(\tfrac{1}{2}e_2\omega e_2,e_2)^{n_2}\\ &=\phi(s-X)\cdot g_0(t)\cdot(g_1)^{n_1}\cdot(g_2)^{n_2}, \end{aligned} \tag{7.26}$$

which is a product of the generators

$$\phi(s):=(s,0), \tag{7.27}$$

$$g_0(t):=(\tfrac{1}{2}t^2e_0\omega e_0,te_0), \tag{7.28}$$

$$g_1:=(\tfrac{1}{2}e_1\omega e_1,e_1), \tag{7.29}$$

$$g_2:=(\tfrac{1}{2}e_2\omega e_2,e_2). \tag{7.30}$$

These generators satisfy the relations

$$\phi(s)\cdot\phi(t)=\phi(s+t), \tag{7.31}$$

$$\phi(1)=1, \tag{7.32}$$

$$g_0(s)\cdot g_0(t)=g_0(s+t), \tag{7.33}$$

$$g_0(1)=\phi(\tfrac{1}{2}z_0), \tag{7.34}$$

$$(g_1)^{\nu_1}=\phi(\tfrac{1}{2}z_1), \tag{7.35}$$

$$(g_2)^{\nu_2}=\phi(\tfrac{1}{2}z_2), \tag{7.36}$$

$$g_1\cdot g_2\cdot g_1^{-1}\cdot g_2^{-1}=\phi(\gamma) \tag{7.37}$$

and other trivial commutators. Here we have defined $\{z_0,z_1,z_2\}$ by

$$z_0 := e_0 \omega e_0 = \left(\frac{1}{D_0} \right)^2 b_1 b_2 b_3, \tag{7.38}$$

$$z_1 := \nu_1^2 e_1 \omega e_1, z_2 := \nu_2^2 e_2 \omega e_2. \tag{7.39}$$

Because e_0 is an integral vector and ω is an integral matrix, z_0 is an integer. Furthermore, (7.12) implies that z_1 and z_2 are also integers.

In reverse order, the generators $\{\phi(s), g_0(t), g_1, g_2\}$ and their relations (7.31)–(7.37) determine the MTG uniquely. These generators with the relations form the MTG in a constructive manner. Consequently, the MTG in T^3 is completely characterized by the set of parameters $(z_0, z_1, z_2, \nu_1, \nu_2, \gamma)$, where $\{z_0, z_1, z_2, \nu_1, \nu_2\}$ are integers and γ is a rational number constrained by the condition (7.18).

Now we discuss the representation theory of the MTG exhaustively. The space of functions $\{f: \mathbf{R}^{n+1} \to \mathbf{C}\}$ provides the regular representation of the group $\mathbf{R} \times_\omega \mathbf{R}^n$ via

$$\begin{aligned} U(v_0, v) f(x_0, x) &:= f((v_0, v)^{-1} \cdot (x_0, x)) \\ &= f((-v_0 + v\,\omega v, -v) \cdot (x_0, x)) \\ &= f(x_0 - v_0 + v\,\omega v - v\,\omega x, x - v). \end{aligned} \tag{7.40}$$

We restrict the representation U on the space of equivariant functions, which are constrained by (3.7) and (3.8). Then we have

$$U(v_0, v) f(x_0, x) = e^{2\pi i (v_0 - v\,\omega v + v\,\omega x)} f(x_0, x - v), \tag{7.41}$$

which reproduces the unitary transformations (2.3) and (2.4) when the twisting matrix (3.11) is taken. Particularly $(v_0, 0)$ is represented by

$$U(v_0, 0) f(x_0, x) = e^{2\pi i v_0} f(x_0, x). \tag{7.42}$$

Hence the representation U induces an isomorphism of (7.31) and (7.32) by

$$U(\phi(s)) = e^{2\pi i s}. \tag{7.43}$$

Moreover, if we put

$$U_0(t) := U(g_0(t)), \quad U_1 := U(g_1), \quad U_2 := U(g_2), \tag{7.44}$$

they satisfy

$$U_0(s) U_0(t) = U_0(s+t), \tag{7.45}$$

$$U_0(1) = e^{\pi i z_0}, \tag{7.46}$$

$$(U_1)^{\nu_1} = e^{\pi i z_1}, \tag{7.47}$$

$$(U_2)^{\nu_2} = e^{\pi i z_2}, \tag{7.48}$$

$$U_1 U_2 U_1^{-1} U_2^{-1} = e^{2\pi i \gamma}, \tag{7.49}$$

since U is a homomorphism of the relations (7.33)–(7.37).

An irreducible representation of $U_0(t)$ is labeled by an integer q_0 and defined by

$$U_0(t) |q_0\rangle = e^{2\pi i (q_0 + (1/2) z_0) t} |q_0\rangle. \tag{7.50}$$

On the other hand, to construct a representation of the algebra generated by U_1 and U_2, we introduce a set of orthogonal vectors $\{|q_1,q_2\rangle | q_1 \in \mathbf{Z}/\nu_1\mathbf{Z}, q_2 \in \mathbf{Z}/\nu_2\mathbf{Z}\}$. We assume identification $|q_1,q_2\rangle = |q_1+k_1\nu_1, q_2+k_2\nu_2\rangle$ for any $k_1,k_2 \in \mathbf{Z}$. Let the operators U_1 and U_2 act on them by

$$U_1|q_1,q_2\rangle = e^{2\pi i(q_1+(1/2)z_1)/\nu_1}|q_1,q_2\rangle, \tag{7.51}$$

$$U_2|q_1,q_2\rangle = e^{2\pi i(q_2+(1/2)z_2)/\nu_2}|q_1+\nu_1\gamma,q_2\rangle. \tag{7.52}$$

The step of q_1 generated by U_2 is

$$\Delta q_1 := \nu_1\gamma = dp_1\frac{\ell}{d} = p_1\ell. \tag{7.53}$$

Then the fundamental relations, (7.47)–(7.49), are satisfied as

$$(U_1)^{\nu_1}|q_1,q_2\rangle = e^{2\pi i(q_1+(1/2)z_1)}|q_1,q_2\rangle = e^{2\pi i(1/2)z_1}|q_1,q_2\rangle, \tag{7.54}$$

$$\begin{aligned}(U_2)^{\nu_2}|q_1,q_2\rangle &= e^{2\pi i(q_2+(1/2)z_2)}|q_1+\nu_1\nu_2\gamma,q_2\rangle \\ &= e^{2\pi i(1/2)z_2}|q_1,q_2\rangle \quad (\text{because } \nu_2\gamma \text{ is an integer}),\end{aligned} \tag{7.55}$$

$$\begin{aligned}U_1U_2U_1^{-1}U_2^{-1}|q_1,q_2\rangle &= U_1U_2U_1^{-1}e^{-2\pi i(q_2+(1/2)z_2)/\nu_2}|q_1-\nu_1\gamma,q_2\rangle \\ &= U_1U_2e^{-2\pi i(q_1-\nu_1\gamma+(1/2)z_1)/\nu_1}e^{-2\pi i(q_2+(1/2)z_2)/\nu_2}|q_1-\nu_1\gamma,q_2\rangle \\ &= U_1\,e^{-2\pi i(q_1-\nu_1\gamma+(1/2)z_1)/\nu_1}|q_1,q_2\rangle = e^{2\pi i\gamma}|q_1,q_2\rangle.\end{aligned} \tag{7.56}$$

Thus the basis $\{|q_1,q_2\rangle\}$ spans a representation space of the algebra generated by U_1 and U_2. This representation space is reducible generally. We can see that the action of U_2 is cyclic. Namely, if we put

$$c := \frac{\mathrm{LCM}\{\Delta q_1,\nu_1\}}{\Delta q_1} = \frac{\mathrm{LCM}\{p_1\ell,p_1 d\}}{p_1\ell} = \frac{\mathrm{LCM}\{\ell,d\}}{\ell} = \frac{d}{\mathrm{GCD}\{\ell,d\}}, \tag{7.57}$$

then

$$c\Delta q_1 = \mathrm{LCM}\{\Delta q_1,\nu_1\} \tag{7.58}$$

is an integral multiple of ν_1 and therefore the U_2 action (7.52) iterated by c times gives

$$(U_2)^c|q_1,q_2\rangle = e^{2\pi i(q_2+(1/2)z_2)c/\nu_2}|q_1+c\Delta q_1,q_2\rangle = e^{2\pi i(q_2+(1/2)z_2)c/\nu_2}|q_1,q_2\rangle. \tag{7.59}$$

Moreover,

$$\frac{\nu_1}{c} = \frac{\nu_1\mathrm{GCD}\{\ell,d\}}{d} = \frac{dp_1\mathrm{GCD}\{\ell,d\}}{d} = p_1\mathrm{GCD}\{\ell,d\}, \tag{7.60}$$

$$\frac{\nu_2}{c} = \frac{\nu_2\mathrm{GCD}\{\ell,d\}}{d} = \frac{dp_2\mathrm{GCD}\{\ell,d\}}{d} = p_2\mathrm{GCD}\{\ell,d\} \tag{7.61}$$

are integers. Therefore, each choice of $q_1 \in \mathbf{Z}$ modulo $(\nu_1/c)\mathbf{Z}$ and $q_2 \in \mathbf{Z}$ modulo $(\nu_2/c)\mathbf{Z}$ specifies one of inequivalent irreducible representations. Consequently, the dimension of the irreducible representation is

$$\text{dimension} = c = \frac{d}{\mathrm{GCD}\{d,\ell\}}. \tag{7.62}$$

On the other hand, the number of inequivalent representations for a fixed q_0 is

$$\#\text{inequivalent irreducible representations} = \frac{\nu_1}{c}\cdot\frac{\nu_2}{c} = p_1 p_2 (\mathrm{GCD}\{d,\ell\})^2. \tag{7.63}$$

These numbers give a number

$$(\text{dimension})^2 \times (\#\text{inequivalent irreducible representations}) = c^2 \times \frac{\nu_1\nu_2}{c^2} = \nu_1\nu_2, \tag{7.64}$$

which coincides with the dimension of the algebra generated by U_1 and U_2, as required by the Peter–Weyl theory on group representation.[29] Thus we have obtained the complete set of irreducible representations of the algebra.

In summary, an irreducible representation of the MTG in the three-dimensional torus is specified by

$$\chi = (q_0, [q_1], [q_2]) \in \mathbf{Z} \times \mathbf{Z}_{(\nu_1/c)} \times \mathbf{Z}_{(\nu_2/c)}. \tag{7.65}$$

Using the decomposition (7.26) we have

$$\begin{aligned} &U(s, te_0 + n_1 e_1 + n_2 e_2)|q_0, q_1, q_2\rangle \\ &\quad = e^{2\pi i(s-X)} U_0(t)(U_1)^{n_1}(U_2)^{n_2}|q_0, q_1, q_2\rangle \\ &\quad = e^{2\pi i\{(s-X)+(q_0+(1/2)z_0)t+(q_1+\gamma\nu_1 n_2+(1/2)z_1)n_1/\nu_1+(q_2+(1/2)z_2)n_2/\nu_2\}}|q_0, q_1+\gamma\nu_1 n_2, q_2\rangle \end{aligned} \tag{7.66}$$

with X evaluated as

$$X = \tfrac{1}{2}(te_0 + n_1 e_1 + n_2 e_2)\omega(te_0 + n_1 e_1 + n_2 e_2) + \tfrac{1}{2}\gamma n_1 n_2. \tag{7.67}$$

B. Examples in the three-dimensional torus

Here we apply the previous method of representation of the MTG to three examples of magnetic fields in T^3.

The first example is a magnetic field parallel to the x_3-axis,

$$b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \nu \end{pmatrix} \tag{7.68}$$

with a positive integer ν. The generators (7.9) of the MTG are chosen as

$$e_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_1 = \frac{1}{\nu}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \frac{1}{\nu}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{7.69}$$

The vectors $\{e_1, e_2\}$ reproduce the discrete magnetic shifts (2.17) in the plane perpendicular to the magnetic field. The cycles (7.12) of e_1 and e_2 are found to be

$$\nu_1 = \nu, \quad \nu_2 = \nu, \tag{7.70}$$

respectively. Using them we evaluate the parameters of the MTG as

$$d=\mathrm{GCD}\{\nu_1,\nu_2\}=\mathrm{GCD}\{\nu,\nu\}=\nu, \tag{7.71}$$

$$\gamma=e_1(\omega-{}^t\omega)e_2=e_1\cdot(e_2\times b)=\frac{1}{\nu}, \tag{7.72}$$

$$\ell=d\gamma=\nu\frac{1}{\nu}=1, \tag{7.73}$$

$$z_0=z_1=z_2=0. \tag{7.74}$$

The size and the number of irreducible representations are

$$\text{dimension}=c=\frac{d}{\mathrm{GCD}\{\ell,d\}}=\frac{\nu}{\mathrm{GCD}\{1,\nu\}}=\frac{\nu}{1}=\nu, \tag{7.75}$$

$$\#\text{inequivalent irreducible representations}=\frac{\nu_1\nu_2}{c^2}=\frac{\nu^2}{\nu^2}=1. \tag{7.76}$$

In this case (7.51) and (7.52) reproduce the representations (2.20) and (2.21) in T^2.

The second example is a magnetic field perpendicular to the x_1-axis and lying in the middle of the x_2- and x_3-axes,

$$b=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}=\begin{pmatrix} 0 \\ \nu \\ \nu \end{pmatrix}, \tag{7.77}$$

with a positive integer ν. The generators of the MTG are chosen as

$$e_0=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad e_1=\frac{1}{\nu}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad e_2=\frac{1}{\nu}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{7.78}$$

The cycles of e_1 and e_2 are

$$\nu_1=\nu, \quad \nu_2=\nu, \tag{7.79}$$

and other parameters of the MTG are also evaluated as

$$d=\nu,\ \gamma=\frac{1}{\nu},\ \ell=1,\ z_0=z_1=z_2=0, \tag{7.80}$$

$$\text{dimension}=c=\nu, \tag{7.81}$$

$$\#\text{inequivalent irreducible representations}=\frac{\nu_1\nu_2}{c^2}=1. \tag{7.82}$$

The third example is a magnetic field in the direction of (1,1,1),

$$b=\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}=\begin{pmatrix} \nu \\ \nu \\ \nu \end{pmatrix} \tag{7.83}$$

with a positive integer ν. A calculation similar to the previous ones gives a series of parameters. Here we show only the results

$$e_0=\begin{pmatrix}1\\1\\1\end{pmatrix},\quad e_1=\frac{1}{\nu}\begin{pmatrix}1\\0\\0\end{pmatrix},\quad e_2=\frac{1}{\nu}\begin{pmatrix}0\\1\\0\end{pmatrix},\tag{7.84}$$

$$\nu_1=\nu,\ \nu_2=\nu,\ d=\nu,\ \gamma=\frac{1}{\nu},\ \ell=1, z_0=\nu, z_1=z_2=0,\tag{7.85}$$

$$\text{dimension}=c=\nu,\tag{7.86}$$

$$\#\text{inequivalent irreducible representations}=\frac{\nu_1\nu_2}{c^2}=1.\tag{7.87}$$

VIII. REPRESENTATION THEORY OF THE MTG IN AN n-TORUS

Here we describe how to characterize the MTGs in an n-dimensional torus. We can choose generators $\{e_1,e_2,\ldots,e_l,f_1,f_2,\ldots,f_l,g_1,g_2,\ldots,g_m\}$ $(2l+m=n)$ of the magnetic shift group Ω^n such that

$$e_i,f_i\in\mathbf{Q}^n,\quad (\omega-{}^t\omega)e_i,(\omega-{}^t\omega)f_i\in\mathbf{Z}^n,\quad e_i(\omega-{}^t\omega)f_j=\gamma_i\delta_{ij}\quad (i,j=1,\ldots,l),\tag{8.1}$$

$$g_k\in\mathbf{Z}^n,\quad (\omega-{}^t\omega)g_k=0\quad (k=1,\ldots,m),\tag{8.2}$$

with nonzero rational numbers $\gamma_i\in\mathbf{Q}$. The vectors $\{g_1,\ldots,g_m\}$ are demanded to be minimal integral vectors in the sense that there is no real number s satisfying

$$0<s<1,\quad sg_k\in\mathbf{Z}^n,\tag{8.3}$$

for each $k=1,\ldots,m$. Let $\{\mu_i,\nu_i\}(i=1,\ldots,l)$ be smallest positive integers such that

$$\mu_ie_i,\nu_if_i\in\mathbf{Z}^n\tag{8.4}$$

and that there are no integers $\{m_i,n_i\}$ satisfying

$$0<m_i<\mu_i,\quad m_ie_i\in\mathbf{Z}^n,\tag{8.5}$$

$$0<n_i<\nu_i,\quad n_if_i\in\mathbf{Z}^n.\tag{8.6}$$

Equations (8.1) and (8.4) imply that $\gamma_i\mu_i$ and $\gamma_i\nu_i$ are integers. Thus, by putting

$$d_i:=\mathrm{GCD}\{\mu_i,\nu_i\},\tag{8.7}$$

we can see that γ_id_i is an integer. Consequently, the group of translations (6.16) is decomposed as

$$\Omega^n=\mathbf{Z}e_1\oplus\cdots\oplus\mathbf{Z}e_l\oplus\mathbf{Z}f_1\oplus\cdots\oplus\mathbf{Z}f_l\oplus\mathbf{R}g_1\oplus\cdots\oplus\mathbf{R}g_m\tag{8.8}$$

and the MTG (6.18) is expressed as

$$S_A\cong S^1\times_\omega(\mathbf{Z}_{\mu_1}\times\cdots\times\mathbf{Z}_{\mu_l}\times\mathbf{Z}_{\nu_1}\times\cdots\times\mathbf{Z}_{\nu_l}\times T^m).\tag{8.9}$$

Finally, we describe an outline of the representation theory of the MTG's in an n-dimensional torus. Let us define integers x_i,y_i,z_j by

$$x_i:=\mu_i^2e_i\omega e_i,\tag{8.10}$$

$$y_i:=\nu_i^2f_i\omega f_i\quad (i=1,\ldots,l),\tag{8.11}$$

$$z_k := g_k \omega g_k \quad (k=1,\ldots,m). \tag{8.12}$$

Generators of the MTG (8.9) are represented by a set of unitary operators

$$T(s) := U(s,0), \quad s \in \mathbf{R}, \tag{8.13}$$

$$U_i := U(\tfrac{1}{2} e_i \omega e_i, e_i), \tag{8.14}$$

$$V_i := U(\tfrac{1}{2} f_i \omega f_i, f_i), \tag{8.15}$$

$$W_k(t) := U(\tfrac{1}{2} t^2 g_k \omega g_k, t g_k), \quad t \in \mathbf{R}. \tag{8.16}$$

They satisfy the equations

$$T(s)T(t) = T(s+t), \tag{8.17}$$

$$T(1) = 1, \tag{8.18}$$

$$(U_i)^{\mu_i} = T(x_i/2), \tag{8.19}$$

$$(V_i)^{\nu_i} = T(y_i/2), \tag{8.20}$$

$$U_i V_i U_i^{-1} V_i^{-1} = T(\gamma_i), \tag{8.21}$$

$$W_k(s) W_k(t) = W_k(s+t), \tag{8.22}$$

$$W_k(1) = T(z_k/2), \tag{8.23}$$

and other trivial commutators. These equations for the n-torus are generalization of the equations (7.31)–(7.37) for the three-torus. A representation space is spanned by the basis vectors

$$|\lambda,p,q,r\rangle = |\lambda,p_1,p_2,\ldots,p_l,q_1,q_2,\ldots,q_l,r_1,r_2,\ldots,r_m\rangle \tag{8.24}$$

labeled by $\lambda \in \mathbf{Z}$, $p_i \in \mathbf{Z}_{\mu_i}$, $q_i \in \mathbf{Z}_{\nu_i}$, and $r_k \in \mathbf{Z}$. The generators act on the basis vectors according to

$$T(s)|\lambda,p,q,r\rangle = e^{2\pi i \lambda s}|\lambda,p,q,r\rangle, \tag{8.25}$$

$$U_i|\lambda,p,q,r\rangle = e^{\pi i \lambda (2p_i + x_i)/\mu_i}|\lambda,p,q,r\rangle, \tag{8.26}$$

$$V_i|\lambda,p,q,r\rangle = e^{\pi i \lambda (2q_i + y_i)/\nu_i}|\lambda,p_1,\ldots,p_i+\gamma_i\mu_i,\ldots,p_l,q,r\rangle, \tag{8.27}$$

$$W_k(t)|\lambda,p,q,r\rangle = e^{\pi i \lambda (2r_k + z_k)t}|\lambda,p,q,r\rangle. \tag{8.28}$$

These are generalization of (7.43) and (7.50)–(7.52). The cycle of V_i is given by

$$c_i := \frac{\mathrm{LCM}\{\gamma_i\mu_i,\mu_i\}}{\gamma_i\mu_i} = \frac{\mathrm{LCM}\{\gamma_i d_i, d_i\}}{\gamma_i d_i} = \frac{d_i}{\mathrm{GCD}\{\gamma_i d_i, d_i\}}. \tag{8.29}$$

Hence an irreducible representation is labeled by

$$\begin{aligned} \chi = (\lambda,[p_1],\ldots,[p_l],[q_1],\ldots,[q_l],r_1,\ldots,r_m) &\in \mathbf{Z}\times\mathbf{Z}_{(\mu_1/c_1)} \\ &\times\cdots\times\mathbf{Z}_{(\mu_l/c_l)}\times\mathbf{Z}_{(\nu_1/c_1)}\times\cdots\times\mathbf{Z}_{(\nu_l/c_l)}\times\mathbf{Z}^m. \end{aligned} \tag{8.30}$$

The dimension of the irreducible representation is

$$\text{dimension} = \prod_{i=1}^{l} c_i , \tag{8.31}$$

and the number of inequivalent representations is

$$\#\text{inequivalent irreducible representations} = \prod_{i=1}^{l} \frac{\mu_i \nu_i}{c_i^2} \tag{8.32}$$

for fixed $(\lambda, r_1, \ldots, r_m) \in \mathbf{Z}^{m+1}$.

IX. CONCLUSION

Let us summarize our discussions. We began this article with a discussion on symmetry of a charged particle in a uniform magnetic field. We saw that the quantum system in T^2 has a discrete noncommutative translation symmetry. The symmetry is characterized by a central extension of a cyclic group.

In the following part of this article we introduced a noncommutative product into $\mathbf{R}^{n+1}$. Using the group structure, we defined the magnetic fiber bundles P_ω^{n+1}, which is a fiber bundle over T^n with a fiber S^1. Then we showed that the set of magnetic fiber bundles is classified by the quotient space of integral matrices $\mathrm{Mat}(n,\mathbf{Z})/\mathrm{Sym}(n,\mathbf{Z})$. We introduced connections into the fiber bundles and classified them by $\mathrm{Mat}(n,\mathbf{Z}) \times \mathbf{R}^n / \mathrm{Sym}(n,\mathbf{Z}) \times \mathbf{Z}^n$ as shown in (5.5). The lifted translations leaving the connection invariant form the magnetic translation group of (6.4). We characterized the MTG by (6.18) with (6.16). This characterization of the MTG is one of main results of this article. We found that the magnetic shift group Ω^n is discrete when the characteristic matrix $(\omega - {}^t\omega)$ is nondegenerated.

In the rest of the article we discussed the representation theory of the MTG for T^3 in detail and applied it to a few examples. The dimensions of an irreducible unitary representation is given by c in (7.62) and each irreducible representation is labeled by χ in (7.65). These results may be useful for application to the electron system in a lattice in an inclined magnetic field. We briefly described the representation theory of the MTG for T^n and summarized the result in (8.30) and (8.31).

Here we would like to mention remaining problems. It is desirable to apply the representation theory of the MTG to spectral analyses of the Laplace and Dirac operators. Originally the spectral problem of the quantum mechanics in a torus motivated this study. For this application the Peter–Weyl theory on group representation will play an essential role. In the next study we would like to pursue the analysis of the Laplace operator in the torus with a magnetic field. Moreover, an equilateral torus admits discrete transformations that exchange vertices of the torus and that leave the metric and the magnetic field invariant. It is also desirable to include such discrete transformations into the MTG for the complete spectral analysis.

By developing the theory of the MTG we will find its applications to physics. Inclusion of the supersymmetry into the MTG is an interesting direction for the future development. Sakamoto, Tachibana, and Takenaga[30,31] have pointed out that breaking of the translation symmetry causes breaking of the supersymmetry because the supersymmetry includes the translation symmetry. Hence the magnetic field may trigger supersymmetry breaking. On the other hand, the MTG in an n-torus is regarded as a generalization of the noncommutative torus, which attracted much attention recently in the string theory.[32] The B-field in a compactified space naturally induces a noncommutative structure, which is described by the MTG. Jackiw[33] also showed that how the noncommutative structure emerges in physical situations. If we turn our attention to solid state physics, we find another interesting application of the MTG also in this area. Tranquada[34] observed spontaneous formation of a charge density wave at a nonzero wave number in a copperoxide superconductor. This ordered state is called the stripe phase, in which the translation symmetry is broken. A similar stripe phase occurs commonly in a quantum Hall system.[35] Application of the MTG may help understanding of the stripe phases.

Note added in proof. After acceptance for publication of this article we obtained more strong results on the magnetic translation group in n dimensions. As concerns γ_i in (8.1) and μ_i, ν_i in (8.4), we proved that $\mu_i = \nu_i = 1/\gamma_i$. Consequently, the definition (8.7) means simply that $d_i = \nu_i$. Eq. (8.29) is also simplified as $c_i = \nu_i$. In (8.31) the dimension of the irreducible representation becomes $\Pi_{i=1}^{l} \nu_i$. Finally, the number of inequivalent reprenentations (8.32) is reduced to one. As a corollary, we can show that $\nu_1 = \nu_2 = d = c = 1/\gamma$ and hence $\ell = d\gamma = 1$ in Sec. VII. More strongly, we can prove that $\nu_1 = \mathrm{GCD}\{b_1, b_2, b_3\}$ for the three-deimensional magnetic field. Proofs of these statements are to be published elsewhere.

ACKNOWLEDGMENTS

I wish to thank Kin-ya Nakamura, whose thesis motivated this work. I wish to thank M. Sakamoto for stimulating discussions. This work originates from collaborations with him. I thank A. Asada, who taught me classification of S^1-fiber bundles by the first Chern class. I appreciate valuable comments by K. Fujii, S. Higuchi, S. Iida, T. Iwai, N. Maeda, N. Miyazaki, J. Nishimura, Y. Ohnuki, and I. Tsutsui. This work was partially supported by Grant-in-Aids for Scientific Research from Ministry of Education, Culture, Sports, Science and Technology of Japan.

[1] E. Brown, Phys. Rev. **133**, A1038 (1964).
[2] J. Zak, Phys. Rev. **134**, A1602 (1964); **134**, A1607 (1964).
[3] J. Zak, Phys. Rev. B **39**, 694 (1989).
[4] N. Ashby and S. C. Miller, Phys. Rev. **139**, A428 (1965).
[5] J. Avron, I. Herbst, and B. Simon, Duke Math. J. **45**, 847 (1978).
[6] J. E. Avron, I. W. Herbst, and B. Simon, Ann. Phys. **114**, 431 (1978).
[7] J. E. Avron, I. W. Herbst, and B. Simon, Commun. Math. Phys. **79**, 529 (1981).
[8] J. Avron and B. Simon, J. Phys. A **18**, 2199 (1985).
[9] B. A. Dubrovin and S. P. Novikov, Sov. Math. Dokl. **22**, 240 (1980).
[10] S. P. Novikov, Sov. Math. Dokl. **23**, 298 (1981).
[11] J. Asch, H. Over, and R. Seiler, J. Geom. Phys. **13**, 275 (1994).
[12] T. Lulek, Rep. Math. Phys. **34**, 71 (1994).
[13] W. Florek, Rep. Math. Phys. **34**, 81 (1994).
[14] D. Lipinski, Rep. Math. Phys. **34**, 97 (1994).
[15] S. Walcerz, Rep. Math. Phys. **34**, 107 (1994).
[16] W. Florek, Rep. Math. Phys. **38**, 235 (1996).
[17] W. Florek, Rep. Math. Phys. **38**, 325 (1996).
[18] R. Kuwabara, J. Math. Tokushima Univ. **29**, 9 (1995); correrction and addendum, J. Math. Tokushima Univ. **30**, 81 (1996).
[19] R. Kuwabara, J. Math. Tokushima Univ. **33**, 33 (1999).
[20] M. J. Gruber, J. Geom. Phys. **34**, 137 (2000).
[21] I. Antoniadis, Phys. Lett. B **246**, 377 (1990).
[22] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B **429**, 263 (1998); hep-ph/9803315.
[23] M. Sakamoto, M. Tachibana, and K. Takenaga, Phys. Lett. B **457**, 33 (1999); hep-th/9902069.
[24] K. Ohnishi and M. Sakamoto, Phys. Lett. B **486**, 179 (2000); hep-th/0005017.
[25] H. Hatanaka, S. Matsumoto, K. Ohnishi, and M. Sakamoto, Phys. Rev. D **63**, 105003 (2001); hep-th/0010283.
[26] S. Matsumoto, M. Sakamoto, and S. Tanimura, Phys. Lett. B **518**, 163 (2001); hep-th/0105196.
[27] M. Sakamoto and S. Tanimura, Phys. Rev. D **65**, 065004 (2002); hep-th/0108208.
[28] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, 3rd English ed. (Springer-Verlag, Berlin, 1966).
[29] J.-P. Serre, *Représentations Linéaires des Groupes Finis*, 2nd edition (Hermann, Paris, 1971). (A Japanese translation by Iwahori and Yokonuma is published from Iwanami.)
[30] M. Sakamoto, M. Tachibana, and K. Takenaga, Phys. Lett. B **458**, 231 (1999); hep-th/9902070.
[31] M. Sakamoto, M. Tachibana, and K. Takenaga, Prog. Theor. Phys. **104**, 633 (2000); hep-th/9912229.
[32] A. Connes, M. R. Douglas, and A. Schwarz, J. High Energy Phys. **9802**, 003 (1998); hep-th/9711162.
[33] R. Jackiw, Nucl. Phys. Proc. Suppl. **108**, 30 (2002); hep-th/0110057.
[34] J. M. Tranquada, J. D. Axe, N. Ichikawa, Y. Nakamura, S. Uchida, and B. Nachumi, Phys. Rev. B **54**, 7489 (1996).
[35] E. Fradkin and S. A. Kivelson, Phys. Rev. B **59**, 8065 (1999).