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Supersymmetric Quantum-Hall Effect on a Fuzzy Supersphere

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Supersymmetric quantum-Hall liquids are constructed on a supersphere in a supermonopole background. We derive a supersymmetric generalization of the Laughlin wave function, which is a ground state of a hard-core $OSp(1|2)$ invariant Hamiltonian. We also present excited topological objects, which are fractionally charged deficits made by super Hall currents. Several relations between quantum-Hall systems and their supersymmetric extensions are discussed.

Supersymmetric quantum-Hall phenomena were realized in the two-dimensional flat space under strong magnetic field. Laughlin derived a wave function that well describes quantum incompressible liquids [4]. His wave function is rotationally symmetric but not translationally symmetric on the plane. Hence, it does not possess all the symmetries in a plane and is not suited for computer simulations. Haldane overcame this problem by constructing quantum-Hall systems on two-spheres in a Dirac magnetic monopole background [5]. He constructed a Laughlin-like wave function, which we call the Laughlin-Haldane wave function, on the sphere that possesses all the rotational symmetries of the two-sphere. The sphere used in by Haldane’s analysis is simply a fuzzy two-sphere. Recently, Zhang and Hu have succeeded in constructing four-dimensional quantum-Hall systems in Matrix models. Their theory has attracted much attention.

Quantum-Hall systems contain noncommutative structures, such as Matrix theories and D-brane physics [1]. They are perhaps the simplest known physical setup of noncommutative geometry and exhibit many of its exotic properties [2]. Therefore, quantum-Hall systems, which can be readily investigated in the laboratory, represent a practical alternative to the physics which string theory attempts to describe, which is still far beyond our realm of experimental capability. This is one reason that quantum-Hall systems are very fascinating. It is expected that ideas developed by investigating quantum-Hall systems will help in furthering our understanding of the high energy physics [3].

Originally, quantum-Hall phenomena were realized in the two-dimensional flat space under strong magnetic field. Laughlin derived a wave function that well describes quantum incompressible liquids [4]. His wave function is rotationally symmetric but not translationally symmetric on the plane. Hence, it does not possess all the symmetries in a plane and is not suited for computer simulations. Haldane overcame this problem by constructing quantum-Hall systems on two-spheres in a Dirac magnetic monopole background [5]. He constructed a Laughlin-like wave function, which we call the Laughlin-Haldane wave function, on the sphere that possesses all the rotational symmetries of the two-sphere. The sphere used in by Haldane’s analysis is simply a fuzzy two-sphere. Recently, Zhang and Hu have succeeded in constructing four-dimensional quantum-Hall systems in Matrix models [6]. The systems they consider are quantum-Hall liquids on fuzzy four-spheres and, intriguingly, possess branelike excitations. Because Matrix theories can be used to describe higher-dimensional spaces and possess extended objects, their quantum-Hall systems are the first discovered “physical” systems that exhibit behavior similar to that described by Matrix models. Their theory has attracted much attention and has been developed by many authors [7]. In particular, on the basis of fuzzy complex projective manifolds, Karabali and Nair have generalized them into even higher-dimensional quantum-Hall systems [8]. Hasebe and Kimura, based on higher-dimensional fuzzy spheres, have found another way to generalize them for an arbitrary even number of dimensions in colored monopole backgrounds [9]. In fact, such developments in the study of quantum-Hall systems have provided information that may be important in obtaining an understanding of D-brane physics. In particular, it has been reported that, with use of the Dirac-Born-Infeld action, the higher-dimensional fuzzy spheres in Matrix models can be identified with dielectric D-branes in colored monopole backgrounds [10].

Recently, it was found that nonanticommutative (NAC) field theory is naturally realized on D-branes in $R-\bar{R}$ field or graviphoton backgrounds [11]. Also, it has been shown that, in supermatrix models, fuzzy superspheres arise as classical solutions, and their fluctuations yield NAC field theories [12]. Some interesting relations between lowest Landau level (LLL) physics and NAC geometry have also been reported [13]. With these recent developments, it would be worthwhile to extend the theory of quantum-Hall systems to a supersymmetric framework. Indeed, the supersymmetric quantum-Hall systems might be the simplest physical setup of NAC geometry. Further, encouraged by previous success in the investigation of higher-dimensional quantum-Hall systems, we may hope that such systems not only possess exotic properties in the NAC world but also reveal yet unknown aspects of supermatrix models.

A supersphere is a geometrical object taking the form of a coset manifold given by $S^{2|2} = OSp(1|2)/U(1)$. By construction, a supersphere manifestly possesses the exact $N = 1$ supersymmetry, which is generated by the $OSp(1|2)$ super Lie group. The fact that the supersymmetry remains exact is an advantage of using the coset manifolds of super Lie groups. The number of degrees of freedom of the supersphere is given by $\dim S^{2|2} = \dim OSp(1|2) - \dim U(1) = 5 - 1 = 4$. Two of these degrees of freedom correspond to the Grassmann even coordinates, and the other two correspond to the Grassmann odd coordinates on the supersphere. The supersphere is embedded in a flat superspace whose coordinates are $x_\alpha (\alpha = 1, 2, 3)$, which are Grassmann even, and $\theta_\alpha (\alpha = 1, 2)$, which are Grassmann odd. The radius of the supersphere is given by $R = \sqrt{x_0^2 + C_{\alpha \beta} \theta_\alpha \theta_\beta}$, where $C_{\alpha \beta}$ is an antisymmetric tensor with $C_{12} = 1$. At the center of the

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supersymmetry, we place a supermonopole whose magnetic charge is 1/2 when I is an integer. The magnetic field of the supermonopole is given by $B = 2\pi I/4\pi R^2 = 1/2R^2$, and the magnetic length is defined as $\ell_B = 1/\sqrt{B} = R/\sqrt{I}$. The thermodynamic limit corresponds to $R, I \to \infty$, with $\ell_B$ fixed to a finite value. For simplicity, in the following we set $R = 1$, which makes all quantities dimensionless.

We now briefly discuss the one-particle state on the supersphere in the supermonopole background. A detailed analysis is found in Ref. [14]. The one-particle Hamiltonian is given by

$$H = \frac{1}{2M} \left( \Lambda_a^2 + C_{ab} \Lambda_a \Lambda_b \right),$$

where $\Lambda_a$ and $\Lambda_a$ are the $O(1|2)$ covariant particle angular momenta, $\Lambda_a = -i \epsilon_{abc} x_c D_b + \frac{1}{2} \theta_a (\sigma_a)_{ab} D_b$ and $\Lambda_a = \frac{1}{2} (C_{ab})_{\lambda a} X_b \theta_{\lambda b}$. The covariant derivatives are given by $D_a = \partial_a + i A_a$ and $D_{\lambda a} = \partial_a + i A_{\lambda a}$, where $\Lambda_a = -\frac{1}{2} i \epsilon_{abc} \Lambda_a X_b$ and $\Lambda_a = -\frac{1}{2} i (\sigma_a)_{ab} \Lambda_b$. The supermonopole field strengths are given by $B_a = \frac{1}{2} X_a \Lambda_a$ and $B_{\lambda a} = \frac{1}{2} \theta_{\lambda a}$. The commutation relations for the covariant angular momenta of the particle are obtained as $[\Lambda_a, \Lambda_b] = i \epsilon_{abc} (\Lambda_c - B_c)$, $[\Lambda_a, \Lambda_a] = \frac{1}{2} (\sigma_a)_{ab} (\Lambda_b - B_b)$, and $[\Lambda_a, \Lambda_b] = \frac{1}{2} (C_{ab})_{\lambda a} \Lambda_b$. Thus, they do not satisfy the $O(1|2)$ commutation relations exactly, due to the presence of the supercurrent field. The total $O(1|2)$ angular momenta are constructed as $L_a = \Lambda_a + B_a$ and $L_\alpha = \Lambda_\alpha + B_\alpha$. The operators $L_a$ play the role of the supercharge in the system. The covariance under $O(1|2)$ transformations is expressed as $[L_a, X_b] = i \epsilon_{abc} X_c$, $[L_a, X_a] = \frac{1}{2} (\sigma_a)_{ab} X_b$ and $[L_a, X_b] = \frac{1}{2} (C_{ab})_{\lambda a} X_b$, where $X_a$ represents $L_a$, $\Lambda_a$, and $B_a$, and $X_b$ represents $L_b$, $\Lambda_b$, and $B_b$.

The $O(1|2)$ Casimir operator for the total $O(1|2)$ angular momenta $L_a$ and $L_\alpha$ are

$$L_a L_b + C_{ab} L_\alpha L_\beta = \frac{1}{2} (\Lambda_a^2 + \Lambda_b^2),$$

where $a = \frac{1}{2} + n$. Here, $n = 0, 1, 2, \ldots$ indicates the Landau level. The supermonopole field is perpendicular to the surface of the supersphere, while the particle moves on the supersphere. Therefore, the particle angular momenta are orthogonal to the supermonopole field: $[\Lambda_a, B_a] = C_{ab} \Lambda_a X_b = B_a \Lambda_a + C_{ab} B_a \Lambda_b = 0$. Observing these relations, we find the energy eigenvalues to be $E_n = \frac{1}{2M} \left[ n(n + \frac{1}{2}) + I(n + \frac{1}{2}) \right]$, thus, in the LL, the energy becomes $E_{LL} = \frac{1}{2} \omega$, where $\omega = B/M$ is the cyclotron frequency.

The number of unit cells, each occupying an area $2\pi \ell_B^2$ on the supersphere, is $N_\Phi = 4\pi/2\pi \ell_B^2 = 1$. It is convenient to define the filling fraction as $\nu = N_\Phi/N_B$. We now give a comment. From the $O(1|2)$ representation theory, the dimension of the irreducible representation $j = 1/2$ is $D = (2j + 1) + (2j)(j + 1/2) = 2j + 1$. Therefore, the number of states in the LL is twice as large as the number of magnetic cells in the large $I$ limit. This implies that in each magnetic cell, there are two degenerate states due to the supersymmetry. Hence, in this system, the value of the filling fraction $N/N_\Phi$ is twice as large as that in the ordinary definition $N/D$.

The supercoherent state $\psi$ is defined as the state that is aligned in the direction of the supermagnetic flux, $(B_a, B_\alpha) \propto (x_a, \theta_\alpha)$; that is, we have $l_\alpha \psi = x_a C_{ab} l_b \psi$, $\theta_\beta = \frac{1}{2} \psi$, where $l_a$ and $l_\alpha$ constitute the fundamental representation of the $O(0|2)$ generators. Explicitly, these are

$$l_a = \frac{1}{2} \left( \begin{array}{c} \sigma_a \ 0 \\ 0 \ 0 \end{array} \right), \quad l_\alpha = \frac{1}{2} \left( \begin{array}{c} \tau_\alpha \ 0 \\ 0 \ \tau_\alpha \end{array} \right), \quad l_\beta = \frac{1}{2} \left( \begin{array}{c} 0 \ 0 \\ -\tau_\beta \ 0 \end{array} \right),$$

where the quantities $\{\sigma_a\}$ are the Pauli matrices, while $\tau_\alpha = (1, 0)^T$ and $\tau_\beta = (0, 1)^T$. Up to a $U(1)$ phase factor, the explicit form of the supercoherent state is found to be

$$\psi = (u, v, \eta) = \left( \sqrt{1 + x_a^2} \right)^2 \left( 1 - \frac{1}{4(1+x_a^2)} \right)^{1/2} \left( 1 + \frac{1}{4(1+x_a^2)} \right)^{1/2} \left( 1 + x_\alpha \theta_\alpha + (x_a + i x_\alpha) \theta_\alpha \right).$$

This is identical to the super Hopf spinor, which satisfies the commutation relations $\psi^\dagger l_\alpha \psi = \frac{1}{2} x_\alpha$ and $\psi^\dagger l_\beta \psi = \frac{1}{2} \theta_\alpha$, where $\psi^\dagger$ denotes the superadjoint, defined as $\psi^\dagger = (u^*, v^*, -\eta^*)$, and $*$ denotes pseudoconjugation, which acts on a Grassmann number $\xi$ as $\xi^\dagger = -\xi, (\xi_1 \xi_2)^\dagger = \xi_1^\dagger \xi_2^\dagger$. The coordinates $(x_a, \theta_\alpha)$ are superreal, in the sense that we have $(x_a, \theta_\alpha) = (x_a, C_{ab} \theta_\beta)$.

The supercoherent state in a supermonopole background can be obtained similarly. The supercoherent state, directed to the point $(\Omega_a, \Omega_\alpha)$, should satisfy the equation

$$[\Omega_a, \Omega_\alpha] = \frac{1}{2} \left( \begin{array}{c} \sigma_a \ 0 \\ 0 \ 0 \end{array} \right),$$

where $\chi$ is a constant super Hopf spinor given by $\chi = (a, b, \xi)$, which is mapped to the point $(\Omega_a, \Omega_\alpha)$ on the supersphere by $\Omega_a(\chi) = 2x_a^\dagger l_a^\dagger$, and $\Omega_\alpha(\chi) = 2x_\alpha^\dagger l_\alpha^\dagger$. The supercoherent state is found to be

$$\psi^\dagger(\chi, u, v, \eta) = (\chi^\dagger \psi)^\dagger = (a^* u + b^* v - \eta^*)^\dagger.$$

Supermonopole harmonics $u_{m,n}$ and $v_{m,n}$ are introduced on the supersphere. They form a basis for the LLL and are eigenstates of $L_z$ with the eigenvalues $\frac{(m-n)}{2}$ and $\frac{(m+n)}{2}$, respectively. Their explicit forms are $u_{m,n} = \sqrt{I/|m_1 m_2|} u^{m_1} v^{m_2}$ and $v_{m,n} = \sqrt{I/|m_1 m_2|} v^{m_1} u^{m_2}$, where $m_1 + m_2 = I$ and $n_1 + n_2 = I - 1$. The degeneracy of $u_{m,n}$ is $I + 1$, while that of $v_{m,n}$ is $I$. Thus, the total degeneracy is $(2I + 1)$, which is exactly the dimension of the Hilbert space of the LLL. Thus, without including any complex variables $\{u^*, v^*, \eta^*\}$, the functions in the LLL are constructed from the variables $\{u, v, \eta\}$. For this reason, the $O(1|2)$ operators are effectively represented as

$$L_a = \psi^\dagger l_a^\dagger \psi \quad \text{and} \quad L_\alpha = \psi^\dagger l_\alpha^\dagger \psi,$$

where $l_a = \left( \frac{\partial}{\partial x_a} \right)$ and $l_\alpha = \left( \frac{\partial}{\partial \theta_\alpha} \right)$ forms a complex representation of $O(1|2)$ with $l_a = -l_a^\dagger$ and $l_\alpha = C_{ab} l_b$. The complex representation in $O(1|2)$ is related to the original by the unitary transformation, $l_a = \mathcal{R}^\dagger l_a \mathcal{R}$, $l_\alpha = \mathcal{R}^\dagger l_\alpha \mathcal{R}$, where $\mathcal{R}$ is given by.

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Thus, the representation of $OSp(1|2)$ is pseudoreal. The properties of $\mathcal{R}$ are as follows: $\mathcal{R}^\dagger = \mathcal{R}^\dagger = \mathcal{R}^{-1}$ and $\mathcal{R}^2 = (\mathcal{R}^\dagger)^2 = \text{diag}(-1, -1, 1)$. Using the matrix $\mathcal{R}$, the complex spinor is given by $\tilde{\psi} = \mathcal{R}(\psi^J)^J = (v^*, -u^*, \eta^*)$. Then, without including the complex spinor $\tilde{\psi}^I$, the $OSp(1|2)$ singlet can be constructed from $\psi$ and $\psi^I$ alone: $\tilde{\psi}^I = \psi^I \mathcal{R} \tilde{\psi}^I = (\psi^I)^J \mathcal{R} \psi^J$.

We next study the two-particle state. The total angular momenta of $OSp(1|2)$ are given by $L_a^\text{tot} = L_a(1) + L_a(2)$ and $L_a^{\text{tot}} = L_a(1) + L_a(2)$, where $L_a(i)$ and $L_a(j)$ are the $OSp(1|2)$ generators of the $i$th particle. The two-particle supercoherent state located at the point $(\Omega_a, \Omega_a)$ satisfies the equation $[\Omega_a(\chi)L_a^\text{tot} + C_{a\beta} \Omega_a(\chi)L_\beta]\psi^{(l,j)}_X = -J\psi^{(l,j)}_X$. The solution is written as

$$\psi^{(l,j)}_X = (u_1, v_1, -v_1 u_2 + \eta_1 \eta_2)^{l-j} \cdot \psi^{(l,j)}_Y((u_1, v_1, \eta_1) \psi^{(l,j)}_Y(u_2, v_2, \eta_2),$$

where the first component on the right-hand side is an $OSp(1|2)$ singlet that determines the distance between two particles. Thus, the two-particle state $\psi^{(l-j)}_X$ represents an extended object whose spin is $J$, whose center-of-mass is located at $(\Omega_a(\chi), \Omega_a(\chi))$, and whose size is proportional to $(l - J)$.

In the LLL, this system reduces to a fuzzy supersphere [14]. Hence, the two-body interaction is reduced to that on a fuzzy supersphere, which is expressed as a supersymmetric extension of Haldane's pseudopotential [5], $\Pi V[\Omega_a(1)\Omega_a(2) + C_{a\beta} \Omega_a(1)\Omega_a(2)]\Pi = \sum_{I=0,1/2,1}^N V^{(l,j)}_I P_J [L_a(1)L_a(2) + C_{a\beta} L_a(1)L_\beta(2)]$, where $P_J$ is the projection operator to the Hilbert space spanned by the states that form an irreducible representation of $OSp(1|2)$. It is noted that the $OSp(1|2)$ spin $J$ takes not only integer values but also half-integer values. Physically, this implies that fermionic particles, as well as bosonic particles, appear as two-particle states in supersymmetric quantum-Hall systems.

In Haldane’s work [5], the Laughlin-Haldane wave function is constructed from the $SU(2)$ singlet state, which is apparently invariant under any rotations of the two-sphere. Therefore, it is natural to use an $OSp(1|2)$ singlet wave function as the supersymmetric extension of the Laughlin-Haldane wave function. Explicitly, it is given by

$$\Psi^{(m)} = \prod_{l<j}^N [-\psi^{(l,j)}_X \mathcal{R} \psi^{(l,j)}_X]^m = \prod_{I<j}^N (u_{ij} v_j - v_i u_j - \eta_i \eta_j)^m.$$

When the product is expanded, it can easily be seen that the sum of the powers of $u_i$ and $v_i$ is $m(N - 1)$. Also, the LLL constraint $m_i + m_2 = N_a$ for the monopole harmonics $u_{m_1, m_2}$ should be satisfied. Hence, the number of particles and the number of magnetic cells are related as $m(N - 1) = \mathcal{N}_b$. Thus, $\Psi^{(m)}$ describes a supersymmetric quantum-Hall liquid at $\nu = 1/m$ in the thermodynamic limit. The supercoherent state for $\Psi^{(m)}$, whose center is located at $(\Omega_a(\chi), \Omega_a(\chi))$, is given by $\Psi^{(m)}(\chi) = \prod V^{(l,j)}_I P_J [L_a(1)L_a(2) + C_{a\beta} L_a(1)L_\beta(2)],$ with energy 0, where $V_J > 0$. This Hamiltonian is a direct generalization of Haldane’s Hamiltonian, with the $OSp(1|2)$ Casimir operator replacing the $SU(2)$ one.

Because of the noncommutative algebra on the fuzzy supersphere, the super Hall currents $I_a = \frac{d}{dt} x_a$ and 206802-3
\[ I_a = \frac{d}{dt} \theta_a \] can be expressed as
\[
I_a = -i[x_a, V] = \alpha \epsilon_{abc} x_b E_c - i \alpha \frac{1}{2} (\sigma_a C)_{\alpha \beta} \theta_a E_\beta.
\]
\[
I_a = -i[\theta_a, V] = -i \alpha \frac{1}{2} (\sigma_a \beta) \theta_a E_\beta - i \alpha \frac{1}{2} (\sigma_a \beta) \theta_a E_\alpha,
\]
where \( \alpha = 2/I \), the superelectric fields \( E_a \) and \( E_\alpha \) are defined as \( E_a = -\partial_a V \) and \( E_\alpha = -C_{\alpha \beta} \theta_\beta V \), and we have used the fact that the Hamiltonian is reduced to the potential term \( V \) in the LLL. It can easily be seen that the super Hall currents satisfy the equation \( E_a I_a + C_{\alpha \beta} E_\alpha I_\beta = 0 \), which expresses the orthogonality of the Hall currents and the electric fields in the supersymmetric sense. When we pierce (eliminate) the supermagnetic flux at some point on the supersphere adiabatically, supersymmetric quantum-Hall liquids are constructed, we should proceed to the investigation of supersymmetric second and third Hopf maps. Once these are constructed, we should proceed to the investigation of their relation to D-brane systems and super Twistor models.

To summarize, we have constructed a low-dimensional supersymmetric quantum-Hall system and investigated its basic properties. Similar to the original quantum-Hall systems, this system raises many interesting questions, for instance, those regarding edge excitations, anyonic objects, effective field theory, and the relation of the present system to integrable models. In a planar limit, this system reduces to a supersymmetric harmonic oscillator system [15], which is related to the Pauli Hamiltonian for a spin-1/2 particle with a gyromagnetic factor of 2 or the Jaynes-Cummings model without interaction terms used in quantum optics. There are many real systems which show supersymmetric properties [16]. It would be worthwhile to investigate the realization of the supersymmetric holonomy, such as the Wilczek-Zee nonabelian holonomy (which generally appears in the presence of degenerate energy levels) and possible relevance to the supersymmetric quantum-Hall effects, in real systems. With regard to high energy physics, one of the most important tasks is to develop a higher-dimensional generalization based on supersymmetric second and third Hopf maps. Once these are constructed, I would like to acknowledge Yusuke Kimura for useful discussions. I am also glad to thank Taichiro Kugo, Hiroshi Kunitomo, Naoki Sasakura, and Tatsuya Tokunaga for conversations. This work was supported by JSPS.

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[7] See, for example, D. Karabali et al., hep-th/0407007, and references there in.