A Second-Order Algorithm for Continuous-Time Nonlinear Optimal Control Problems

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Abstract—**A second-order algorithm is presented for the solution of continuous-time nonlinear optimal control problems. The algorithm is an**

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adaptation of the trust region modifications of Newton's method and solves at each iteration a linear-quadratic control problem with an additional constraint. Under some assumptions, the proposed algorithm is shown to possess a global convergence property. A numerical example is presented to illustrate the method.

I. Introduction

Strongly motivated by the success of the trust region approach in the finite-dimensional optimization [4], [10], [11], [13], this note deals with a globally convergent second-order algorithm for continuous-time optimal control problems. Second-order algorithms which are essentially equivalent to Newton's method in the control function space have been presented in the early literature [1], [3], [8], [9]. Unfortunately, these algorithms are only locally convergent, and hence suffer from difficulties in choosing a good initial guess of an optimal control. Bullock and Franklin [2] proposed a modified second-order method which might considerably enlarge the region of convergence. The basic idea underlying the latter is quite similar to that of the trust region approach presented in this note. Their method is, however, based on some heuristic considerations and no convergence analysis is provided. As far as the second-order methods are concerned, there seem to have been relatively few attempts to devise efficient algorithms for solving optimal control problems [12].

II. Trust Region Method for Optimal Control Problems

Consider the following unconstrained optimal control problem. Let $x(t)$ and $u(t)$ be an n-dimensional state vector and an m-dimensional control vector, respectively, which are related by the nonlinear dynamical equations

$$\dot{x}(t)=f(x(t),\ u(t),\ t), \qquad x(0)=x_0 \tag{2.1}$$

over the time interval $[0,\ T]$. The cost functional to be minimized is defined by

$$J(u)=F(x(T))+\int_0^T L(x(t),\ u(t),\ t)\ dt. \tag{2.2}$$

The terminal time T is assumed to be fixed.

We assume that the functions f: $R^{n+m+1} \rightarrow R^n$, F: $R^n \rightarrow R$ and L: $R^{n+m+1} \rightarrow R$ have continuous second derivatives. Also we assume that the optimal control problem (2.1), (2.2) is solved in the space $L^2[0,\ T]$ of square-integrable control functions on $[0,\ T]$. For any $u \in L^2[0,\ T]$, the norm of u is denoted simply by $\|u\|$.

Let $u \in L^2[0,\ T]$. Then we may obtain the corresponding trajectory x by integrating the system equations (2.1). Let λ be an n-dimensional Lagrange multiplier function and define the Hamiltonian H: $R^n \times R^m \times R \rightarrow R$ by

$$H(x(t),\ u(t),\ \lambda(t),\ t)=L(x(t),\ u(t),\ t)+\lambda(t)'f(x(t),\ u(t),\ t). \tag{2.3}$$

Then, for any control u and the corresponding trajectory x, we may obtain $\lambda(t)$, $t \in [0,\ T]$, by the backward integration of the adjoint equations

$$\dot{\lambda}(t)=-H_x(x(t),\ u(t),\ \lambda(t),\ t), \qquad \lambda(T)=F_x(x(T)). \tag{2.4}$$

We assume that, for any given u, the system equations (2.1) and the adjoint equations (2.4) are uniquely solvable.

The following formula for the second variation of the cost functional J around the nominal control u is well known and has played an important role in the second-order algorithms such as [2], [9]:

$$\begin{aligned} Q(v)=&\int_0^T H_u(t)'v(t)\ dt+\frac{1}{2}y(T)'F_{xx}(x(T))y(T)\\ &+\frac{1}{2}\int_0^T [v(t)'H_{uu}(t)v(t)+2v(t)'H_{ux}(t)y(t)\\ &+y(t)'H_{xx}(t)y(t)]\ dt \end{aligned} \tag{2.5}$$

where v and y are understood to take the place of variations δu and δx, respectively, and satisfy the linearized system equations

$$\dot{y}(t)=f_x(t)y(t)+f_u(t)v(t), \qquad y(0)=0. \tag{2.6}$$

In the above expressions, the simplified notation such as $H_u(t)$ and $f_x(t)$ has been used to denote the respective functions evaluated at the nominal control u and the corresponding x and λ. It is assumed throughout that the above formulas (2.5), (2.6) give a second-order approximation of (2.1), (2.2) in the sense that the following relation holds:

$$J(u+v)-J(u)=Q(v)+o(\|v\|^2). \tag{2.7}$$

The modified Newton's method presented in this paper consists of successively solving auxiliary problems of the following form:

$$\text{minimze } Q(v) \text{ subject to } \|v\| \leq \Delta \tag{2.8}$$

where Δ is a positive parameter which specifies a neighborhood, called the *trust region,* around u. It is noted that problem (2.8) implicitly includes (2.6) that provides the dependence relationship between y and u in the expression (2.5) of Q. If the constraint $\|v\| \leq \Delta$ is ignored, then problem (2.8) reduces to the auxiliary problem of the classical second variation method [9]. A merit of introducing this constraint is that problem (2.8) is very likely to possess an optimal solution even without strong conditions such as the positive definiteness of H_{uu}, which must be assumed to hold for successive auxiliary problems in the classical second variation methods. Thus, we shall simply assume the existence of an optimal solution of problem (2.8), rather than examining existence conditions.

Suppose we have obtained an optimal solution $\bar{v}$ of problem (2.8). If the functional Q affords a good approximation of the original cost functional in the region $\|v\| \leq \Delta$, then a substantial improvement over the nominal control u would be obtained by the control $u + \bar{v}$. On the other hand, if Δ is so large that the approximation is inaccurate at $\bar{v}$, then the control $u + \bar{v}$ may not be expected to yield a significant reduction in the cost functional value. Therefore, the parameter Δ should be controlled so as to take account of the accuracy of the approximation Q at $\bar{v}$. This may be done systematically on the basis of the ratio

$$\rho=\{J(u+\bar{v})-J(u)\}/Q(\bar{v}) \tag{2.9}$$

where the numerator and the denominator on the right-hand side are the actual reduction in J caused by $\bar{v}$ and the corresponding predicted reduction, respectively, [4], [10], [11], [14]. Clearly, the closer ρ is to unity, the more accurate the approximation is. It is to be noted that the denominator is negative, unless $v = 0$ solves problem (2.8).

Now we may explicitly state the algorithm. Let $0 < \eta_1 < \eta_2 < 1$ and $0 < \gamma_1 < 1 < \gamma_2$ be prespecified constants. (The values $\eta_1 = 0.25$, $\eta_2 = 0.75$, $\gamma_1 = 0.25$, and $\gamma_2 = 2$ are suggested in [4].)

Step 0: Choose a nominal control u^0 and $\Delta^0 > 0$. Set $k = 0$.

Step 1: Calculate the nominal trajectory x^k and the Lagrange multiplier function λ^k corresponding to u^k by integrating (2.1) and (2.4), respectively.

Step 2: Define the second-order approximation Q^k around u^k by (2.5), (2.6). Solve the auxiliary problem

$$\text{minimize } Q^k(v) \text{ subject to } \|v\| \leq \Delta^k \tag{2.10}$$

and let the solution be v^k.

Step 3: If $v^k = 0$, then terminate. Otherwise, using v^k, evaluate ρ^k by (2.9).

Step 4: If $\rho^k < \eta_1$, then set $\Delta^{k+1} = \gamma_1\Delta^k$ and $u^{k+1} = u^k$; if $\eta_1 \leq \rho^k < \eta_2$, then set $\Delta^{k+1} = \Delta^k$ and $u^{k+1} = u^k + v^k$; if $\rho^k \geq \eta_2$, then set $\Delta^{k+1} = \gamma_2\Delta^k$ and $u^{k+1} = u^k + v^k$. Increase k by one and return to Step 1.

This algorithm requires the solution of (2.10) [or (2.8)] on each iteration. For finite-dimensional problems, Moré and Sorensen [10], [11], [14] propose a procedure which ingeniously replaces the auxiliary problem by a sequence of systems of linear equations. In the present case, this amounts to successively solving the two-point boundary value problems

$$\dot{y}(t)=f_x(t)y(t)+f_u(t)v(t),\ y(0)=0, \tag{2.11}$$

$$\dot{\mu}(t) = -H_{xx}(t)y(t) - H_{xu}(t)v(t) - f_x(t)\mu(t), \ \mu(T) = F_{xx}(x(T))y(T), \tag{2.12}$$

$$H_u(t) + H_{ux}(t)y(t) + (H_{uu}(t) + \alpha I)v(t) + f_u(t)\mu(t) = 0 \tag{2.13}$$

in such a way that v and α satisfy the conditions

$$\alpha \geq 0, \ \|v\| \leq \Delta \text{ and } \alpha(\|v\| - \Delta) = 0, \tag{2.14}$$

$$H_{uu}(t) + \alpha I \geq 0. \tag{2.15}$$

Note that (2.11)–(2.15) are derived from the optimality conditions [6] for problem (2.8). In practice, it is more convenient to transform (2.11)–(2.13) into an equivalent system which contains a matrix Riccati differential equation. Also computation of α satisfying (2.14) and (2.15) can be carried out using Newton's method for solving nonlinear equations. Because of the space limitation, however, we shall not elaborate here a computational procedure for finding the solution of (2.11)–(2.15). A detailed description of the procedure may be found in [5].

III. Convergence Analysis

In this section, we shall analyze convergence properties of the algorithm presented in Section II. Let $\{u^k\}$ be a sequence of control functions generated by the algorithm. It follows directly from the construction of $\{u^k\}$ that the sequence $\{J(u^k)\}$ of cost values is monotonically nonincreasing. Moreover, we shall shortly see that the sequence $\{H_u^k\}$ converges to zero, where H_u^k denotes H_u evaluated at the nominal control u^k. This result is important because H_u^k can be regarded as the gradient of the cost functional J at u^k.

First, we show that, if the algorithm happens to terminate after finitely many iterations, then the last iterate satisfies the first and the second-order necessary conditions for optimality.

Theorem 3.1: Suppose that the algorithm terminates at iteration k. Then we have

$$H_u^k = 0$$

and

$$H_{uu}^k \geq 0.$$

Proof: The algorithm terminates only if $v^k = 0$. We then have $y^k = 0$, $\mu^k = 0$, and $\alpha^k = 0$ by (2.11), (2.12), and (2.14), respectively. Therefore, the desired results follow from (2.13) and (2.15). □

In what follows, we shall suppose that the algorithm generates an infinite sequence $\{u^k\}$. Also, we assume that the coefficient matrices appearing in (2.5), (2.6) are uniformly bounded with respect to u. Then, since $v(t)$ and $y(t)$ in (2.5) satisfy (2.6), the function y may be related to v by a bounded linear operator on $L^2[0, T]$, and hence the last integral in (2.5) may be considered to be quadratic with respect to v. Moreover, we may deduce from the uniform boundedness of the coefficient matrices that this integral is uniformly bounded. That is, there exists a constant $M > 0$ such that for any v

$$\int_0^T [v(t)'H_{uu}(t)v(t) + 2v(t)'H_{ux}(t)y(t) + y(t)'H_{xx}(t)y(t)] \ dt \leq M\|v\|^2 \tag{3.1}$$

where $y(t)$ is, of course, related to $v(t)$ by (2.6). Under this assumption, we may obtain a bound on the predicted reduction in J which results from the solution of the auxiliary problem (2.10).

Lemma 3.2: Let v^k solve the auxiliary problem (2.10). Then, we obtain the bound

$$Q^k(v^k) \leq -\frac{1}{2} \|H_u^k\| \min \{\Delta^k, \|H_u^k\|/M\} \tag{3.2}$$

where M is a constant satisfying (3.1).

A result similar to Lemma 3.2 has been established by Powell [13] for the finite-dimensional case. Detailed proof of this lemma is omitted here because the proof given in [13] extends to the present case in a straightforward manner.

We are now ready to demonstrate that, for any initial control u^0, the algorithm generates a sequence $\{u^k\}$ which asymptotically satisfies the first-order necessary conditions for optimality.

Theorem 3.3: Suppose that the cost functional J is bounded from below and H_u is uniformly continuous with respect to u. If the algorithm generates an infinite sequence $\{u^k\}$, then we have

$$\lim_{k\to\infty} \|H_u^k\| = 0.$$

Proof: Suppose that there exists a $\gamma > 0$ such that $\|H_u^k\| \geq \gamma$ for all k sufficiently large.

Let us assume that $\lim\inf_{k\to\infty} \Delta^k = 0$. Then, there must be a subsequence such that $\rho^k < \eta_1$ and $\Delta^k \to 0$. On this subsequence, we obtain from (3.2)

$$Q^k(v^k) \leq -\frac{1}{2}\gamma\Delta^k \tag{3.3}$$

for every k large enough. Since $\|v^k\| \leq \Delta^k$, it is not difficult to deduce from (2.8), (2.9), and (3.3) that $|\rho^k - 1| \to 0$, contradicting $\rho^k < \eta_1$. Thus, $\lim\inf_{k\to\infty} \Delta^k > 0$, so that there exists an infinite subsequence for which $\rho^k \geq \eta_1$ and $u^{k+1} = u^k + v^k$ hold. Since $Q^k(v^k) < 0$, the inequality $\rho^k \geq \eta_1$ together with (2.9) and (3.2) yield

$$J(u^k) - J(u^{k+1}) \geq \frac{\eta_1}{2} \|H_u^k\| \min \{\Delta^k, \|H_u^k\|/M\}. \tag{3.4}$$

However, since $\{J(u^k)\}$ is monotonically nonincreasing, it follows from the boundedness of J that the left-hand side of (3.4) tends to zero as k increases. This is a contradiction because $\lim\inf_{k\to\infty} \Delta^k > 0$ and $\|H_u^k\| \geq \gamma$. Consequently, we have $\lim\inf_{k\to\infty} \|H_u^k\| = 0$.

It is further possible to show that the entire sequence $\{\|H_u^k\|\}$ actually converges to zero. However, the proof is somewhat involved, and hence omitted here. A complete proof may be found in [5]. □

Theorem 3.4: In addition to the hypotheses of Theorem 3.3, suppose that H_{uu} is continuous with respect to u. If the sequence $\{u^k\}$ generated by the algorithm converges to a limit u^*, then we have

$$H_u^* = 0 \tag{3.5}$$

and

$$H_{uu}^* \geq 0 \tag{3.6}$$

where H_u^* and H_{uu}^* denote H_u and H_{uu}, respectively, evaluated at u^*.

Proof: Since $u^k \to u^*$, (3.5) readily follows from Theorem 3.3. To establish (3.6), let us assume that the sequence $\{\alpha^k\}$ is asymptotically bounded away from zero, where α^k denotes the optimal Lagrange multiplier α that satisfies the optimality conditions (2.10)–(2.15) for the auxiliary problem (2.10). Then, by (2.14), we have $\|v^k\| = \Delta^k$ for all k sufficiently large. Since $u^k \to u^*$ implies $v^k \to 0$, it follows that $\Delta^k \to 0$. In view of (2.7) and (2.9), however, $v^k \to 0$ implies $\rho^k \to 1$, and hence there exists an integer k_0 such that $\Delta^k \geq \Delta^{k_0}$ for all $k \geq k_0$. Since this is a contradiction, we must have $\lim\inf_{k\to\infty} \alpha^k = 0$. Consequently, (3.6) follows from (2.15) by taking the limit of a subsequence such that $\alpha^k \to 0$. This completes the proof. □

IV. A Numerical Example

We have applied the above method to the Van der Pol equation

$$\dot{x}_1 = (1 - x_2^2)x_1 - x_2 + u,$$

$$\dot{x}_2 = x_1; \ x_1(0) = x_2(0) = 1.5$$

with cost functional

$$J(u) = \frac{1}{2}\int_0^5 (x_1^2 + x_2^2 + u^2) \ dt.$$

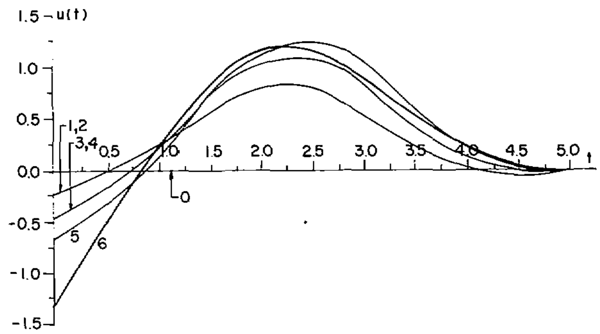

Fig. 1.

TABLE I

Iteration	$J(u)$	$J(u+v)$	$Q(v)$	ρ	Δ
0	9.586	6.476	- 3.024	1.028	1.0
1	6.476	10.256	- 6.146	- 0.615	2.0
2	6.476	4.957	- 1.583	0.960	0.5
3	4.957	4.893	0.357	- 0.181	1.0
4	4.957	4.504	- 0.548	0.838	0.25
5	4.504	4.468	- 0.073	0.494	0.5

The initial control has been set to be $u_0 = 0$, and the initial value of Δ is taken to be 1. (This problem is taken from [7] where a more detailed comparison of various methods has been made.) The numerical integration was carried out with the step size of 0.05 via the Runge–Kutta–Gill method. The convergence of the control function is exhibited in Fig. 1, and the values of $J(u)$, $J(u + v)$, $Q(v)$, ρ, Δ at each iteration are shown in Table I. It has been observed that after 12 iterations, the value of cost functional is decreased to 4.418 showing a very fast convergence; also, the same optimal control as in [7] has been obtained at this stage. Observe that at the third iteration the value of $Q(v)$ is positive. This is because the solution of the matrix Riccati equation for the linearized equation does not give a local minimum of $Q(v)$. The reason for this is that the Hessian matrix H_{xx} is actually negative around $t = 1$. However, this difficulty is bypassed by (automatically) shrinking Δ as shown in Table I—exhibiting an advantage of the trust region method.

V. Conclusion

A second-order algorithm for optimal control problems has been described and its convergence properties have been investigated. The results obtained in Section III indicate that the proposed algorithm enjoys very desirable convergence properties. In particular, Theorem 3.2 shows that the algorithm is globally convergent in the sense that the generated sequence is asymptotically stationary for any choice of an initial control. Furthermore, Theorems 3.1 and 3.4 show that the limit of the generated sequence satisfies the first-order and the second-order optimality conditions. This property, which is inherited from the trust region methods in finite-dimensional optimization, is considered to be a result of making effective use of the second-order information of the problem. In the finite-dimensional case, it has been proven that the trust region modifications of Newton's method have the quadratic rate of convergence, if the limit of the generated sequence satisfies the second-order sufficient conditions for optimality [4], [14]. We feel it reasonable to expect that the last statement remains valid for the present problems.

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