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Gravitational waves in cosmological models of Hořava-Witten theory

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We study the behavior of gravitational waves and their back reaction on the background in cosmological solutions of the five-dimensional Hořava-Witten theory. As a dynamical background, we consider two cosmological solutions with spatially flat expanding FRW branes, called (↑) and (↓) solutions, in which the orbifold size increases and decreases in time, respectively. For these background solutions, the wave equation for the tensor perturbation can be solved by the method of separation of variables, and the mode functions are classified by a separation constant which can be regarded as a graviton mass. We show that the spatial behavior of the mode functions is the same for both background solutions but the temporal behavior is significantly different. We further show that for the (↑) solution the background bulk geometry is unstable against the back reaction of the perturbation, while for the (↓) solution the back reaction on the bulk geometry can be neglected. We also show that, in contrast with the effect to the bulk geometry, the back reaction of the perturbation significantly alters the intrinsic geometry of the brane for the (↓) solution.

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I. INTRODUCTION

Over the past few years, a considerable number of studies have been made on the brane-world scenario in which our Universe is realized as a boundary of a higher dimensional spacetime [1–16]. In particular, inspired by the recent progress in heterotic M theory [2], five-dimensional brane-world models in which 3-branes are embedded in an effective five-dimensional spacetime compactified on $S^1/\mathbb{Z}_2$ [3,4] have attracted much attention. For example, cosmological solutions in the five-dimensional Hořava-Witten theory have been discovered [5–9]. The five-dimensional models of Randall and Sundrum, which were proposed to solve the hierarchy problem [12] and to demonstrate an alternative to compactification [13], also have many similarities to the five-dimensional Hořava-Witten theory.

In the brane-world scenario, all ordinary matter fields are confined on the brane, while a graviton can propagate in the fifth dimension. Hence, in order to test the idea of the brane world, one needs to study the nature of gravity in this scenario. The behavior of linearized gravity in the Randall-Sundrum models has recently been studied in detail. It has been shown that massless modes of the metric perturbation are decoupled from massive modes and Einstein gravity is recovered at low energy scales (see, e.g., [12–15]). However these investigations have been done only in the highly symmetric background models such that four-dimensional maximally symmetric branes, i.e., Minkowski branes [12,13] and de Sitter branes [15], are embedded in a five-dimensional anti–de Sitter spacetime, which also is locally maximally symmetric in five dimensions.

On the other hand, many people have discussed the possibility of a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) brane world in the Hořava-Witten theory [5–7] and in the Randall-Sundrum scenario [16]. However, there have so far been few attempts to investigate perturbations on such a dynamical cosmological brane world, although the four-dimensional real Universe has inhomogeneous fluctuations as is shown by the observations of the cosmic microwave background (CMB) anisotropy.

As pointed out in Ref. [17], it is expected that all gravitons on the brane become massive in a dynamical brane-world model. Then, the excitation of massive graviton would become a crucial defect in the brane-world scenario, or provide a new model of dark matter in the brane-world cosmology. It is therefore important to study the evolution of perturbations on a dynamical brane model in order to explore the cosmological consequences of the brane-world idea.

Recently, formalisms for cosmological perturbations on the brane world have been developed by several authors [17–22]. In particular, for general cosmological brane-world models, the evolution equations for the metric and matter perturbations in the bulk and the boundary conditions for them at the brane have been established by Kodama *et al.* [17] in terms of gauge-invariant variables. The perturbations on a brane are inevitably coupled to the perturbations on the bulk. The evolution equations and boundary conditions for cosmological perturbations, in particular, the scalar and vector perturbations on the cosmological brane are too complicated to solve. On the other hand, as shown in [17], when the anisotropic stress perturbation vanishes, the tensor perturbations decouple from the matter perturbation and the boundary condition becomes a Neumann type. Hence, as far as the tensor perturbations are concerned, the problem is easier to deal with.

In this paper, as a first step to investigate the perturbations on a dynamical brane world, we study the behavior of tensor perturbations on two cosmological brane-world models in five-dimensional Hořava-Witten theory. In general, the equations for perturbations are no longer separable for a dynamical 3-brane which is not maximally symmetric as a hypersurface. Fortunately, for the cosmological solutions in the Hořava-Witten model found in [5,6], the evolution equation of tensor perturbations becomes separable. However, the evolution equation is still rather complicated to solve exactly. So in this paper, we consider the late time behavior of the tensor perturbation by using WKB approximation to analyze the evolution equation. Further, we discuss the back

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reaction problem and the stability of the background solutions as well as the brane motion using the second-order perturbation theory. We shall show that one of the cosmological solutions is unstable if the back reaction of the tensor perturbation is taken into account. We shall also show that the back reaction from massive modes significantly alters the brane motion, i.e., the evolution of the brane universe, for the solution whose bulk geometry is stable against the tensor perturbation.

The present paper is organized as follows. In the next section we briefly recapitulate the five-dimensional Hořava-Witten theory and the cosmological solutions which are used as the background for perturbation. In Sec. III we give the equations of motion for the tensor perturbation and the boundary condition for them, and analyze the behavior of the tensor perturbation by using the WKB method. Then in Sec. IV we discuss the back reaction of the perturbation on the background. Section V is devoted to conclusion and discussion.

II. COSMOLOGICAL SOLUTIONS OF HOŘAVA-WITTEN THEORY

Hořava and Witten have shown that the strongly coupled limit of the $E_8 \times E_8$ heterotic string theory has been identified with the M theory compactified on a $S^1/\mathbb{Z}_2$ orbifold with E_8 gauge fields on each orbifold fixed plane [2]. After a compactification on a Calabi-Yau threefold, the fields of the standard model can be confined to the 3-brane [3]. Thus one has an effective model in which our four-dimensional Universe is a 3-brane embedded in an effective five-dimensional spacetime compactified on $S^1/\mathbb{Z}_2$. As was shown by Lukas, Ovrut, Stelle, and Waldram [4], the bosonic sector of this effective model contains the five-dimensional metric g_{MN}, a modulus V describing the variation of the Calabi-Yau volume along the orbifold, and the $U(1)$ gauge field $\mathcal{A}_M$ and two charged scalars (σ,ξ) parametrizing the antisymmetric tensor field. If we assume that the gauge field and the two charged scalar fields vanish, the five-dimensional effective action for the bosonic sector of the Hořava-Witten theory is given by [4]

$$S=\frac{1}{2\kappa_5^2}\left[\int_{M_5}\sqrt{-g}\left(R-\frac{1}{2V^2}(\nabla V)^2-\frac{\alpha^2}{3V^2}\right)+2\sqrt{2}\int_{M_4^{(1)}}\sqrt{-g}\,\frac{\alpha}{V}-2\sqrt{2}\int_{M_4^{(2)}}\sqrt{-g}\,\frac{\alpha}{V}\right], \qquad (1)$$

where κ_5 is the five-dimensional gravitational constant, α is a constant, and $\mathcal{M}_5$ is the five-dimensional spacetime bounded by the branes $\mathcal{M}_4^{(1)}$ and $\mathcal{M}_4^{(2)}$.

The cosmological solutions with flat FRW branes for this effective action have been constructed by Lukas *et al.* [5] and been generalized to include closed and open FRW branes by Reall [6]. In the present paper, as a simple case, we shall consider only the model in which expanding flat FRW branes are embedded in the five-dimensional bulk.

Let $x^4 \equiv y$ be a coordinate in the orbifold direction with $y\in[-\pi\rho,\pi\rho]$ and $\mathbb{Z}_2$ acting on S^1 by $y\to -y$. The orbifold fixed planes are located at $y=0,\pi\rho$. Then, starting from the effective five-dimensional action (1) with the metric ansatz

$$ds^2=a_0^2e^{2A(t)}C(y)(-dt^2+\delta_{ij}dx^idx^j)+e^{2B(t)}D(y)dy^2, \qquad (2)$$

one obtains the cosmological solutions of five-dimensional Hořava-Witten theory with flat FRW branes [5]

$$C(y)=D(y)^{1/4}=\frac{\sqrt{2}}{3}\alpha|y|+1, \qquad (3)$$

$$e^{2A}=t^{1-\delta}, \quad e^{2B}=t^{2\delta}, \qquad (4)$$

where a_0 are constants and $\delta=\pm\sqrt{3}/2$. The field V for these solutions is given by

$$V=e^{B(t)}C(y)^3. \qquad (5)$$

Hereafter we shall refer to the upper and lower choices of sign as the $(\uparrow)$ and $(\downarrow)$ solutions, respectively.

The $(\downarrow)$ solution describes the model that the four-dimensional FRW universe expands while the orbifold space shrinks. On the other hand, the $(\uparrow)$ solution describes the model in which both the four-dimensional FRW universe and the orbifold space expand, and the latter expands faster than the former does.

For a while, we shall focus our attention on the four-dimensional brane at $y=0$. In terms of the cosmic proper time τ defined by

$$\tau\equiv a_0\int e^A dt=\frac{2a_0}{3-\delta}t^{(3-\delta)/2}, \qquad (6)$$

the Hubble parameter is given by

$$H(t)\equiv\frac{1}{a}\frac{da}{d\tau}=\frac{1-\delta}{2a_0}t^{-(3-\delta)/2}. \qquad (7)$$

Then the wave number k_H whose wavelength corresponds to the horizon scale is given by

$$k_H\equiv aH=\frac{1-\delta}{2t}. \qquad (8)$$

The scale factor of the four-dimensional FRW brane is written as

$$a(\tau)=a_0\left(\frac{3-\delta}{2a_0}\tau\right)^{(1-\delta)/(3-\delta)}. \qquad (9)$$

Comparing this scale factor with that of the no-extra-dimension cosmology, we find that the brane expands as if it were a standard four-dimensional flat FRW universe which contains a perfect fluid obeying the equation of state $p=w\rho$ with

$$w=\frac{3+\delta}{3(1-\delta)}, \qquad (10)$$

although the brane-world solutions considered here are vacuum solutions. In this picture, the FRW brane in the (↑) solution looks like an unphysical universe because the dominant energy condition violates $w>1$. On the other hand, in the (↓) case, the FRW brane describes a physical universe in the sense that the dominant energy condition holds, $0<w\approx 0.37<1$.

III. BEHAVIOR OF TENSOR PERTURBATIONS

A. Wave equation for tensor perturbations

Gravitational perturbations in the bulk are decomposed into components of the three types: scalar, vector, and tensor, and each component can be expanded by tensor harmonics of the same type on the 3-space of constant curvature. Then, the tensor perturbations represent gravitational wave modes in the four-dimensional FRW brane.

For the action (1), the anisotropic stress perturbations vanish both in the bulk and in the brane. Then, for the background metric Eqs. (2) and (3), the equations of motion for the tensor perturbation and the boundary condition are given, respectively, by [17]

$$\ddot{H}_T+[2\dot{A}(t)+\dot{B}(t)]\dot{H}_T-\frac{a_0^2e^{2(A(t)-B(t))}}{C(y)^3}H_T''+k^2H_T=0, \tag{11}$$

$$H_T'=0, \quad \text{at } y=0,\pi\rho, \tag{12}$$

where H_T is the expansion coefficient of the metric perturbation $\delta g_{ij}=2a_0^2e^{2A}C(y)H_T\mathbb{T}_{ij}$, in terms of the tensor-type harmonic tensor $\mathbb{T}_{ij}$ on the flat 3-space [17]. Here, $-k^2$ is an eigenvalue of the Laplacian on the flat 3 space, and dots and primes denote derivatives with respect to t and y, respectively.

Note that the boundary condition (12) is simply written as the derivative with respect to y as in the static brane case. However, the evolution equation (11) contains an additional friction term $\dot{B}\dot{H}_T$, which does not exist when the background orbifold space is static, and the y-derivative term (the third term), which gives graviton's mass as we shall see below. So, these two terms reflect the effect of the dynamics of the background branes model on gravitational waves.

Provided that $H_T(t,y)=T(t)Y(y)$, the equations of motion (11) are reduced to the following set of equations for $Y(y)$ and $T(t)$,

$$Y_l''+\left(\frac{m}{a_0}\right)^2C(y)^3Y_l=0, \tag{13}$$

$$\ddot{T}_l+[2\dot{A}(t)+\dot{B}(t)]\dot{T}_l+k^2T_l+m^2e^{2[A(t)-B(t)]}T_l=0, \tag{14}$$

where the dimensionless constant m is defined as follows:

$$m^2\equiv\left(\frac{\sqrt{2}\alpha l a_0}{3}\right)^2. \tag{15}$$

Here, l represents the level of inhomogeneity in the orbifold direction and m takes discrete values labeled by an integer n, as we shall see in the next subsection. Note that the signature of the second term in the left-hand side of Eq. (13) is chosen so that the solutions of Eq. (13) satisfy the boundary condition (12).

In the case of $\dot{B}=0$, the last term in Eq. (14) provides the eigenvalue (times e^{2A}/a_0^2) of the d'Alembertian on the four-dimensional brane. Therefore the $m=0$ modes behave as the massless mode in the brane when the orbifold space is static. In this sense, we shall refer to $m=0$ $(l=0)$ and $m\neq 0$ $(l\neq 0)$ modes as "massless" and "massive" modes, respectively.

B. Solutions of the *y*-dependent part

In this subsection, we shall give the solutions of Eq. (13) which satisfy the boundary condition (12).

For $m=0$ modes, Eq. (13) reads

$$Y_l(y)=C_1+C_2y, \tag{16}$$

where C_1 and C_2 are integration constants. From the boundary condition (12), we find that C_2 must vanish. Therefore, the zero-mode $(l=0)$ solution for the y-dependent part is

$$Y_l(y)=C_1. \tag{17}$$

On the other hand, for $m\neq 0$ modes, the solutions of Eq. (13) for $y\geqslant 0$ are given by

$$Y_l(y)=C(y)^{1/2}\left[C_1H_{1/5}^{(1)}\left(\frac{2l}{5}C(y)^{5/2}\right)+C_2H_{1/5}^{(2)}\left(\frac{2l}{5}C(y)^{5/2}\right)\right], \tag{18}$$

where C_1 and C_2 are constants.

From the boundary condition at $y=0$, we find that the ratio of C_1 and C_2 becomes

$$\frac{C_2}{C_1}=-\left.\frac{H_{-4/5}^{(1)}(z)}{H_{-4/5}^{(2)}(z)}\right|_{z=2l/5} \tag{19}$$

for each l. From the boundary condition at $y=\pi\rho$, we obtain

$$\left.\frac{H_{-4/5}^{(1)}(z)}{H_{-4/5}^{(2)}(z)}\right|_{z=2l/5}=\left.\frac{H_{-4/5}^{(1)}(z)}{H_{-4/5}^{(2)}(z)}\right|_{z=2l(\sqrt{2}\alpha\pi\rho/3+1)^{5/2}/5}. \tag{20}$$

This gives the value of l for each excited mode.

For the case of $2l/5\gg 1$, l can be written as

$$l\approx\frac{5}{2}\frac{n\pi}{(\sqrt{2}\alpha\pi\rho/3+1)^{5/2}-1}, \tag{21}$$

where $n=1,2,\ldots$. $\alpha\pi\rho\ll 1$ is satisfied in the context of the five-dimensional Hořava-Witten theory, except for inflationary epoch [8]. In this case, Eq. (21) reads

$$m \approx \frac{a_0 n}{\rho}. \tag{22}$$

Thus the reduced KK mass defined by

$$m_{KK,l} \equiv \frac{m}{a_0 e^{B(t)} \sqrt{C(y)}} \tag{23}$$

takes a typical value of the orbifold energy scale.

The norm squared of $Y_l(y)$ is then given by

$$|Y_l(y)|^2 \approx \cos^2\left[\frac{2l}{5}\{(\sqrt{2}\alpha|y|/3+1)^{5/2}-1\}\right]. \tag{24}$$

Therefore $|Y_l(y)|^2$ has a peak at each boundary (at $y=0$ and $y=\pi\rho$).

C. Solutions of t-dependent parts

In this subsection, we shall examine the behavior of solutions to Eq. (14). For simplicity, we omit the suffix l hereafter.

For the $m=0$ modes, Eq. (14) is exactly solved to yield

$$T = D_1 H_0^{(1)}(kt) + D_2 H_0^{(2)}(kt), \tag{25}$$

where D_1 and D_2 are constants. In contrast, for the $m \neq 0$ modes, Eq. (14) cannot be solved exactly. Therefore, we analyze the behavior of solutions by means of the WKB method by rewriting Eq. (14) as

$$(t^{1/2}T)^{\cdot\cdot} + S(t)^2 t^{1/2} T = 0, \tag{26}$$

where $S(t)$ is defined by

$$S(t) \equiv \sqrt{k^2 + \frac{1}{4t^2} + m^2 t^{1-3\delta}}. \tag{27}$$

In the region where $|\dot{S}| \ll S^2$ holds, we can use the WKB method to obtain the approximate solution

$$T(t) \approx D_1(k)(tS(t))^{-1/2} \exp\left[-i\int^t S(t')dt'\right] + D_2(k)(tS(t))^{-1/2} \exp\left[i\int^t S(t')dt'\right]. \tag{28}$$

Since $\dot{S}/S^2$ is written as

$$\frac{|\dot{S}|}{S^2} = \frac{2|-1+2(1-3\delta)m^2 t^{3(1-\delta)}|}{(1+4(kt)^2+4m^2 t^{3(1-\delta)})^{3/2}}, \tag{29}$$

the WKB approximation is good in the region where $t \gg m^{-2/(3(1-\delta))}$ or $t \gg 1/(2k)$. The former relation, which is equivalent to $m_{\mathrm{KK}}\tau \gg 1$, is satisfied when the time scale is larger than the orbifold radius, while the latter is satisfied when the wavelength is shorter than the Horizon radius, i.e., $k \gg k_H$.

In the (↑) background case, since $t^{1-3\delta}$ is a monotonically decreasing function, the mass term becomes negligible compared with the k^2 term on the right-hand side of Eq. (27). Therefore, solutions to Eq. (26) are well approximated by the solutions (25) in the massless case in a sufficient late time for any fixed k. In contrast, in the (↓) background case, the $m^2 t^{1-3\delta}$ term increases with time, and the solutions deviate from those for the massless case in late times. In particular, for $4m^2 t^{3(1-\delta)} \gg 1+4(kt)^2$, the WKB solution is given by

$$T \approx (mt^{3(1-\delta)/2})^{-1/2}\left(D_1(k)\exp\left[-i\frac{2mt^{3(1-\delta)/2}}{3(1-\delta)}\right] + D_2(k)\exp\left[i\frac{2mt^{3(1-\delta)/2}}{3(1-\delta)}\right]\right), \tag{30}$$

after an appropriate redefinition of the constants D_1 and D_2. Note that the late time solution (30) does not depend on the wave number k explicitly, and the argument of the exponential function in Eq. (30) is proportional to $m_{KK}\tau$. In particular, from Eq. (25) and Eq. (30) or from Eq. (27) and Eq. (28), we see that the ratio of the amplitude of a massive mode to that of a massless mode behaves as

$$\frac{T_{m\neq 0}}{T_{m=0}} \propto S(t)^{-1/2} \sim e^{[B(t)-A(t)]/2} = t^{(3\delta-1)/4} \tag{31}$$

for $m_{KK} \gg k/a$. This apparently shows that the massive modes become negligible in late times. However, if we consider their back reaction, the conclusion changes significantly, as we will see in the next section.

IV. BACK REACTION OF THE PERTURBATION

In this section, we study the back reaction of the tensor perturbation on the bulk background geometry and on the intrinsic geometry of the brane with the help of the second-order perturbation theory.

First note that if we expand the deviation of the bulk geometry from the background in terms of some small parameter, the second order part $\delta_2 g$ satisfies the equation

$$(\mathcal{L}^{(1)}\delta_2 g)_{MN} = \kappa_5^2 ({T^{\mathrm{GW}}}_{MN} + \delta_2 T_{MN}), \tag{32}$$

where $\mathcal{L}^{(1)}$ is the differential operator for the metric perturbation obtained from the linear perturbation of the Einstein equations, ${T^{\mathrm{GW}}}_{MN}$ is the effective energy-momentum tensor for the linear perturbation $\delta_1 g$ of geometry, which is quadratic in $\delta_1 g$, and $\delta_2 T_{MN}$ is the second-order perturbation with respect to the field V of the bulk energy-momentum tensor

$$\kappa_5^2 T_{MN} = \frac{1}{2V^2}\partial_M V \partial_N V - \frac{1}{2} g_{MN}\left(\frac{1}{2V^2}(\nabla V)^2 + \frac{\alpha^2}{3V^2}\right). \tag{33}$$

The explicit expression for ${T^{\mathrm{GW}}}_{MN}$ is given in the Appendix. In contrast to the linear perturbation, the spatial average in the three-dimensional sense does not vanish in general and produces a spatially homogeneous contribution to $\delta_2 g$. This contribution can be regarded as the back reaction of the perturbation on the background geometry.

In particular, for the tensor perturbation in the models considered in the present paper, the effective energy density $\rho_{\rm GW}$ is given by

$$2\kappa_5^2\rho_{\rm GW}\equiv -2\kappa_5^2\langle T^{\rm GW}{}^0_0\rangle \tag{34}$$

$$=(a_0^2e^{2A(t)}C(y))^{-1}(|\dot H_T|^2+k^2|H_T|^2) + e^{-2B}|H_T'|^2+\cdots, \tag{35}$$

under the normalization of the tensor harmonics as $\langle \mathbb{T}_{ij}\mathbb{T}^{ij}\rangle =1$. The leading term of $\rho_{\rm GW}$ is given by the first term, which behaves as $e^{-2A(t)}\dot H_T^2 \sim t^{-2+\delta}S$. Meanwhile, the leading term for the energy density of the V field determining the bulk background geometry is given by the potential energy $\alpha^2/(3V^2)\propto t^{-2\delta}$, and its second-order perturbation is given by

$$-\kappa_5^2\delta_2T^0_0=2\frac{\delta_2V}{V}\kappa_5^2T^0_0+\frac{\dot B^2}{2a^2}\frac{(\delta_2\dot V)}{\dot V}+\frac{\alpha^2}{V^2}\frac{\delta_2V'}{V'}$$

$$\sim 2\frac{\delta_2V}{V}\kappa_5^2T^0_0. \tag{36}$$

From these equations, in the $(\downarrow)$ case we find that the ratio of $\rho_{\rm GW}$ to the V field energy density decreases in proportion to $t^{-3(1+|\delta|)/2}$. Further, since the field equation for V is given by

$$\Box_{g+\langle\delta_2 g\rangle}[\ln(V+\langle\delta_2V\rangle)]+\frac{2\alpha^2}{3(V+\langle\delta_2V\rangle)^2}=0, \tag{37}$$

up to the second order, $\langle\delta_2V\rangle/V$ is small if $\langle\delta_2 g\rangle$ is negligible. Hence, in this case the back reaction can be neglected in late times. In contrast, for the $(\uparrow)$ model, the decrease of $\rho_{\rm GW}$ is slower than $-T^0_0$. Hence, this model is unstable against the back reaction of the tensor perturbation. Of course the linear perturbation around the original background solution gives a good approximation for the behavior of the system up to some time determined by the initial amplitude of the tensor perturbation, and the instability becomes important only after that time.

Next, we consider the back reaction effect on the intrinsic geometry of the brane. Since a full treatment of this problem is very difficult, we only make a rough estimate using the Hamiltonian constraint along the brane,

$${}^{(4)}R=-2\kappa_5^2T_{\perp\perp}+K^2-K^{\mu\nu}K_{\mu\nu}, \tag{38}$$

where ${}^{(4)}R$ is the Ricci scalar of the four-dimensional metric $g_{\mu\nu}$ of the brane, $T_{\perp\perp}$ is the component of the energy-momentum tensor along the unit normal to the brane, and $K_{\mu\nu}$ is the extrinsic curvature of the brane. As explained above, if we take into account the back reaction of the tensor perturbation on the brane geometry, $T_{\perp\perp}$ should be replaced by $T^5_5+T^{\rm GW}{}^5_5+\delta_2(T_{\perp\perp})$ in the second-order perturbation framework. Here, K^μ_ν is related to the intrinsic energy-momentum tensor ${}^{(4)}T^\mu_\nu$ of the brane by the junction condition

$$K^\mu_\nu=\frac{1}{2}\kappa_5^2\left({}^{(4)}T^\mu_\nu-\frac{1}{3}{}^{(4)}T\delta^\mu_\nu\right)=\mp\frac{\sqrt{2}\alpha}{6V}\delta^\mu_\nu. \tag{39}$$

Further, the boundary condition on V at the brane is expressed as

$$\nabla_\perp V=\pm\sqrt{2}\alpha. \tag{40}$$

Hence we obtain

$${}^{(4)}R(g+\langle\delta_2 g\rangle)={}^{(4)}R(g)-2\kappa_5^2\langle T^{\rm GW}{}^5_5\rangle + \frac{\dot B^2}{a^2}\left(\frac{\langle\delta_2V\rangle}{V}-\frac{\langle\delta_2\dot V\rangle}{\dot V}\right), \tag{41}$$

where from the Appendix $\langle T^{\rm GW}{}^5_5\rangle$ is given by

$$\kappa_5^2\langle T^{\rm GW}{}^5_5\rangle=-\frac{3}{2a^2}(\dot H_T^2-k^2H_T^2)+G^5_5H_T^2+\frac{2}{a^2}(\dot A+\dot B)H_T\dot H_T - \frac{2}{b^2}H_TH_T'', \tag{42}$$

with $b=e^{B(t)}C^2$ and

$$G^5_5=\frac{1}{4b^2}\frac{(V')^2}{V^2}+\frac{1}{4a^2}\frac{\dot V^2}{V^2}-\frac{\alpha^2}{6V^2}=\frac{\alpha^2}{3C^6}t^{-2\delta}+\frac{\delta^2}{4a_0^2C}t^{\delta-3}. \tag{43}$$

By putting the asymptotic estimates for H_T into this expression, we find that in the $(\downarrow)$ background case, the $\dot H_T^2/a^2$ term dominates and decays as $\tau^{-(3+\delta)/(3-\delta)} =\tau^{-1+2|\delta|/(3+|\delta|)}$. Since the background value of ${}^{(4)}R(g)$ decreases in proportion to $1/\tau^2\sim t^{\delta-3}$, $\langle T^{\rm GW}{}^5_5\rangle$ decreases more slowly than ${}^{(4)}R(g)$. Thus in this case $\langle T^{\rm GW}{}^5_5\rangle$ dominates the background value for ${}^{(4)}R(g)$ in the late stage and in order for the FRW nature of the brane to be preserved, δ_2V/V must become much larger than unity. This implies that the back reaction of the tensor perturbation significantly modifies the evolutionary behavior of the four-dimensional universe on the brane.

V. CONCLUSION AND DISCUSSION

In the present paper we have studied the evolution of gravitational wave perturbations in the dynamical FRW brane-world models of the five-dimensional Hořava-Witten theory. As the background spacetime, we have used two cosmological solutions, i.e., the $(\uparrow)$ and $(\downarrow)$ solutions, in which the branes represent the spatially flat expanding FRW universes. The most important feature of these solutions was the fact that we can solve the evolution equation for the tensor perturbation with help of the method of separation of variables in spite of the dynamical nature of the brane. Thus we were able to study the spatial and the temporal behavior of the tensor perturbation separately.

Since the model is compact in the fifth dimension, we obtained a discrete spectrum for the separation constant which can be interpreted as the graviton mass, and wave functions for the massless modes and for the massive modes were decoupled as in the case of the static brane solutions. However, we found that the spatial behavior of the wave functions for the massive modes is different from that in the Randall-Sundrum model [13]: in our case they have maxima at the branes, but in the RS model they have minima at the branes when expressed in terms of the variable H_T adopted in the present paper. This suggests that the coupling between the massless modes and the massive modes on the branes may become important when we consider nonlinear corrections in the models considered in the present paper.

Although the spatial behavior of perturbations for the two solutions was exactly the same, their temporal behavior was quite different. Namely, we have found that in the $(\uparrow)$ background the temporal behavior of massive modes approaches that of massless modes in a late time, while in the $(\downarrow)$ background the massive modes decay more rapidly than the massless modes. We can understand this difference as being caused by the difference in the behavior of $\exp[A(t)-B(t)]$, i.e., the difference between the expansion rate of the four-dimensional FRW brane and that of the orbifold space. Roughly speaking, waves become massless when $\lambda\, m_{KK} \ll 1$, while they become massive when $\lambda\, m_{KK} \gg 1$, where $\lambda \equiv a_0 e^{A(t)}\sqrt{C(y)}k^{-1}$ is the reduced proper wavelength. Therefore, in the $(\uparrow)$ background, since $\exp[A(t)-B(t)]$ is a decreasing function and $\lambda\, m_{KK} \to 0$, every mode becomes effectively massless in the late time. On the other hand, in the case of the $(\downarrow)$ background, $\exp[A(t)-B(t)]$ is an increasing function, and the modes become more and more massive with time. Then, as the WKB approximation shows, they suffer from an extra damping in proportion to $1/(\lambda m_{KK})$.

This result shows that in both models the tensor perturbation is dominated by massless modes. In models such as the Randall-Sundrum model in which the bulk geometry is determined by a cosmological constant, massless modes of the tensor perturbation are expected to have no important effect on the bulk geometry. In contrast, in the $(\uparrow)$ solution of the Hořava-Witten theory, the energy density of the bulk spacetime decreases in time. Hence, the back reaction of the energy density of the tensor perturbation may become important. In fact, we have shown that in the second-order perturbation framework, the contribution of the tensor perturbation supersedes the original background energy density determining the bulk geometry in the $(\uparrow)$ case. Hence, this background solution is unstable against nonlinear corrections. Although this result was obtained for a special brane motion obtained under the assumption that the brane contains no matter apart from the ϕ field, it is expected to hold also for a more realistic brane which contains ordinary matter, it is because the essential feature of the temporal behavior of the tensor perturbation does not depend on the boundary condition at the brane.

We have also examined the second-order back reaction of the tensor perturbation on the intrinsic geometry of the branes, and have found that the back reaction effect significantly alters the time evolution of the brane geometry for the $(\downarrow)$ solution, although the back reaction on the bulk geometry is negligible for this solution. This result is consistent with the naive expectation that the massive modes of the tensor perturbation behave as dark matter.

These results suggest that the stability against the back reaction can be used as a criterion to physically acceptable brane-world models and to discuss cosmological implications of models. Thus it will be interesting to analyze the nonlinear stability of the Randall-Sundrum models as well as of more realistic solutions in the Hořava-Witten theory, in which matter fields in the bulk and on the brane may play significant roles in the stability problem.

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APPENDIX

In this appendix, we give the expression for the second order part with respect to the perturbations of the $(m+n)$-dimensional Einstein-Hilbert action

$$S=\frac{1}{2\kappa^2}\int d^{m+n}x\sqrt{-g}(R-2\Lambda), \tag{A1}$$

and its energy-momentum tensor, where κ and R are the $(m+n)$-dimensional gravitational constant and Ricci scalar, respectively, and Λ is the cosmological constant.

By decomposing the metric $g_{\mu\nu}$ into the background $\bar{g}_{\mu\nu}$ and the perturbation $h_{\mu\nu}$ as

$$g_{\mu\nu}=\bar{g}_{\mu\nu}+h_{\mu\nu}, \tag{A2}$$

and substituting it into Eq. (A1), we obtain the following expression for the second-order part with respect to the perturbation $h_{\mu\nu}$ of the $(m+n)$-dimensional Einstein-Hilbert action:

$$\begin{aligned} S_2=\frac{1}{2\kappa^2}\int d^{m+n}x\sqrt{-\bar{g}}\{&\tfrac{1}{4}[h^{\mu\nu}{}_{;\rho}(2h^{\rho}{}_{\mu;\nu}-h_{\mu\nu}{}^{;\rho})\\ &+h_{;\mu}(h^{;\mu}-2h^{\mu\nu}{}_{;\nu})]+\tfrac{1}{8}(h^2-2h^{\mu\nu}h_{\mu\nu})\bar{R}\\ &+\tfrac{1}{2}(2h^{\mu\rho}h_{\rho}{}^{\nu}-hh^{\mu\nu})\bar{R}_{\mu\nu}-\tfrac{1}{4}\Lambda(h^2-2h^{\mu\nu}h_{\mu\nu})\}. \end{aligned} \tag{A3}$$

By taking the variation of this action with respect to the background metric $\bar{g}_{\mu\nu}$,

$$\delta S_2=\int d^{m+n}x\sqrt{-\bar{g}}\tfrac{1}{2}T^{\mu\nu}\delta\bar{g}_{\mu\nu}, \tag{A4}$$

we obtain the following energy-momentum tensor $T_{\mu\nu}$:

$$
\begin{aligned}
4\kappa^2 T_{\mu\nu} = & -g_{\mu\nu}[\tfrac{1}{2}h^{\rho\sigma}{}_{;\lambda}h_{\rho\sigma}{}^{;\lambda}-\tfrac{1}{2}h_{;\rho}h^{;\rho}-h_{;\rho\sigma}h^{\rho\sigma}+h^{\rho\sigma}{}_{;\sigma}h_{\rho}{}^{\lambda}{}_{;\lambda}+2h^{\rho\sigma}{}_{;\sigma\lambda}h_{\rho}{}^{\lambda}+\tfrac{1}{2}\Box(h^2-2h^{\rho\sigma}h_{\rho\sigma})+(h^{\rho\lambda}h_{\lambda}{}^{\sigma}-hh^{\rho\sigma})_{;\rho\sigma} \\
& +R_{\sigma}{}^{\lambda}{}_{\rho\alpha}h^{\sigma\alpha}h^{\rho}{}_{\lambda}-(h^{\rho\lambda}h_{\lambda}{}^{\sigma}-hh^{\rho\sigma})R_{\rho\sigma}]-\tfrac{1}{2}(h^2-2h^{\rho\sigma}h_{\rho\sigma})(R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R+g_{\mu\nu}\Lambda)+h^{\rho\sigma}{}_{;\mu}h_{\rho\sigma;\nu}-2h_{\nu}{}^{\rho}\Box h_{\rho\mu} \\
& -2h^{\rho\sigma}{}_{;\mu\rho}h_{\nu\sigma}-2h^{\rho\sigma}{}_{;\mu}h_{\nu\sigma;\rho}+2h^{\rho}{}_{\mu;\nu\sigma}h^{\sigma}{}_{\rho}+2h^{\rho}{}_{\mu;\nu}h^{\sigma}{}_{\rho;\sigma}-h_{;\mu}h_{;\nu}+h_{\mu\nu}\Box h+2h_{;\mu}h^{\rho}{}_{\nu;\rho}-h^{;\rho}h_{\mu\nu;\rho}-2h^{;\rho}{}_{;\nu}h_{\rho\mu} \\
& -2h_{\mu}{}^{\rho}{}_{;\rho}h_{\nu}{}^{\sigma}{}_{;\sigma}+2h^{\rho\sigma}{}_{;\sigma}h_{\mu\nu;\rho}+4h^{\rho\sigma}{}_{;\sigma\mu}h_{\rho\nu}+6R_{\mu}{}^{\rho}{}_{\sigma\lambda}h_{\nu}{}^{\lambda}h^{\sigma}{}_{\rho}+2(h^{\rho}{}_{\mu}h^{\sigma}{}_{\nu}-h_{\mu\nu}h^{\rho\sigma})_{;\rho\sigma}-(R-2\Lambda)(hh_{\mu\nu} \\
& -2h_{\mu\rho}h^{\rho}{}_{\nu})+\tfrac{1}{2}(h^2-2h^{\rho\sigma}h_{\rho\sigma})_{;\mu\nu}-4R_{\mu\rho}(h_{\nu}{}^{\sigma}h_{\sigma}{}^{\rho}-hh^{\rho}{}_{\nu})-2R_{\rho\sigma}(h^{\rho}{}_{\nu}h_{\mu}{}^{\sigma}-h^{\rho\sigma}h_{\mu\nu})-\Box(h_{\mu}{}^{\rho}h_{\rho\nu}-hh_{\mu\nu}) \\
& +2(h^{\rho\lambda}h_{\lambda\nu}-hh^{\rho}{}_{\nu})_{;\mu\rho}, \qquad \text{(A5)}
\end{aligned}
$$

where $\Box$ is the $(m+n)$-dimensional d'Alembertian.

Under the notation adopted in [17], the unperturbed background geometry in brane-world models is expressed as

$$d\overline{s^2}=\overline{g}_{\mu\nu}dz^{\mu}dz^{\nu}=g_{ab}(y)dy^a dy^b+r^2(y)d\sigma_n^2, \qquad \text{(A6)}$$

where the metric

$$d\sigma_n^2=\gamma_{ij}(x)dx^i dx^j \qquad \text{(A7)}$$

is that of the n-dimensional space with a constant sectional curvature K. The tensor mode of the metric perturbation is expanded as

$$h_{ab}=0, \quad h_{ai}=0, \quad h_{ij}=2r^2 H_{Tij}. \qquad \text{(A8)}$$

For this tensor perturbation, the spatial average of the energy-momentum tensor (A5) in the spatially flat $(K=0)$ case is given by

$$
\begin{aligned}
\kappa^2\langle T^{GW}_{ab}\rangle = & g_{ab}\left[\frac{3}{2}\left\{(DH_T)^2+\frac{k^2}{r^2}H_T^2\right\}\right. \\
& \left.+\frac{2}{r}H_T Dr\cdot DH_T+\Lambda H_T^2\right]-D_aH_TD_bH_T \\
& -2H_TD_aD_bH_T-\frac{4}{r}H_TD_arD_bH_T+G_{ab}H_T^2, \qquad \text{(A9)}
\end{aligned}
$$

$$\kappa^2\langle T^{GW}_{ai}\rangle=0, \qquad \text{(A10)}$$

$$
\begin{aligned}
\kappa^2\langle T^{GW}_{ij}\rangle = & r^2\delta_{ij}\left[\frac{3}{2}\left\{(DH_T)^2+\frac{k^2}{r^2}H_T^2\right\}+\left(\Lambda+\frac{1}{n}G^k_k\right)H_T^2\right] \\
& -\left[2\{r^2(DH_T)^2+k^2H_T^2\}\right. \\
& \left.+4r^2\left(\frac{1}{n}G^k_k+\Lambda\right)H_T^2\right]C_{ij}+k_ik_jH_T^2, \qquad \text{(A11)}
\end{aligned}
$$

where we have normalized as $\langle \mathbb{T}_{ij}\mathbb{T}^{ij}\rangle=1$ and $\langle \mathbb{T}_{ik}\mathbb{T}^k{}_j\rangle=C_{ij}$.

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