# Iterative Learning Control of Hamiltonian Systems: I/O Based Optimal Control Approach

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***Abstract*—In this note, a novel iterative learning control scheme for a class of Hamiltonian control systems is proposed, which is applicable to electromechanical systems. The proposed method has the following distinguished features. This method does not require either the precise knowledge of the model of the target system or the time derivatives of the output signals. Despite the lack of information, the tracking error monotonously decreases in $L_2$ sense and, further, perfect tracking is achieved when it is applied to mechanical systems. The self-adjoint related properties of Hamiltonian systems proven in this note play the key role in this learning control. Those properties are also useful for general optimal control. Furthermore, experiments of a robot manipulator demonstrate the effectiveness of the proposed method.**

***Index Terms*—Mechanical systems, optimal control, tracking.**

## I. Introduction

Hamiltonian control systems are the systems described by well known Hamilton's canonical equations with controlled Hamiltonians [5]. They are introduced mainly to characterize variational properties of dynamical systems and are used for optimal control. Those systems were also utilized to describe physical systems, and the related geometric methods of controlling this class of systems supplied fruitful results in control engineering, e.g., [13] and [18]. Furthermore, this control framework was generalized in order to handle electromechanical systems, as well as conventional mechanical ones, and several control methods are proposed for them [9], [15], [18]. Thus, a scope of this note contains control of a class of physical systems such as mechanical, electrical and electromechanical systems.

The main objective of this note is to achieve iterative learning control of Hamiltonian control systems. The simplest problem setting of iterative learning control is as follows. Consider the target nonlinear operator $\Sigma : u \mapsto y$ with a prescribed desired output $y^d$. Iterative learning control is to find an input $u = u^d$ which achieves the desired output $\Sigma(u^d) = y^d$ by an iteration law

$$u_{(i+1)} = u_{(i)} + k\left(y^d - y_{(i)}\right). \tag{1}$$

Here, $u_{(i)}$ and $y_{(i)}$ denote the input and output at the $i$th operation (in laboratory experiment). The objective is to find an appropriate iteration law $k(\cdot)$ such that $y_{(i)} \to y^d$ as $i \to \infty$.

The original result on iterative learning control by Arimoto *et al.* [1] adopted the iteration law $k(\cdot)$ in a proportional derivative (PD) controller like simple form which does not require the precise knowledge of the target system $\Sigma$. This method is widely used since it can generate the desired input $u^d$ without using *a priori* information of the target system. However, it also has defects that we need to use high order time derivatives of output (error) signals which often cause instability of the convergence of the iteration in the presence of measurement noises, and that the tracking error does not monotonously decrease along the iteration. Several methods were proposed to improve this approach, e.g. [11], [12], and [17]. See also [2], [3], and [14] for the survey of the development of iterative learning control.

Alternatively, Yamakita and Furuta [19] proposed an iteration algorithm similar to iterative learning control, in which the adjoint of the target system $\Sigma$ is taken as the iteration law $k(\cdot)$. Of course, this is not a standard iterative learning control because it does require the precise model of the target system in order to construct its adjoint, that is, it is not *learning* in fact. Since this algorithm is based on optimization theory and optimal control, however, the convergence to the desired input $u^d$ is fast and numerically stable.

In this note, we propose a novel iterative learning control scheme based on the framework of optimal control. Since it is based on optimal control, it employs adjoint operators in a similar way to Yamakita's approach. However, our result does not require the precise knowledge of the target system in constructing the adjoint operators, that is, our approach is based on input–output (I/O)-based optimal control. To this end, we are going to utilize qualitative properties of physical (Hamiltonian) systems rather than quantitative ones. More precisely, we prove the self-adjoint related properties of Hamiltonian control systems. These properties allow one to obtain the I/O mapping of the adjoint by only using the I/O data of the target system, that is, the optimal control based iterative learning control scheme can be implemented by I/O data only. In the end, we can obtain a novel iterative learning control scheme for Hamiltonian control systems which does not require either the precise knowledge of the target system nor the high order time derivatives of the output (error) signal. Furthermore, since our approach is based on optimal control, the trajectory tracking error monotonously decreases in $L_2$ sense. Moreover, this scheme achieves perfect tracking when it is applied to simple mechanical systems. The authors believe that the proposed method is the first result on iterative learning control that achieves both perfect tracking and monotonously decreasing tracking error without using time derivatives nor the precise model of the target system. The self-adjoint property is useful for general optimal control as well as iterative learning control, and the results in this note will provide a new basis for model-free optimal control.

## II. Self-Adjoint Properties of Hamiltonian Systems

This section proves some properties on the self-adjoint related structure of the variationals of Hamiltonian systems which is one of the main results in this note. The properties proven here are quite useful for general optimal control as well as iterative learning control.

First of all, let us recall the Fréchet derivative of a dynamical system. Consider an operator $\Sigma : X \times U \to X \times Y$ with Hilbert spaces $X$, $U$ and $Y$ with a state-space realization

$$(x^1, y) = \Sigma(x^0, u) : \begin{cases} \dot{x} = f(x, u, t), & x(t^0) = x^0 \\ y = h(x, u, t) \\ x^1 = x(t^1) \end{cases} \tag{2}$$

defined on a time interval $t \in [t^0, t^1]$. Typically, $X = \mathbb{R}^n$, $U = L_2^m[t^0, t^1]$ and $Y = L_2^r[t^0, t^1]$. A simpler notation $\Sigma^{x^0} : U \to Y$ with

$$y = \Sigma^{x^0}(u) : \begin{cases} \dot{x} = f(x, u, t), & x(t^0) = x^0 \\ y = h(x, u, t) \end{cases}$$

is also employed. The following lemma gives that the Fréchet derivative of the operator $\Sigma$ on $\mathbb{R}^n \times L_2$ with the state-space realization in (2) which is a generalized version of [5].

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*Lemma 1:* [7] The state-space realization of the Fréchet derivative of the operator $\Sigma$ with the state-space realization (2) is given by the variational system of $\Sigma$ defined by

$$\left(x_v^1, y_v\right) = \mathrm{d}\Sigma\left((x^0,u),\left(x_v^0,u_v\right)\right)$$
$$: \begin{cases} \dot x = f(x,u,t), & x(0)=x^0\\ \binom{\dot x_v}{y_v} = \frac{\partial}{\partial(x,u)}\binom{f(x,u,t)}{h(x,u,t)}\binom{x_v}{u_v}, & x_v(0)=x_v^0 \\ x_v^1 = x_v(x^1) \end{cases}.$$

The Fréchet derivative $\mathrm{d}\Sigma^{x^0}(u)(\mathrm{d}u)$ of $\Sigma^{x^0}(u)$ is given by

$$y_v = \mathrm{d}\Sigma^{x^0}(u,u_v) : \begin{cases} \dot x = f(x,u,t), & x(0)=x^0\\ \binom{\dot x_v}{y_v} = \frac{\partial}{\partial(x,u)}\binom{f(x,u,t)}{h(x,u,t)}\binom{x_v}{u_v}, & x_v(0)=0 \end{cases}.$$

By its construction the Fréchet derivative $\mathrm{d}\Sigma(u,\mathrm{d}u)$ is a locally linear approximation to $\Sigma(u)$, i.e.,

$$\mathrm{d}\Sigma(u,v) \approx \Sigma(u+v) - \Sigma(u) \tag{3}$$

holds when $v$ is small.

Consider a Hamiltonian system with dissipation and a controlled Hamiltonian $H(x,u,t)$ described by

$$(x^1,y) = \Sigma(x^0,u)$$
$$: \begin{cases} \dot x = (J-R)\frac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0)=x^0 \\ y = -\frac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}} \\ x^1 = x(t^1) \end{cases}. \tag{4}$$

Here, the structure matrices $J \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{n\times n}$ are skew-symmetric and symmetric positive semidefinite, respectively. The matrix $R$ represents dissipative elements such as friction of mechanical systems and resistance of electric circuits. For this system, the following theorem holds.

*Theorem 1:* Consider the Hamiltonian system with dissipation and the controlled Hamiltonian $\Sigma$ in (4). Suppose that $J$ and $R$ are constant and that there exists a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ satisfying

$$\begin{aligned} J &= -TJT^{-1}\\ R &= TRT^{-1}\\ \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2} &= \begin{pmatrix} T & 0\\ 0 & I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix} T^{-1} & 0\\ 0 & I\end{pmatrix}. \end{aligned} \tag{5}$$

Then the Fréchet derivative of $\Sigma$ is described by another Hamiltonian system

$$\left(x_v^1, y_v\right) = \mathrm{d}\Sigma\left((x^0,u),\left(x_v^0,u_v\right)\right)$$
$$: \begin{cases} \dot x = (J-R)\frac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0)=x^0\\ \dot x_v = (J-R)\frac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}}, & x_v(t^0) = x_v^0\\ y_v = -\frac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}} \\ x_v^1 = x_v(t^1) \end{cases} \tag{6}$$

with a controlled Hamiltonian $H_v(x,u,x_v,u_v,t)$

$$H_v(x,u,x_v,u_v,t) = \frac{1}{2}\begin{pmatrix} x_v\\ u_v\end{pmatrix}^{\mathrm{T}}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix} x_v\\ u_v\end{pmatrix}.$$

Suppose, moreover, that $J-R$ is nonsingular. Then the adjoint $(x_a^1,u_a) \mapsto (x_a^0,y_a)(\mathrm{d}\Sigma(x^0,u))^*(x_a^1,u_a)$ is given by the same state-space realization

$$\begin{cases} \dot x = (J-R)\frac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0)=x^0\\ \dot x_v = -(J-R)\frac{\partial H_v(x,u,x_v,u_a,t)}{\partial x_v}^{\mathrm{T}}, & x_v(t^1) = -(J-R)Tx_a^1\\ y_a = -\frac{\partial H_v(x,u,x_v,u_a,t)}{\partial u_a}^{\mathrm{T}} \\ x_a^0 = -T^{-1}(J-R)^{-1}x_v(t^0) \end{cases}. \tag{7}$$

That is, $(\mathrm{d}\Sigma)^*$ coincides with the time-reversal version of $\mathrm{d}\Sigma$. Even if $J-R$ is singular, the adjoint of the variational with zero initial state $u_a \mapsto y_a = (\mathrm{d}\Sigma^{x^0}(u))^*(u_a)$ is given by the same state-space realization (7) with the zero terminal state $x_v(t^1)=0$.

*Proof:* First of all, let us calculate the variational system of $\Sigma$ according to Lemma 1

$$\begin{cases} \dot x = (J-R)\frac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0)=x^0\\ \begin{pmatrix}\dot x_v\\ y_v\end{pmatrix} = \frac{\partial}{\partial(x,u)}\begin{pmatrix} (J-R)\frac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}} \\ -\frac{\partial H(x,u,t)}{\partial u}^{\mathrm{T}}\end{pmatrix}\begin{pmatrix} x_v\\ u_v\end{pmatrix}, & x_v(t^0) = x_v^0\\ x_v^1 = x_v(t^1) \end{cases}.$$

We obtain

$$\begin{aligned} \begin{pmatrix}\dot x_v\\ y_v\end{pmatrix} &= \begin{pmatrix} J-R & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}^{\mathrm{T}}\begin{pmatrix} x_v\\ u_v\end{pmatrix}\\ &= \begin{pmatrix} J-R & 0\\ 0 & -I\end{pmatrix}\\ &\quad\times\left[\frac{\partial}{\partial(x_v,u_v)}\left\{\frac{1}{2}\begin{pmatrix} x_v\\ u_v\end{pmatrix}^{\mathrm{T}}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}^{\mathrm{T}}\begin{pmatrix} x_v\\ u_v\end{pmatrix}\right\}\right]^{\mathrm{T}}\\ &= \begin{pmatrix} J-R & 0\\ 0 & -I\end{pmatrix}\frac{\partial H_v(x,u,x_v,u_v,t)}{\partial(x_v,u_v)}^{\mathrm{T}}\\ &= \begin{pmatrix} (J-R)\frac{\partial H_v(x,u,x_v,u_v,t)}{\partial x_v}^{\mathrm{T}}\\ -\frac{\partial H_v(x,u,x_v,u_v,t)}{\partial u_v}^{\mathrm{T}}\end{pmatrix} \end{aligned}$$

which equals to (6). Next, we calculate its adjoint as

$$\begin{cases} \dot x = (J-R)\frac{\partial H(x,u,t)}{\partial x}^{\mathrm{T}}, & x(t^0)=x^0\\ \begin{pmatrix}\dot x_a\\ y_a\end{pmatrix} = \begin{pmatrix} -I & 0\\ 0 & I\end{pmatrix} \\ \qquad\times\left(\begin{pmatrix} J-R & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\right)^{\mathrm{T}}\begin{pmatrix} x_a\\ u_a\end{pmatrix}, & x_a(t^1) = x_a^1\\ x_a^0 = x_a(t^0) \end{cases}.$$

Here, let us define a (possibly singular) coordinate transformation $\bar{x_a} = -(J-R)Tx_a$. Then, we obtain

$$\begin{aligned} \begin{pmatrix}\dot{\bar x}_a\\ y_a\end{pmatrix} &= \begin{pmatrix} -(J-R)T & 0\\ 0 & I\end{pmatrix}\begin{pmatrix}\dot x_a\\ y_a\end{pmatrix}\\ &= \begin{pmatrix} (J-R)T & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix} -I & 0\\ 0 & I\end{pmatrix}\\ &\quad\times\left(\begin{pmatrix} J-R & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\right)^{\mathrm{T}}\begin{pmatrix} x_a\\ u_a\end{pmatrix}\\ &= \begin{pmatrix} -(J-R) & 0\\ 0 & -I\end{pmatrix}\begin{pmatrix} T & 0\\ 0 & I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\\ &\quad\times\begin{pmatrix} T & 0\\ 0 & I\end{pmatrix}^{-1}\begin{pmatrix} (J-R)Tx_a & 0\\ 0 & -u_a\end{pmatrix}\\ &= \begin{pmatrix} -(J-R) & 0\\ 0 & -I\end{pmatrix}\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}\begin{pmatrix}\bar x_a\\ u_a\end{pmatrix}. \end{aligned}$$

Furthermore, if $J-R$ is nonsingular then the behavior of the state $x_a(t)$ can be recovered by $x_a(t) = T^{-1}(J-R)^{-1}\bar{x}_a(t)$. This implies (7) and completes the proof. □

Note that many electric circuits satisfy (5) in Theorem 1. Unfortunately, however, most mechanical and electromechanical systems do not satisfy it. Section IV will discuss how to overcome this problem. It is also noted that if the system is a *gradient system* [4] which is a nonlinear generalization of a linear symmetric system, that is, $J=0$, then the assumption (5) in Theorem 1 is automatically satisfied with $T = I$ (provided $R$ is constant). On the other hand, if the system is conservative, that is, $R=0$ then it is self-adjoint in the usual sense [5] in $L_2$ spaces which can be extended to the signal space $\mathbb{R}^n \times L_2$ using the technique similar to Theorem 1 (provided $J$ is constant) [10]. Furthermore, another characterization of nonlinear adjoint operators can be found in [8].

*Remark 1:* Note that the state of the dynamics in (7) is the time-reversal version of that in (6). Suppose the input $u$ is given such that the time history of the Hessian of the Hamiltonian with respect to $(x, u)$ is symmetrical with respect to the middle of the time interval $(t^0+t^1)/2$, i.e.,

$$\frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}(t-t^0) = \frac{\partial^2 H(x,u,t)}{\partial(x,u)^2}(t^1-t) \qquad \forall t \in [t^0, t^1].$$

Then, $\mathrm{d}\Sigma$ has a (pseudo) self-adjoint state-space realization. This condition often occurs in a point-to-point control of robot manipulators. A useful alternative condition for mechanical systems will be discussed in Section IV.

Under the circumstances in Remark 1, Theorem 1 implies that the time-reversal system of the adjoint $(\mathrm{d}\Sigma)^*$ coincides with the variational $\mathrm{d}\Sigma$. Combined with the property of the variational system (3), we can calculate the I/O mapping of the adjoint by only using the I/O data of the original system as follows when $v$ is small:

$$(\mathrm{d}\Sigma(u))^*(v) = \mathcal{R}\circ\mathrm{d}\Sigma(u)\circ\mathcal{R}(v) \approx \mathcal{R}\circ(\Sigma(u+\mathcal{R}(v)) - \Sigma(u)) \quad (8)$$

where $\mathcal{R}$ is a time-reversal operator defined by

$$\mathcal{R}(u)(t-t^0) = u(t^1-t) \qquad \forall t \in [t^0, t^1]. \quad (9)$$

This means that the adjoint of the variational of the original system can be easily obtained via the aforementioned procedure.

## III. Iterative Learning Control and Optimal Control

This section explains how to apply the self-adjoint related property proven in Section II to iterative learning control.

Let us consider the Hamiltonian system in (4). As briefly explained in Section I, iterative learning control of a nonlinear system $\Sigma$: $u \mapsto y$ with a prescribed desired output $y^d$ (defined on a time interval $[t^0, t^1]$) is to obtain the input $u^d$ producing the desired output $\Sigma(u^d) = y^d$ by iteration law (1). Recall that $u_{(i)}$ and $y_{(i)}$ denote the input and output in the $i$th iteration in laboratory experiment.

Clearly, the objective is to find an appropriate $k(\cdot)$ satisfying $y_{(i)} \to y^d$ as $i \to \infty$ with respect to some norm space. In conventional iterative learning control [1], a special norm (so called $\lambda$ norm) was adopted which is different from the standard norms such as $L_n$ hence the tracking error $y^d - y_{(i)}$ can be pretty large in the sense of the standard norm before converging to zero. Here, we take a cost function similar to the standard $L_2$ norm

$$\Gamma(y) = \int_{t^0}^{t^1} \left(y(t) - y^d(t)\right)^{\mathrm{T}} \Gamma_y \left(y(t) - y^d(t)\right) \mathrm{d}t \quad (10)$$

with a positive–definite matrix $\Gamma_y \in \mathbb{R}^{m\times m}$, and try to let it converge to zero.

Let us calculate its Fréchet derivative

$$\begin{aligned}\mathrm{d}\Gamma(y)(\mathrm{d}y) &= -2\left\langle \Gamma_y(y^d-y), \mathrm{d}y\right\rangle_{L_2} \\ &= -2\left\langle \Gamma_y(y^d-y), \mathrm{d}\Sigma(u)(\mathrm{d}u)\right\rangle_{L_2} \\ &= -2\left\langle (\mathrm{d}\Sigma(u))^*\,\Gamma_y(y^d-y), \mathrm{d}u\right\rangle_{L_2}.\end{aligned}$$

Therefore, the steepest decent method implies that we should change the input $u$ in the direction of $(\mathrm{d}\Sigma(u))^*\Gamma_y(y^d-y)$ in order to reduce the cost function $\Gamma$ (since the choice $\mathrm{d}u = (\mathrm{d}\Sigma(u))^*\Gamma_y(y^d-y)$ makes the derivative $\mathrm{d}\Gamma$ negative). Hence, the iteration law (1) should be taken as

$$u_{(i+1)} = u_{(i)} + K_{(i)}\left(\mathrm{d}\Sigma^{x^0}\left(u_{(i)}\right)\right)^* \Gamma_y\left(y^d - y_{(i)}\right) \quad (11)$$

with an appropriate positive–definite *gain* matrix $K_{(i)} > 0$, in order to reduce the cost function $\Gamma$ effectively.

Let us recall the fact that we can utilize the self-adjoint property proven in Theorem 1 since our target system $\Sigma$ is a Hamiltonian system (4). In order to use (8) in Remark 1, we formally employ the following assumption.

*Assumption A1:* It is assumed that the desired trajectory $x^d(t)$ and input $u^d(t)$ satisfy

$$\left.\frac{\partial^2 H(x,u)}{\partial(x,u)^2}\right|_{\substack{x=x^d(t-t^0)\\u=u^d(t-t^0)}} = \left.\frac{\partial^2 H(x,u)}{\partial(x,u)^2}\right|_{\substack{x=x^d(t^1-t)\\u=u^d(t^1-t)}},\ \forall t \in [t^0, t^1].$$

Under Assumption A1, (8) (and Remark 1) reduces (11) down to

$$\begin{aligned}u_{(i+1)} &= u_{(i)} + K_{(i)}\left(\mathrm{d}\Sigma^{x^0}\left(u_{(i)}\right)\right)^*\Gamma_y\left(y^d - y_{(i)}\right) \\ &= u_{(i)} + K_{(i)}\mathcal{R}\circ\left(\mathrm{d}\Sigma^{x^0}\left(u_{(i)}\right)\right)\circ\mathcal{R}\left(\Gamma_y\left(y^d - y_{(i)}\right)\right) \\ &= u_{(i)} + \bar{K}_{(i)}\mathcal{R}\circ\mathrm{d}\Sigma^{x^0}\left(u_{(i)}\right)\circ\mathcal{R}\left(\kappa_{(i)}\Gamma_y\left(y^d - y_{(i)}\right)\right) \\ &\approx u_{(i)} + \bar{K}_{(i)}\mathcal{R} \\ &\quad \circ\left(\Sigma^{x^0}\left(u_{(i)} + \mathcal{R}\left(\kappa_{(i)}\Gamma_y\left(y^d - y_{(i)}\right)\right)\right) - y_{(i)}\right)\end{aligned}$$

with a sufficiently small $\kappa_{(i)} > 0$ and $\bar{K}_{(i)} := K_{(i)}/\kappa_{(i)}$ by definition of Fréchet derivative. Renaming $\bar{K}_{(i)}$ into $K_{(i)}$ again yields the following two steps iteration law which can be implemented by only using the I/O data of the target system $\Sigma$.

*Procedure 1:* Consider the Hamiltonian system (4) with a given desired trajectory $x^d(t)$ of $x(t)$ defined on $[t^0, t^1]$. Suppose the assumptions in Theorem 1 and Assumption A1 hold. Then, the iterative learning control law is given by

$$\begin{aligned}u_{(2i+1)} &= u_{(2i)} + \mathcal{R}\left(\kappa_{(i)}\Gamma_y\left(y^d - y_{(2i)}\right)\right) \\ u_{(2i+2)} &= u_{(2i)} + K_{(i)}\mathcal{R}\left(y_{(2i+1)} - y_{(2i)}\right)\end{aligned} \quad (12)$$

for $i = 0, 1, 2, \cdots$. Here, $\Gamma_y$ defines the cost function $\Gamma$ in (10). The parameters $\kappa_{(i)} > 0 \in \mathbb{R}$ and $K_{(i)} > 0 \in \mathbb{R}^{m \times m}$ are small enough design parameters. $\mathcal{R}$ denotes the time-reversal operator defined in (9).

Note that the iteration procedure does *not* require any physical parameters such as $T$, $J$, $R$, and $H$. Only the requirement is the fact that the I/O mapping $\Sigma$: $u \mapsto y$ is described by the Hamiltonian system. This result will provide a basis of a new iterative learning control for a class of physical (Hamiltonian) systems. Unfortunately, this iteration procedure only guarantees the convergence to a local minimum of the cost function (10), that is, perfect tracking is not guaranteed in general.[1] It will be shown in the following section that perfect tracking is always achieved when it is applied to mechanical systems.

## IV. Iterative Learning Control of Mechanical Systems

A typical mechanical system can be described by a Hamiltonian system

$$\Sigma : \begin{cases} \begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & -R_p \end{pmatrix} \begin{pmatrix} \frac{\partial H(q,p,u)}{\partial q}^{\mathrm{T}} \\ \frac{\partial H(q,p,u)}{\partial p}^{\mathrm{T}} \end{pmatrix} \\ y = \frac{\partial H(q,p,u)}{\partial u}^{\mathrm{T}} = q \end{cases} \tag{13}$$

with the Hamiltonian

$$H(q, p, u) = \frac{1}{2} p^{\mathrm{T}} M(q)^{-1} p + V(q) - u^{\mathrm{T}} q$$

where a positive–definite matrix $M(q) > 0 \in \mathbb{R}^{m \times m}$ denotes the inertia matrix, a positive–semidefinite matrix $R_p$ denotes the friction coefficients, and a scalar function $V(q)$ denotes the potential energy of the system.

Unfortunately, however, this system does not satisfy the assumptions in Theorem 1 since there does not exist the matrix $T$ satisfying the matching condition (5). The procedure in the sequel enables the system to satisfy this condition approximately.

Typically, feedback controllers are employed to control the system (13) even when iterative learning control is applied, since it is marginally stable without any feedback. This subsection discusses feedback system design for the proposed iterative learning control method. It is known that that a simple PD feedback preserves the structure of the Hamiltonian system (13). Further discussions on controller design preserving the structure of general Hamiltonian systems can be found in [9], [15], and [18]. Let us consider a PD controller

$$u = \bar{u} - K_q\, q - K_p \dot{q} \tag{14}$$

where $\bar{u}$ is a new input and $K_q$, $K_p > 0 \in \mathbb{R}^{m \times m}$ are symmetric positive–definite matrices. Applying a coordinate transformation

$$q = \varepsilon \bar{q} \tag{15}$$

with a positive constant $\varepsilon > 0$ converts the system into another Hamiltonian system

$$\begin{cases} \begin{pmatrix} \dot{\bar{q}} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\varepsilon} I \\ -\frac{1}{\varepsilon} I & -(R_p + K_p) \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{H}(\bar{q},p,\bar{u})}{\partial \bar{q}}^{\mathrm{T}} \\ \frac{\partial \bar{H}(\bar{q},p,\bar{u})}{\partial p}^{\mathrm{T}} \end{pmatrix} \\ y = -\dfrac{\partial \bar{H}(\bar{q}, p, \bar{u})}{\partial \bar{u}}^{\mathrm{T}} = \varepsilon \bar{q} = q \end{cases} \tag{16}$$

with a new Hamiltonian

$$\bar{H}(\bar{q}, p, \bar{u}) = \frac{1}{2} p^{\mathrm{T}} M(\varepsilon \bar{q})^{-1} p + V(\varepsilon \bar{q}) + \frac{\varepsilon^2}{2} \bar{q}^{\mathrm{T}} K_q \bar{q} - \varepsilon \bar{u}^{\mathrm{T}} \bar{q}.$$

Let us choose the parameter matrix in Theorem 1 as

$$T = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{17}$$

and check the matching condition (5). The former two equations hold straightforwardly and the left- and right-hand sides of the last equation become

$$\begin{aligned} &\frac{\partial^2 \bar{H}(\bar{q}, p, \bar{u})}{\partial(\bar{q}, p, \bar{u})^2} \\ &\quad = \begin{pmatrix} \varepsilon^2 \left( \frac{\partial^2 H_0(q,p)}{\partial q^2} + K_q \right) & \varepsilon \frac{\partial M(q)^{-1} p}{\partial q}^{\mathrm{T}} & -I \\ \varepsilon \frac{\partial M(q)^{-1} p}{\partial q} & M(q)^{-1} & 0 \\ -I & 0 & 0 \end{pmatrix} \\ &\begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \frac{\partial^2 \bar{H}(\bar{q}, p, \bar{u})}{\partial(\bar{q}, p, \bar{u})^2} \begin{pmatrix} T^{-1} & 0 \\ 0 & I \end{pmatrix} \\ &\quad = \begin{pmatrix} \varepsilon^2 \left( \frac{\partial^2 H_0(q,p)}{\partial q^2} + K_q \right) & -\varepsilon \frac{\partial M(q)^{-1} p}{\partial q}^{\mathrm{T}} & -I \\ -\varepsilon \frac{\partial M(q)^{-1} p}{\partial q} & M(q)^{-1} & 0 \\ -I & 0 & 0 \end{pmatrix}. \end{aligned}$$

Hence, if the "$P$ gain" $K_q$ is chosen large enough and the parameter $\varepsilon$ is taken small enough accordingly, then the relation

$$\frac{\partial^2 \bar{H}(\bar{q}, p, \bar{u})}{\partial(\bar{q}, p, \bar{u})^2} \approx \begin{pmatrix} T & 0 \\ 0 & I \end{pmatrix} \frac{\partial^2 \bar{H}(\bar{q}, p, \bar{u})}{\partial(\bar{q}, p, \bar{u})^2} \begin{pmatrix} T^{-1} & 0 \\ 0 & I \end{pmatrix}$$

holds, that is, (5) in Theorem 1 is satisfied approximately. Note that the "D gain" $K_p$ should also be chosen large enough to let the matrix $R_q + K_q$, which describes the dissipation behavior of the system (16) in the coordinate $(\bar{q}, p)$, sufficiently large compared with the matrix $I/\varepsilon$, which denotes the oscillation behavior. This should be done for numerical stability of the iterative learning procedure. Here, we adopt the following assumptions corresponding to Assumption A1.

*Assumption B1:* It is assumed that the desired trajectory $x^d(t) = (q^d(t), p^d(t)) = (q^d(t), M(q^d(t))^{-1} q^d(t))$ satifies

$$\left. \frac{\partial^2 H_0(q,p)}{\partial(q,p)^2} \right|_{x = x^d(t - t^0)} = \left. \frac{\partial^2 H_0(q,p)}{\partial(q,p)^2} \right|_{x = x^d(t^1 - t)}, \ \forall t \in [t^0, t^1].$$

*Assumption B2:* PD gains $K_q$ and $K_p$ are large enough.

When the desired trajectory $q^d(t)$, $t \in [t^0, t^1]$ does not satisfy Assumption B1, we can produce a desired trajectory fulfilling B1 by simply reproducing the same trajectory in the time domain $t \in [t^1, 2t^1 - t^0]$ as

$$q^d_{\mathrm{new}}(t) = \begin{cases} q^d(t), & t \in [t^0, t^1] \\ q^d(2t^1 - t^0 - t), & t \in [t^1, 2t^1 - t^0] \end{cases}.$$

Since Assumptions B1 and B2 imply Assumption A1 and (5) in Theorem 1, we can readily apply the iterative learning scheme in Procedure 1 to our mechanical system (13) with the output $y = q$. In practice, whether Assumption B2 holds or not can be checked by observing the history of the cost function, since if it holds then there exists a small gain $K_{(i)}$ which decreases the cost function.

This iterative learning control scheme can be depicted as in Fig. 1. It is very simple in the sense that it does not employ any physical parameters of the target system. Compared with Arimoto's method [1] which is also simple, the proposed method is expected to be numerically more stable because our approach does not employ time deriva-

[1]In particular, one may fail to achieve perfect tracking in the case where the variational system $\mathrm{d}\Sigma$ has unstable zero-dynamics.

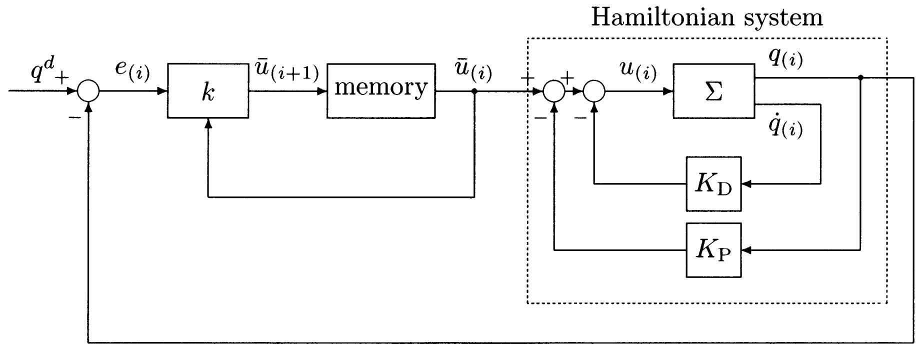


Fig. 1. Iterative learning control of mechanical systems.

tives[2] whereas Arimoto's method requires second order time derivative of $q$ for mechanical systems as in (13). Moreover, the trajectory tracking error $y^d - y_{(i)}$ monotonously decreases in the sense of $L_2$ in our approach.

Furthermore, we can prove the convergence to the global minimum, i.e., *perfect tracking*, of this iteration procedure, though Procedure 1 only guarantees the convergence to a local minimum in general.

*Theorem 2:* Consider the mechanical Hamiltonian system (13). Suppose that Assumptions B1 and B2 hold and there exists a positive constant $\epsilon$ satisfying

$$\kappa_{(i)} K_{(i)} \geq \epsilon I > 0 \qquad \forall i. \tag{18}$$

Then, for any initial input $\bar{u}_{(0)}$, the iterative learning control law (12) converges to an optimal input $\bar{u}^d$.

*Proof:* The variational system $\mathrm{d}\Sigma^{x^0}$ of the mechanical Hamiltonian system (13) can be described by

$$\begin{cases} \begin{pmatrix} \dot{q}_v \\ \dot{p}_v \end{pmatrix} = \begin{pmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{pmatrix} \begin{pmatrix} q_v \\ p_v \end{pmatrix} + \begin{pmatrix} 0 \\ I \end{pmatrix} u_v \\ y_v = q_v \end{cases}$$

with appropriate matrices $A_{ij}$'s. Let us now calculate the zero-dynamics of this system. Take $y_v \equiv 0$. Then, it follows that

$$0 \equiv \dot{q}_v = A_{11} q_v + A_{12} p_v = A_{12} p_v = M(q)^{-1} p_v.$$

Therefore, we prove $p_v \equiv 0$. Finally, we obtain

$$0 \equiv \dot{p}_v = A_{21} p_v + A_{22} q_v + u_v = u_v.$$

This implies that the variational system has no zero-dynamics, that is, when the output of this operator converges to zero then so does the input signal. Therefore, the iteration law (12) and (18) imply

$$\bar{u}_{(2i)} \to \bar{u}_{(2i+2)} \quad \Rightarrow \quad q_{(2i)} \to q^d$$

that is, the control law converges to an optimal input $\bar{u}^d$. This completes the proof. □

*Remark 2:* Note that the sets of coordinate and feedback transformations preserving the structure of Hamiltonian systems such as the set of (14) and (15) are called *generalized canonical transformations*. The iterative learning control for mechanical systems can be obtained due to this transformation. It is always possible to combine the generalized canonical transformation and the proposed iterative learning method in a similar way. All the class of such transformations and the related control results can be found in [9].

[2]The time derivative in the PD controller can also be replaced by a linear causal operator by the technique in [6], in the case where the velocity $\dot{q}$ is not measurable.

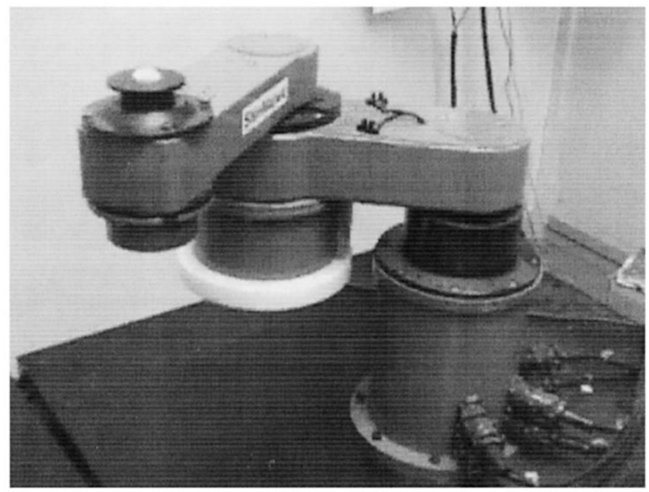

Fig. 2. Two-link manipulator.

## V. Experimental Evaluation

The procedure given in the previous section is now applied to a two-link robot manipulator depicted in Fig. 2, whose height is 0.55 [m]. Each joint is driven by a direct drive motor, and each link rotates on the horizontal plane. This system is a typical example of Hamiltonian systems and its dynamics can be described by (13). Here, $q = (q_1, q_2)$ and $u = (u_1, u_2)$ and the momentum $p$ is defined by $p = M(q)\dot{q}$ with the inertia matrix $M(q)$ given by

$$\begin{aligned} M(q) &= \begin{pmatrix} \rho_1 + \rho_2 + 2l_1\rho_3\cos\theta_2 & \rho_2 + l_1\rho_3\cos\theta_2 \\ \rho_2 + l_1\rho_3\cos\theta_2 & \rho_2 \end{pmatrix} \\ \rho_1 &:= I_1 + m_1 l_{g1}^2 + m_2 l_1^2 \\ \rho_2 &:= I_2 + m_2 l_{g2}^2 \quad \rho_3 := m_2 l_{g2}. \end{aligned}$$

The friction matrix is given by $R_p = \mathrm{diag}(r_1, r_2)$ and the potential energy is $V = 0$ because the links move on the horizontal plane. The parameters are defined as follows: $u_i$ (Nm) denotes the input torque for joint $i$, $m_i$ (kg) denotes the mass of link $i$, $l_i$ (m) denotes the length of link $i$, $l_{gi}$ [m] denotes the length from the center to joint of link $i$, $I_i$ (kgm$^2$) denotes the inertia of link $i$, $r_i$ [Nms/rad] denotes the friction coefficient of joint $i$ and $q_i$ [rad] denotes the rotation angle of link $i$. The concrete values of the parameters are $l_1 = 0.25, l_2 = 0.30, \rho_1 = 2.55$, $\rho_2 = 0.72$, $\rho_3 = 2.60$, $r_1 = 0.2415$ and $r_2 = 0.2457$. Note that the learning control system design does *not* require those parameters at all; see [16] for details of this apparatus.

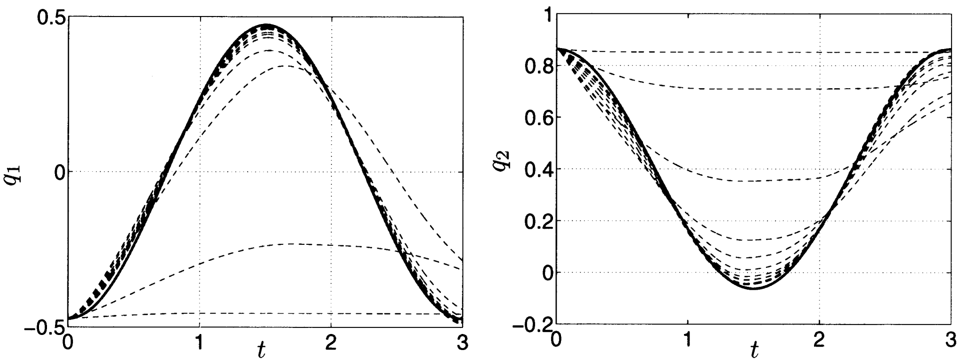


Fig. 3. Responses of the angles $q_1$ and $q_2$ of the links 1 and 2.

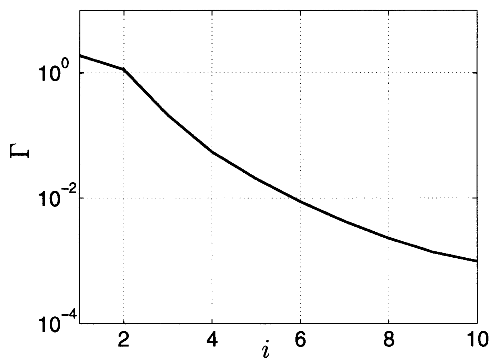


Fig. 4. Cost function $\Gamma$.

The design parameters of the iterative learning control scheme in Procedure 1 are chosen as follows: $\Gamma_y = I$, $\kappa_{(i)} = 1$, $K_{(i)} \equiv 1400\,I$, $K_q = 30\,I$ and $K_p = 20\,I$. The desired trajectory $y^d(t) = q^d(t)$, $t \in [0, 3]$ is given by

$$q^d(t) = \begin{pmatrix} -0.473\,451\cos(0.01\pi t) \\ 0.463\,212\cos(0.01\pi t) + 0.4 \end{pmatrix}.$$

The experimental results of ten times iteration are given in Figs. 3 and 4. Fig. 3 shows the responses of the angles $q_1$ and $q_2$ of links 1 and 2. In the figure, the (thick) solid lines denote the desired trajectories $q_1^d$ and $q_2^d$, the thin dashed lines denote the responses of $q_{1(i)}$ and $q_{2(i)}$ at the $i$th operation, and the thick dashed lines denote the response of $q_{1(10)}$ and $q_{2(10)}$ at the tenth iteration. Fig. 4 depicts the history of the cost function $\Gamma$ in (10) in the log scale at each iteration.

The figures show that the output trajectories converge to the desired ones smoothly. In particular, Fig. 4 shows that the convergence is sufficiently fast. These experimental results show that the proposed method works quite well. Utilizing the qualitative property of physical systems intensively, we can thus obtain a simple and effective iterative learning control scheme in this note.

## VI. Conclusion

This note has discussed iterative learning control of Hamiltonian systems. A novel iterative learning control scheme has been proposed based on the self-adjoint related structure of their variational systems. This method does not require either the knowledge of the precise model of the target system nor the time derivatives of the output signals. Despite the lack of the information, the tracking error monotonously decreases in $L_2$ sense and, further, perfect tracking is achieved when it is applied to simple mechanical systems. Furthermore, experiments of a robot manipulator have demonstrated the effectiveness of the proposed method. The self-adjoint properties are useful for general optimal control as well as iterative learning control and the results in this note will provide a basis for new learning methodologies.

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