Dynamics of Two-Sign Point Vortices in Positive and Negative Temperature States

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Dynamics of two-sign point vortices in positive and negative temperature states is examined by a special-purpose supercomputer (MDGRAPE-2). Temperature T is determined by random sampling of 10^7 states for microcanonical ensemble in which system energy and angular impulse are kept constant. Distributions of the vortices in time-asymptotic equilibrium state are obtained by time development simulations for various values of T. When $T > 0$, both-sign vortices mix with each other and neutralize. When $T < 0$, a part of the vortices condenses and forms the clumps exclusively consisting of the same-sign vortices, while the other part of the vortices distributes outside the clumps uniformly. It is found that the vortices inside the clumps gain energy and the ones outside the clumps lose energy to keep the total energy constant. It suggests the common and essential role of the background vortices in the energy-conserving system that assists the formation of the clumps as well as the crystallization and generation of symmetric configuration observed in the nonneutral plasma experiments.

§1. Introduction

An existence of negative temperature state was first predicted by Onsager.[1)] The negative temperature state appears in systems that occupies only finite amount of phase space volume. It means that the density of states W for such systems goes down to zero in the limit of infinite energy. In other words, W has at least a peak at some $E = E_c$. In the energy region of $E > E_c$, entropy decreases with increase of energy, and inverse temperature $\beta = dS/dE = d\log W/dE$ becomes negative. Equilibrium properties in negative temperature states were discussed by Kida and Smith et al. in axisymmetric and nonaxisymmetric configurations.[2),3)] The first demonstration of negative temperature state by numerical simulations was done by Joyce and Montgomery.[4)] Since then, much research effort in numerical simulations has been devoted to understand the negative temperature states.[5)–8)] However, the number of particles and states are not sufficient in statistical meaning for lack of computational power.

In this paper, we demonstrate a dynamics of two-sign point vortices in negative and positive temperature states by massive simulations using a special-purpose supercomputer for molecular dynamics simulations called MDGRAPE-2.[9)] An objective of this paper is to characterize the equilibrium states of the two-sign vortex system by inverse temperature β. Thus, we first determine temperature of the system as a function of system energy E and angular impulse I. In other words, we

obtain a density of states $W(E, I)$ for the microcanonical ensemble consisting of randomly generated 10^7 states. Then, inverse temperature of the system is determined by $\beta = dS/dE = d\log W(E, I)/dE$ where S is entropy and the Boltzmann constant is normalized to unity. There is a peak at $(E, I) = (E_c, 0)$ in the obtained density of states. When $E > E_c$ and $I = 0$, temperature becomes negative.

To reveal the characteristics of the equilibrium state at positive and negative β, distributions of the vortices in time-asymptotic state are obtained by time development simulations. Because the vortex system is energy conservative, the temperature can be determined by the initial configuration of the vortices that specifies the total energy. When $T > 0$, both-sign vortices mix with each other and neutralize. On the other hand, when $T < 0$, a part of the vortices condenses and forms the clumps. The produced clumps seem to be very stable. Thus, the obtained configuration should be equilibrium one. Condensing the same-sign vortices needs positive energy, so that the other part of the vortices must lose the energy to keep the total energy constant. As a result of the energy balance, there remain a positive and a negative clumps and uniform background distribution in the time-asymptotic equilibrium state.

In the following, we present the density of states in §2, time asymptotic equilibrium state obtained by the time development simulations in §3. Finally, discussion is given in §4.

§2. Density of states

Let us consider a system of $N/2$ positive and $N/2$ negative point vortices with circulation $\pm\Gamma_0$ confined in a circular boundary of radius R. Constants of motion of this system are Hamiltonian

$$H = -\frac{1}{4\pi}\sum_{i}^{N}\sum_{j\neq i}^{N}\Gamma_i\Gamma_j \ln|\boldsymbol{r}_i-\boldsymbol{r}_j| + \frac{1}{4\pi}\sum_{i}^{N}\sum_{j}^{N}\Gamma_i\Gamma_j \ln|\boldsymbol{r}_i-\bar{\boldsymbol{r}}_j| - \frac{1}{4\pi}\sum_{i}^{N}\sum_{j}^{N}\Gamma_i\Gamma_j \ln\frac{R}{|\boldsymbol{r}_j|}, \tag{2·1}$$

and angular impulse

$$I = \sum_{i}^{N}\Gamma_i|\boldsymbol{r}_i|^2, \tag{2·2}$$

where $\boldsymbol{r}_i \equiv (x_i, y_i)$ and Γ_i are the position vector and the circulation of the ith point vortex, respectively. The circular boundary of radius R is represented by the image vortices at $\bar{\boldsymbol{r}}_i = R^2\boldsymbol{r}_i/|\boldsymbol{r}_i|^2$.

To determine temperature of the system, we employ a random sampling of 10^7 states for microcanonical ensemble and obtain number of states (density of states) associated with a cell bounded by $E \sim E + dE$ and $I \sim I + dI$. In calculating energy for each state, Hamiltonian (2·1) must be evaluated that needs calculation time proportional to N^2. To accelerate the calculation of Hamiltonian, a special-purpose computer called MDGRAPE-2 is used, which was originally designed for molecular dynamics simulations. The obtained number of states is given in Fig. 1. The peak of the number of states is located at $(E_c, I_c) = (29.0, 0.0)$. The profile is symmetric about $I = 0$ but asymmetric about $E = E_c$. Temperature of the system

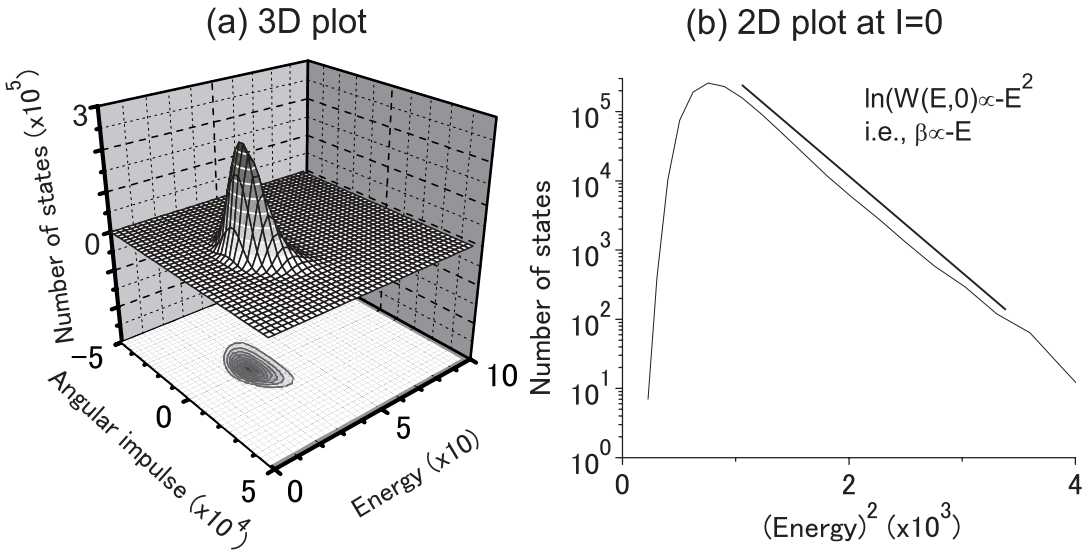

Fig. 1. Density of states is plotted against (a) E and I, and (b) E^2 with $I = 0$.

becomes negative when $E > E_c$ in $I = 0$ plane. In the negative temperature region, $W(E, I)$ is approximately proportional to $-E^2$. Thus, the scaling of $\beta \propto -E$ is obtained, so that β goes down with increase of E. Note that β is proportional to $-1/E$ in the ideal gas at positive temperature.

§3. Time evolution of the vortices

Now the temperature of the system is obtained from the density of states by specifying initial energy, i.e., initial configuration of the vortices, because the system energy is conserved. To characterize the time-asymptotic state by β, we carried out the time development simulations starting from various values of β. The governing equation for ith point vortex $\boldsymbol{r}_i = (x_i, y_i)$ is given by

$$\Gamma_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \Gamma_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}, \tag{3·1}$$

or in explicit form,

$$\frac{d\boldsymbol{r}_i}{dt} = -\frac{1}{2\pi} \sum_{j \neq i}^{N} \Gamma_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2} + \frac{1}{2\pi} \sum_{j}^{N} \Gamma_j \frac{\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j}{|\boldsymbol{r}_i - \bar{\boldsymbol{r}}_j|^2}. \tag{3·2}$$

Equation (3·2) is the discretized Biot-Savart integral. Its second term in right-hand side represents the boundary effect by the image vortices. In the simulations, velocity of each point vortex is evaluated exactly on the position of the vortex and no approximation, e.g., vortex-in-cell method, is introduced. Typical results for $\beta > 0$ and $\beta < 0$ are shown in Fig. 2. In Fig. 2 (a), positive β case is shown. In the initial distribution, the centers of the clumps are arranged on the circle of radius $R_c = 0.2R$

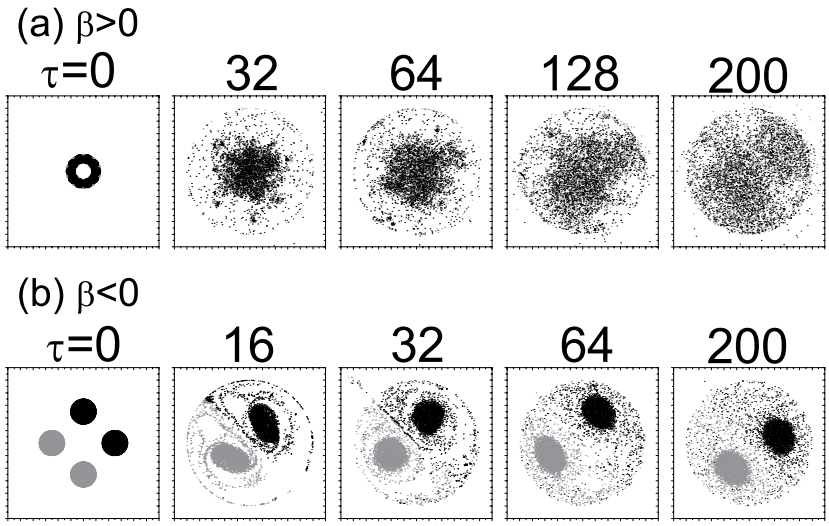

Fig. 2. Time evolution of the vortices in (a) positive and (b) negative temperature states are shown. The values of the energy and angular impulse are (a) $E = 24.1$, $I = 0$ and (b) $E = 2.69 \times 10^4$, $I = 0$.

at regular angular intervals. Each clump consists of the same-sign 346 vortices, and the polarity of the clump changes in turn in clockwise direction. The number of clumps is 10 for positive and 10 for negative clumps. The total number of vortices $N = 6920$ is chosen at the same order as the one used for the random sampling to determine the density of states. Parameters are initial radius of a clump $r_c = 0.07R$, and time scale of self-rotation of a clump in unbounded plane $\tau_c = 9.3$. Because the clumps partially overlap each other, we see only the outline of the clumps at $\tau = 0$. In this case, positive and negative vortices mix with each other. In the final state at $\tau = 200$, the distribution is almost neutralized and velocity at some position inside the circular boundary vanishes. Thus, the time scale of the vortex motion is long compared with the negative β case shown in Fig. 2 (b).

In Fig. 2 (b), negative β case is shown. Each clump consists of the same-sign 1681 vortices. The right and the upper clumps consist of the positive vortices, and the left and the lower clumps consist of the negative vortices. The number of total vortices is 6724. The values of the parameters are $r_c = 0.2R$ and $\tau_c = 15.6$. At $\tau = 16$, merger of the same-sign vortices occurs and clumps consisting of exclusively single-sign vortices are created. The condensation of the same-sign vortices is a remarkable feature at $\beta < 0$. The two oppositely polarized clumps continue orbiting inside the circular boundary and further merger does not occur until the end of the simulation $\tau = 200$. It is likely that this state is equilibrium one.

Energy belonging to the vortices inside the clumps increases and the one belonging to the vortices in the background decreases during the condensation keeping the total energy constant.[9] Thus, the background vortices enable the condensation of the same-sign vortices. This observation indicates the common and essential role of the background vortices in the condensation of the same-sign vortices as well as in

the crystallization observed in the nonneutral plasma experiments.[10)–12)] The size of the clumps in the final state decreases as the system energy increases. Thus, another specific feature at $\beta < 0$ is summarized as the energy concentration in a subsystem. Let us briefly examine the energy concentration to the clumps at $\beta < 0$ by employing a simple model. Consider a system consisting of a positive point vortex at $(r_0, 0)$ and a negative point vortex at $(-r_0, 0)$. In this model, no background vortices are considered because their contribution to the total energy vanishes in the case of uniform distribution. The circular boundary is represented by image vortices at $(R^2/r_0, 0)$ and $(-R^2/r_0, 0)$. The total energy of four-vortex system is written as a function of r_0 and is maximized when $r_0 = (\sqrt{5} - 2)^{1/2} R \approx 0.49R$. While the two clumps continue orbiting motion after the merger, time-averaged configuration of the two clumps in Fig. 2(b) resembles the maximum energy configuration discussed above. We conclude that the vortices that are going to form the clump absorb the system energy as much as possible. The amount of energy belonging to the vortices in the clumps depends on the amount of energy released by the rest of the vortices. The vortex condensation is enabled on the existence of the background vortices in the energy conserving system.

§4. Discussion

We have demonstrated the dynamics of two-sign point vortices confined in the circular boundary by the massive numerical simulations using the special-purpose supercomputer called MDGRAPE-2. The density of states $W(E, I)$ is determined by the large-scale numerical sampling, and the existence of the negative temperature state has been confirmed. Time-asymptotic equilibrium configurations of the vortices are obtained by the time-development simulation. A characteristic feature at $\beta < 0$ is the condensation of the same-sign vortices. A part of the vortices gains the energy to form the clumps, while the other part of the vortices loses the energy to keep the total energy constant. It elucidates the important role of the background vortices in the energy conserving system.

There remains mush issue in the two-dimensional vortex system. It would be interesting to understand the probability of each metastable state in crystallization under the constraint of given energy and angular impulse. The vortex system may continue giving impetus to the nonlinear statistical physics.

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