Renormalized Phase Dynamics in Oscillatory Media

Naofumi Tsukamoto, Hirokazu Fujisaka, and Katsuya Ouchi

Department of Applied Analysis and Complex Dynamical Systems, Graduate School of Informatics, Kyoto University, Kyoto 606-8501, Japan
Kobe Design University, Kobe 651-2196, Japan

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Based on the complex Ginzburg-Landau equation (CGLE), a new mapping model of oscillatory media is proposed. The present dynamics is fully determined by an effective phase field renormalized by amplitude. The model exhibits phase turbulence, amplitude turbulence, and a frozen state reported in the CGLE. In addition, we find a state in which the phase and amplitude have spiral structures with opposite rotational directions. This state is found to be observed also in the CGLE. Thus, one concludes that the behaviors observed in the CGLE can be described by only the phase dynamics appropriately constructed.

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Various spatiotemporal patterns are observed in nonequilibrium systems, such as fluid systems, chemical reaction, and nonlinear optics [1]. Such dynamics in spatially extended systems are often described by nonlinear partial differential equations. The complex Ginzburg-Landau equation (CGLE) is one of the most-studied nonlinear partial differential equations. The CGLE is a universal equation which describes slow spatiotemporal variation near a threshold in extended systems, such as fluid systems, chemical reactions, and pattern formation [3–5].

The time evolution of the local part is described by the Stuart-Landau equation. Thus, the limit cycle or the unstable fixed point coincides with Eq. (1) in the limit $\tau \rightarrow 0$. Therefore, Eq. (4) for small $\tau$ is expected to show approximately the dynamics in the CGLE. Actually, the spatiotemporal dynamics in the CGLE have been investigated by using mapping models similar to Eq. (4) as an approximation [3,8–11].

In this Letter, we consider a mapping model of the CGLE in order to extract the essential nature of the CGLE. We use the mapping model constructed by method proposed in Refs. [12,13], which is similar to those in Refs. [3,8–11]. There are two steps to obtain mapping models: (i) splitting the time evolution of the CGLE into two parts and (ii) recombining them. One of the divided two parts is a local part which consists of the first two terms in the right-hand side of Eq. (1), and the other is a nonlocal part which is the spatial coupling term in Eq. (1). The time evolution of each part can be solved analytically as shown below.

The time evolution of the local part is described by the Stuart-Landau equation $A = \dot{A} = (1 + i c_2)|A|^2 A + D(1 + i c_1)\nabla^2 A$, which is obtained by omitting the spatially coupling term in Eq. (1). Integrating the Stuart-Landau equation over time width $\tau$ and setting $\psi(t) = e^{ic_2\tau} A(t)$, we obtain

$$\psi(t + \tau) = F_\tau(\psi(t)).$$

where $F_\tau(\psi) = \psi(1 - e^{-2\tau})|\psi|^2 + e^{-2\tau}e^{-i(1+ic_2)/2}$. The time evolution of the nonlocal part is described by the complex diffusion equation $\dot{\psi} = D(1 + ic_1)\nabla^2 \psi$, which is obtained by omitting the local parts in the right-hand side of Eq. (1). Integrating the equation over time width $\tau$, we have

$$\psi(r, t + \tau) = \mathcal{L}_{D\tau}\psi(r, t),$$

where $\mathcal{L}_{D\tau}$ is the linear operator defined by $\mathcal{L}_{D\tau}\psi(\mathbf{r}) = \int K_{D\tau}(\mathbf{r} - \mathbf{r}')f(\mathbf{r}')d^d\mathbf{r}'$. Here $K_{D\tau}(\mathbf{r}) = \{4\pi D\tau(1 + i c_1)|\mathbf{r}|^{-d/2}e^{-|\mathbf{r}|^2/4D\tau(1 + i c_1)}\}$, and $d$ is the spatial dimensionality.

Recombining the divided time evolutions, we obtain the mapping model

$$\psi_{n+1}(r) = \mathcal{L}_{D\tau}F_\tau(\psi_n(r)).$$

This equation is a model based on the CGLE with a control parameter $\tau$ and coincides with Eq. (1) in the limit $\tau \rightarrow 0$. Therefore, Eq. (4) for small $\tau$ is expected to show approximately the dynamics in the CGLE. Actually, the spatiotemporal dynamics in the CGLE have been investigated by using mapping models similar to Eq. (4) as an approximation [3,8–11].

In this Letter, we take the opposite limit $\tau \rightarrow \infty$ by keeping $D\tau = 1$ fixed. In this limit, the local map $F_\tau(\psi)$ reduces to

$$F(\psi) = F_\infty(\psi) = \begin{cases} |\psi|^2(1 + ic_2) & (\psi \neq 0), \\ 0 & (\psi = 0). \end{cases}$$

which maps an arbitrary state $\psi$ onto either a state on the limit cycle $(|\psi| = 1)$ or the unstable fixed point $(\psi = 0)$ of the Stuart-Landau equation. Thus, the limit $\tau \rightarrow \infty$ removes the relaxation process to the limit cycle. Setting $\mathcal{L} = \mathcal{L}_{D\tau = 1} = e^{(1+ic_1)|\nabla|^2}$, the above procedure leads to the complex Ginzburg-Landau map (CGLM)

$$\psi_{n+1}(r) = \mathcal{L} F(\psi_n(r)).$$

We will show that the CGLM (6) exhibits the spatiotem-
poral behaviors reported in the CGLE and discuss the essence of the CGLE dynamics.

It should be noted that the limit \( \tau \to \infty \) allows us to renormalize the amplitude component into the phase variable as follows. Introducing the renormalized phase variable \( \theta_n = \arg \psi_n - c_2 \log |\psi_n| \) for \( \psi_n \neq 0 \), we define the phase field with the phase singular point as \( z_n(r) = F(\psi_n(r)) \), that is, \( z_n(r) = e^{i\theta_n(r)} \) for \( \psi_n(r) \neq 0 \) and \( z_n(r) = 0 \) for \( \psi_n(r) = 0 \). Here \( z_n = 0 \) represents the phase singular point, and \( \theta_n \) describes an isochron of the Stuart-Landau equation. Equation (6) can be written as \( \psi_{n+1}(r) = L z_n(r) \), which indicates that the time evolution of \( \psi_n \) is determined by \( z_n \). In addition, we obtain the mapping system of the phase field with the phase singular point as

\[
z_{n+1}(r) = F(L z_n(r)), \tag{7}
\]

which is the phase description of the CGLM. Because there is no approximation through the derivation of Eq. (7) from Eq. (6), this phase description is valid even when the CGLM exhibits a state in which the amplitude plays an important role in dynamics. In the case that there is no phase singular point, Eq. (7) can be reduced to the phase map

\[
e^{i\theta_{n+1}(r)} = L e^{i\theta_n(r)} / |L e^{i\theta_n(r)}|^{1+i c_2}. \tag{8}
\]

First, we carry out the linear stability analysis of plane wave solutions of the CGLM

\[
\hat{\psi}_n(r) = e^{-i|q|^2} \exp[i(q \cdot r - (c_1 - c_2)|q|^2 n)], \tag{9}
\]

with a constant vector \( q \). Because the solutions have no phase singular point (\( \hat{\psi}_n \neq 0 \)), we use the phase map (8), in which the plane wave solutions can be written in the form \( \hat{\psi}_n(r) = q \cdot r - (c_1 - c_2)|q|^2 n + \text{const} \). By substituting \( \theta_n = \hat{\psi}_n + \Delta_n \) into Eq. (8), the linearized equation for \( \Delta_n \) is obtained as \( \Delta_{n+1}(r) = \mathcal{N}[[1 - i c_2 e^{i|q|^2}]\mathcal{L} + (1+i c_2)\mathcal{L}^2 \mathcal{V}^2 \mathcal{V}] \Delta_n(r) \). Therefore, the Fourier transform \( \tilde{\Delta}_n(k) \) of \( \Delta_n \) obeys

\[
\tilde{\Delta}_{n+1}(k) = e^{i\Lambda_q(k)k} e^{i\Delta_n(k)}, \tag{10}
\]

where \( \Lambda_q(k) = -|k|^2 - 2i c_1 q \cdot k + \ln[[1 + i c_2]/2] \times e^{ic_1 k^2} + [(1 - i c_2)/2] e^{-ic_1 k^2} - 2q^2 k \). If \( \mathcal{N}[[\Lambda_q(k)] > 0, the plane wave with the wave number \( q \) is linearly unstable against the perturbation with a wave number \( k \). Expansion of \( \Lambda_q(k) \) to fourth order in \( |k| \) leads to

\[
\Lambda_q(k) = -i V_q |k|^2 - D_2 |k|^4 + i \kappa \gamma |k|^3 - D_4 |k|^4, \tag{11}
\]

with \( V_q = 2(c_1 - c_2)q_k, D_2 = 1 + c_1 c_2 - 2(1 + c_2^2)q_k^2, \gamma = \frac{1}{3} \frac{(1 + c_2^2)}{4 c_1 c_2 q_k}, \) and \( D_4 = \frac{1}{6} (1 + c_2^2) \times [3c_1^2 - 2c_1 c_2 q_k + 8(1 + c_2^2)q_k^2] \). Here \( q_k = q \cdot k / |k| \). In the case that \( D_2 < 0 \), that is, the wave number \( q \) satisfies \( |q|^2 > q_k^2 \), the plane wave solution is linearly unstable against long-wavelength perturbations. This instability is identical to the Eckhaus instability observed in the CGLE [4]. In particular, the spatially uniform state loses its stability at \( 1 + c_1 c_2 = 0 \), which is the same as the Benjamin-Feir-Newell criterion in the CGLE, and the Benjamin-Feir instability occurs for \( 1 + c_1 c_2 < 0 \). Then \( D_2 < 0 \) is satisfied for arbitrary wave numbers, and thus all plane wave solutions are linearly unstable. It is expected that the absolute instability of the plane wave solutions [14] can be also investigated.

Next, we show numerical results of the CGLM in a 2D system with the linear size \( L = 128 \pi \). We have numerically confirmed that Eq. (7) exhibits the same behavior as that in Eq. (6). Numerical cost to solve Eq. (6) is almost the same as that for Eq. (7), and \( z_n \) can be obtained from \( \theta_n \) but not vice versa. Therefore, we use Eq. (6) instead of Eq. (7) for the numerical simulations.

Slightly below the critical point \( 1 + c_1 c_2 = 0 \) of the Benjamin-Feir instability, the phase turbulence arises from initial conditions \( \psi_n = 1 \). As shown in Fig. 1(a), the amplitude \( |\psi_n| \) has the cellular structure as well as in the CGLE [15]. The phases \( \arg \psi_n \) and \( \theta_n \) have the spatial structure similar to that of the amplitude, and its spatial mean slowly increases.

Now we prove that the phase turbulence observed in the CGLM coincides with that in the CGLE. Because this turbulence has no phase singular point, we can use the phase map (8) in this proof. As shown in the linear stability analysis given above, only the long-wavelength modes are destabilized slightly below the critical point \( 1 + c_1 c_2 = 0 \). Therefore, the spatial scale of the variation of the phase variable \( \theta_n \) is expected to be large near the critical point. By letting the spatial scale be of order \( \delta^{-1/2} \) with a smallness parameter \( \delta \), the term \( \nabla^2 \theta_n \) is estimated to be of order \( \delta \), and the linear operator \( \mathcal{L} \) is expanded as \( \mathcal{L} = e^{i1/\delta} \nabla^2 + 1 + i c_1 \delta \nabla^2 + 1 + i c_1 \delta \nabla^2 \mathcal{V}^2 \mathcal{V} \). Substituting this expansion into Eq. (8) and setting \( \theta_{n+1} = \theta_n \to \partial \theta_n / \partial n \), we obtain the Kuramoto-Sivashinsky (KS) equation:

\[
\frac{\partial \theta_n}{\partial n} = (1 + c_1 c_2) \nabla^2 \theta_n + (c_2 - c_1) (\nabla \theta_n)^2 - \frac{1}{2} (c_1^2 - 2c_1 c_2 - 1) \nabla^2 \nabla^2 \theta_n. \tag{12}
\]

It should be noted that the coefficients of \( \nabla^2 \theta_n \) and \( (\nabla \theta_n)^2 \) in Eq. (12) coincide with those in the KS equation derived.
shows that pairs of defects are created and annihilate in dashed line indicates the average number $N$ evolution of the number of defects in the whole space. The dashed line indicates the average number $\langle N \rangle \approx 572.9$.

from the CGLE by the phase reduction method, while the coefficients of $\nabla^2 \nabla^2 \theta_n$ derived from the CGLM and the CGLE are different [2,16]. This deviation may be carried by the difference of the definitions of the “phases,” $\theta_n$ for the CGLM and $\arg \psi$ for the CGLE. Note that the deviation vanishes at the critical value $1 + c_1 c_2 = 0$ and that, in the limit $1 + c_1 c_2 \to 0$, the amplitude fluctuation vanishes ($\langle |\psi_0| \rangle \to 1$), and thus $\theta_n$ approaches $\arg \psi_n$. These facts imply that the phase turbulences in the CGLM and the CGLE are quantitatively the same near the critical point. One may find that the coefficient of $\nabla^2 \nabla^2 \theta_n$ is different from $D_4$ in Eq. (11) for $q = 0$. This is because the growth rate of Eq. (10), $\Delta_n + 1 - \Delta_n = (e^{-\lambda_1} - 1) \Delta_n$, is not $\Delta_q(k)$ but $[e^{\Delta_q(k)} - 1]$, whose long-wavelength expansion for $q = 0$ gives the coefficients of Eq. (12).

In addition to the phase turbulence, we observe the amplitude turbulence for $(c_1, c_2) = (-1, 0.6)$ [Fig. 1(b)], which is characterized by the exponential decay of the spatial correlation [Fig. 2(a)]. In this state, a lot of phase singular points (defects) exist over the whole space. The temporal evolution of the number $N$ of defects [Fig. 2(b)] shows that pairs of defects are created and annihilate in time. We find that the correlation length $\lambda_\text{c}$ defined in Fig. 2(a) is approximately equal to the mean distance $d_{nm}$ between a defect and its nearest neighbor: $\lambda_\text{c} \approx d_{nm} = 9.0$. Because the defect turbulence observed in the CGLE also has these characteristics [17], the amplitude turbulence in the CGLM is expected to be identical to the defect turbulence in the CGLE.

For $(c_1, c_2) = (-1, 0.4)$, after the transient amplitude turbulent state, spiral waves tend to appear and eventually cover the whole space. The amplitude shows no temporal evolution, as shown in Fig. 3(a), while the phase exhibits the spiral waves, which rotate at a constant speed. This state is called either the frozen state (FS) or the vortex glass state [9]. For $(c_1, c_2) = (-3, 0.4)$, although spiral waves also occur, the amplitude is not frozen, and both the phase and the amplitude have spiral structures [Fig. 3(b)]. Hereafter we call this state the amplitude spiral state (ASS).

Figure 4 depicts details of the spatial structures near the spiral core in the FS and the ASS. There are at least three qualitative differences between the FS and the ASS. First, the FS has an ordered spiral structure in the phase [Fig. 4(a)] and a rotationally symmetric hole structure in the amplitude [Fig. 4(b)]. However, the ASS has a distorted spiral in the phase [Fig. 4(c)] and an ordered spiral in the amplitude [Fig. 4(d)]. In the ASS case, the spiral structures in the amplitude and the phase have opposite rotational directions. Second, the position of the spiral core is motionless in the FS, while the core position rotates in the ASS. For example, the core in Fig. 4(d) rotates counterclockwise. Third, far from the spiral core, the FS exhibits the plane wave described by Eq. (9), while the ASS exhibits the plane wave with amplitude modulations. As shown below, the latter seems identical to the modulated amplitude wave (MAW) observed in the 1D CGLE, which is described as $\psi(x, t) = a(x - vt)e^{i(x + \omega t)}$ [18,19]. In the 1D CGLE, we found a solution [Fig. 5(a)] satisfying the relation $\psi_n(x) = a(x - \mu n)e^{i(x + \omega n)}$, which was demonstrated by the fact that $\psi_n(x)e^{-i(x + \omega n)}$ exhibits the translational motion with keeping its profile [Fig. 5(b)].

FIG. 2. Statistics of amplitude turbulence for $(c_1, c_2) = (-1, 0.6)$. (a) Spatial correlation $\langle \Re\langle \psi_0(r)\psi_0(0)\rangle \rangle/\langle |\psi_0(0)|^2 \rangle^2$, where $\langle \rangle$ denotes time averaging. The correlation length $\lambda_\text{c}$ is defined in the same way as in Ref. [17]. (b) Temporal evolution of the number $N$ of defects in the whole space. The dashed line indicates the average number $\langle N \rangle \approx 572.9$.

FIG. 3. Snapshots of $|\psi_n|$ of (a) the FS for $(c_1, c_2) = (-1, 0.4)$ and (b) the ASS for $(c_1, c_2) = (-3, 0.4)$.

FIG. 4 (color online). Snapshots of (a),(b) the FS for $(c_1, c_2) = (-2, 0.4)$ in a subsystem of the linear size $l = 12.5\pi$ and (c),(d) the ASS for $(c_1, c_2) = (-3, 0.5)$ in a subsystem of the linear size $l = 31.25\pi$. (a),(c) Phase field $\arg \psi_n$; (b),(d) amplitude field $|\psi_n|$ with isophase curves (red: $\Re\psi_n = 0$; green: $\Im\psi_n = 0$).
amplitude and the phase have spiral structures [23]. To investigate the relevance of these states with the ASS is a future problem.

Therefore, we believe that the relaxation process does not be well described by only the phase dynamics appropriately constructed. These facts may give new insight into the understanding of oscillatory media.

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*tsuka@acs.i.kyoto-u.ac.jp
†Deceased.

[20] We observed the ASS in the 2D CGLE with the initial condition for the $\psi$ field of the ASS obtained in the CGLE. The parameter values used in the numerical simulation of Eq. (1) are $c_1 = -3.0$, $c_2 = 0.5$, and $D = 1.0$. For these parameters, “usual” initial conditions lead to the FS.