Three-Dimensional Representation of Japanese Phonemes

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Summary

The spectra, which represent nine consonants /p, t, k, b, d, g, m, n, ŋ/, were defined at the boundary between following vowel. Nine consonants of 20-dimensional vectors, which are projected orthogonally on three-dimensional subspace (that is created by the three directions which promote separation in manner and place of articulation), produce nearly a triangular prism. Although spatial representation of vowels has early been tried, physical and analytic expressions of consonants based on actual speech are supposed to be, for the first time, tried here.

Nine Consonants were uttered as C_2 of two-syllable $C_1V_1C_2V_2$ (consonant-vowel-consonant-vowel) words. In order to inspect the existence of physical spectra characterizing each consonant, multivariate statistical analysis of variance for four-factor design with repeated measurements was performed, considering C_1, V_1, C_2 and V_2 as four factors.

1. Introduction

We have so far tried to express speech sound on a coordinate plane or in three-dimensional space on the basis of parameters obtained by analyzing speech sound, and to observe their mutual relations. F_1–F_2 formant plane, in which the first and the second formant frequencies are regarded as parameters, is a most popular example of them. Klein, Plomp and Pols(1) have tried to plot 12 kinds of vowels of 50 speakers in space by spectral information. Considering spectrum represented by the amplitude outputs of 18-channel 1/3-octave filters as 18-dimensional vector, they performed the principal-component analysis. They then insisted that if we present vowels on the plane determined by the first two principal components, its configuration is similar to the configuration in F_1–F_2 formant plane.

Thus by representing phoneme sound by relation of spatial disposition, it is possible to give support, so called in phonetics, to a classified table of phonemes and to a cardinal vowel figure from a physical point of view if the results coincide. Moreover, there would be much more profitable aspect since the measure of distance was introduced.

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When it comes to consonants, however, expressing examples in this way would be rare. The main reason is that it is not easier to analyze consonants than vowels, for there are some difficulties on catching spectra of consonants at their instant of explosion in such case as plosive consonants.

This study took nine phonemes /p, t, k, b, d, g, m, n, and ŋ/ as a preliminary to this trial on spatial expression of phonemes including consonants. We expressed these phonemes in the three-dimensional space after defining their spectra. It is spontaneously indicated that this expression has a profound statistical meaning.

2. CVCV UTTERANCES AND THEIR SPECTRA

It is actually rare that each phoneme is individually uttered. Particularly in Japanese, it is common for consonants to be uttered with vowels like /ka/. We pay attention, here, to C_2 of two-syllable $C_1V_1C_2V_2$ (consonant-vowel-consonant-vowel) word in order to consider consonants which are placed in much more complicated circumstances than in a monosyllable CV word.

Using /p, t, k, b, d, g, m, n, ŋ/ for C_1, C_2 and /a, e, o/ for V_1, V_2, an adult man spoke each word of all the combinations—729 kinds—three times at random. He accented all the C_1V_1-mora dispite the fact that there are a lot of meaningless words in the combinations above. The sounds were uttered in the simplified nonreverberant room within one day and the total number of words was 2187. The reason for choosing /a, e, o/ as vowels was that we can have observations in all possible combinations of any element of C_1, V_1, C_2 and V_2 by eliminating /i and u/, since /ti, tu, di, and du/ do not occur in modern Japanese.

We will give a definition of speech spectra and procedure of segmentation below.

When let $b_1(t), \ldots\ldots, b_p(t)$ $(p=20)$ represent, in order, amplitude outputs of 20-channel 1/4-octave filters (bank of 20 filters whose center frequencies cover 210 up to 5660 Hz) at time t, they are considered to represent speech spectra at that time. After normalizing the square sum of these components at 1, we established p-dimensional vector $\boldsymbol{x}(t)$ by taking logarithm of its components. That is,

$$x_i(t) = \log\{b_i(t) / \sqrt{\sum_{j=1}^{p} b_j^2(t)}\}. \tag{1}$$

We defined p-dimensional vector $\boldsymbol{x}(t) = (x_1(t), \ldots, x_p(t))$ with Eq. (1) which would be used for the analysis $(p=20)$. Amplitude outputs of the filter analyzer are A-D converted at intervals of 10 ms, then put into the computer in real time. On the paper of the line-printer we represent the speech spectrum patterns of the input with which we observed to determine the stationary parts or transition ones.

This study defines times $t_1, t_2, \ldots, t_{13}$ as Fig. 1, corresponding to the stationary or transition parts in each word.

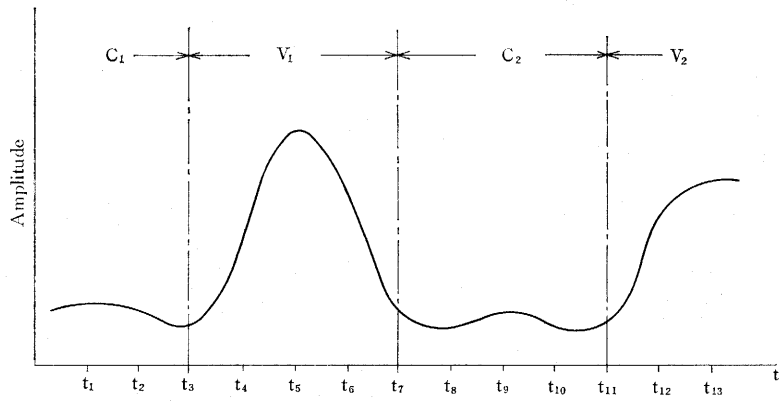

Fig. 1. $C_1V_1C_2V_2$ word and definition of t_i.
$t=t_1$: stationary part of C_1,
$t=t_3$: boundary of C_1-V_1,
$t=t_5$: stationary part of V_1,
$t=t_7$: boundary of V_1-C_2,
$t=t_9$: stationary part of C_2,
$t=t_{11}$: boundary of C_2-V_2,
$t=t_{13}$: stationary part of V_2.
If i is even, $t_i=(t_{i+1}+t_{i-1})/2$.

3. Consonant Spectra

It is a important problem whether consonants within different words show characteristics determined independently of their phonemic environment or not. If the characteristics of consonants differ in accordance with their circumstances, we cannot physically measure them but only by auditory psychological means.

In order to inspect the physical independence in detail, we performed multivariate analysis of variance for four-factor design with repeated measurements*) on the speech spectra $\boldsymbol{x}(t)$ of $C_1V_1C_2V_2$ words, assigning four phonems C_1, V_1, C_2 and V_2 to four factors A, B, C, and D, respectively, in the following model ($a=c=9$, $b=d=3$, $e=3$).

The linear model of multivariate analysis of variance for four-factor design with repeated measurements (See Appendix A)

The model is defined as follows (t is omitted):

$$
\begin{aligned}
\boldsymbol{x}_{ijklm}(1\times p)=&\,\boldsymbol{\kappa} && \text{general level}\\
&+\boldsymbol{\alpha}_i+\boldsymbol{\beta}_j+\boldsymbol{\gamma}_k+\boldsymbol{\delta}_l && \text{main effect}\\
&+\boldsymbol{\varepsilon}_{ij}+\boldsymbol{\zeta}_{ik}+\boldsymbol{\eta}_{il}+\boldsymbol{\theta}_{jk}+\boldsymbol{\lambda}_{jl}+\boldsymbol{\mu}_{kl} && \text{two-factor interaction}\\
&+\boldsymbol{\nu}_{ijk}+\boldsymbol{\rho}_{ijl}+\boldsymbol{\sigma}_{ikl}+\boldsymbol{\tau}_{jkl} && \text{three-factor interaction}\\
&+\boldsymbol{\varphi}_{ijkl} && \text{four-factor interaction}\\
&+\boldsymbol{\chi}_{ijklm}, && \text{residual}
\end{aligned}
\qquad (2)
$$

* As for detailed discussion about how to apply the multivariate analysis of variance to speech spectra, see our report (2) or (3).

where $1 \leqq i \leqq a$, $1 \leqq j \leqq b$, $1 \leqq k \leqq c$, $1 \leqq l \leqq d$ and $1 \leqq m \leqq e$. (See Table A.1 in Appendix A.) The constants a, b, c and d are the numbers of levels of the factor A, B, C and D, respectively. The constant e is the number of repetitions.

The general level vector $\boldsymbol{\kappa}(1 \times p)$ is determined in order that effect vectors $\boldsymbol{\alpha}_i(1 \times p) \sim \boldsymbol{\varphi}_{ijkl}(1 \times p)$ satisfy the following conditions:

$$\begin{aligned}
&\sum_{i=1}^{a} \boldsymbol{\alpha}_i = \mathbf{0},\ \sum_{j=1}^{b} \boldsymbol{\beta}_j = \mathbf{0},\ \sum_{k=1}^{c} \boldsymbol{\gamma}_k = \mathbf{0},\ \sum_{l=1}^{d} \boldsymbol{\delta}_l = \mathbf{0},\\
&\sum_{i=1}^{a} \boldsymbol{\varepsilon}_{ij} = \sum_{j=1}^{b} \boldsymbol{\varepsilon}_{ij} = \mathbf{0},\ \sum_{i=1}^{a} \boldsymbol{\zeta}_{ik} = \sum_{k=1}^{c} \boldsymbol{\zeta}_{ik} = \mathbf{0},\ \sum_{i=1}^{a} \boldsymbol{\eta}_{il} = \sum_{l=1}^{d} \boldsymbol{\eta}_{il} = \mathbf{0},\\
&\sum_{j=1}^{b} \boldsymbol{\theta}_{jk} = \sum_{k=1}^{c} \boldsymbol{\theta}_{jk} = \mathbf{0},\ \sum_{j=1}^{b} \boldsymbol{\lambda}_{jl} = \sum_{l=1}^{d} \boldsymbol{\lambda}_{jl} = \mathbf{0},\ \sum_{k=1}^{c} \boldsymbol{\mu}_{kl} = \sum_{l=1}^{d} \boldsymbol{\mu}_{kl} = \mathbf{0},\\
&\sum_{i=1}^{a} \boldsymbol{\nu}_{ijk} = \sum_{j=1}^{b} \boldsymbol{\nu}_{ijk} = \sum_{k=1}^{c} \boldsymbol{\nu}_{ijk} = \mathbf{0},\ \sum_{i=1}^{a} \boldsymbol{\rho}_{ijl} = \sum_{j=1}^{b} \boldsymbol{\rho}_{ijl} = \sum_{l=1}^{d} \boldsymbol{\rho}_{ijl} = \mathbf{0},\\
&\sum_{i=1}^{a} \boldsymbol{\sigma}_{ikl} = \sum_{k=1}^{c} \boldsymbol{\sigma}_{ikl} = \sum_{l=1}^{d} \boldsymbol{\sigma}_{ikl} = \mathbf{0},\ \sum_{j=1}^{b} \boldsymbol{\tau}_{jkl} = \sum_{k=1}^{c} \boldsymbol{\tau}_{jkl} = \sum_{l=1}^{d} \boldsymbol{\tau}_{jkl} = \mathbf{0},\\
&\sum_{i=1}^{a} \boldsymbol{\varphi}_{ijkl} = \sum_{j=1}^{b} \boldsymbol{\varphi}_{ijkl} = \sum_{k=1}^{c} \boldsymbol{\varphi}_{ijkl} = \sum_{l=1}^{d} \boldsymbol{\varphi}_{ijkl} = \mathbf{0}.
\end{aligned} \tag{3}$$

Besides, assume that

$$\boldsymbol{\chi}_{ijklm} \sim N(\mathbf{0}, \Lambda), \tag{4}$$

that is, $\boldsymbol{\chi}_{ijklm}$ is assumed to be independently distributed according to the p-dimensional normal distribution $N(\mathbf{0}, \Lambda)$.

The likelihood ratio test for null hypothesis.

Let us consider a test of the hypothesis for the main effect of factor A that all the effects of A's levels are equal (there is no effect of A);

$$H_A(t):\ \boldsymbol{\alpha}_1 = \dots = \boldsymbol{\alpha}_a = \mathbf{0} \tag{5}$$

We can test the hypothesis since it is possible to prove*) that the likelihood ratio criterion

$$\nu = \{n - l_2 - \frac{1}{2}(p + l_1 + 1)\} \log \frac{|Q_1 + R|}{|R|} \tag{6}$$

is distributed asymptotically according to χ^2-distribution with pl_1 degrees of freedom under the conditinos—$n = a \cdot b \cdot c \cdot d \cdot e$, $l_1 = a - 1$, $l_1 + l_2 = abcd$, and $n - abcd \geqq p$—when n is sufficiently large. Where,

$$\begin{aligned}
&Q_1 = bcde \sum_{i=1}^{a} (\boldsymbol{x}_{i\cdots\cdot} - \boldsymbol{x}_{\cdots\cdots})'(\boldsymbol{x}_{i\cdots\cdot} - \boldsymbol{x}_{\cdots\cdots}),\\
&R = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \sum_{l=1}^{d} \sum_{m=1}^{e} (\boldsymbol{x}_{ijklm} - \boldsymbol{x}_{ijkl\cdot})'(\boldsymbol{x}_{ijklm} - \boldsymbol{x}_{ijkl\cdot}),
\end{aligned} \tag{7}$$

and

$$\boldsymbol{x}_{\cdots\cdots} = \frac{1}{abcde} \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{c} \sum_{l=1}^{d} \sum_{m=1}^{e} \boldsymbol{x}_{ijklm},$$

$$\boldsymbol{x}_{i\cdots\cdot} = \frac{1}{bcde} \sum_{j=1}^{b} \sum_{k=1}^{c} \sum_{l=1}^{d} \sum_{m=1}^{e} \boldsymbol{x}_{ijklm},$$

* Mr. Masatoshi Kubo, a student of Kyoto University co-operated in accomplishment of the mathematical proof of this model. So that only the result of it is described here, refering to his graduation thesis for the bachelor degree of Kyoto University so as to realize the proof.(4)

$$x_{ijkl\cdot}=\frac{1}{e}\sum_{m=1}^{e} x_{ijklm} \quad .$$

Result of Analysis of Variance for Four-Factor Design

We performed analysis of variance at each t ($=t_3$, t_4, t_5, t_6, t_7, t_9, t_{11}, t_{12}, t_{13}) according to the model explained above. Namely, we computed respectively the test criterion ν (Eq. (6)) for the hypothesis that there are no effect of each factor or no interaction between each factor. We normalized ν by the value of significant level as the following equation, because the degrees of freedom corresponding to main effects and interactions are different from each other and so are the values of 1% significant level of χ^2 test different from each other. (See Table A. 2 in Appendix A.)

$$\nu'=\frac{\nu}{\text{(Value of 1\% significant level of } \chi^2 \text{ test corresponding to the degrees of freedom of } \nu)} \tag{8}$$

The illustration of value for test criterion ν' normalized above is as Fig. 2. In the figure, C_1, V_1, C_2, V_2 represent main effects, and V_1C_2, C_2V_2 the two-factor interactions, and $V_1C_2V_2$ the three-factor interactions, and so on.

The following is revealed from this figure:

(1) In the stationary parts of each phoneme, the main effect of the phoneme is maximum:

(2) The interaction between two contiguous phonemes is, of course, smaller than the each main effect, but larger than the main effect of any phoneme other than the two contiguous phonemes:

(3) The two-factor interaction between the two phonemes, that are not just contiguous to each other, is considerably smaller than the interaction of two contiguous phonemes. Therefore, the influence (on co-articulation) specific to the combinations of the phonemes which put a few other phonemes between them is small:

(4) There is hardly any interaction in the case of more than three factors. We did not illustrate any three-factor interaction whose ν' is less than 1, but only the effects of three-factor interactions of three-phoneme $V_1C_2V_2$ and $C_1V_1C_2$ (which are contiguous to each other) are significant:

(5) (1) and (2) above show the substantial proportion of the total variance. Accordingly, in the case of adopting an rough model upon omitting consideration of interaction, we have only to take account of the effects of the just preceding and following phonemes. We may not think of the effect of the phonemes apart farther than them.

(6) Influences of the preceding and following vowels of V_1 and V_2 on C_2 are almost the same.

Let us discuss, again, whether consonants are physically independent of their phonemic environment or not.

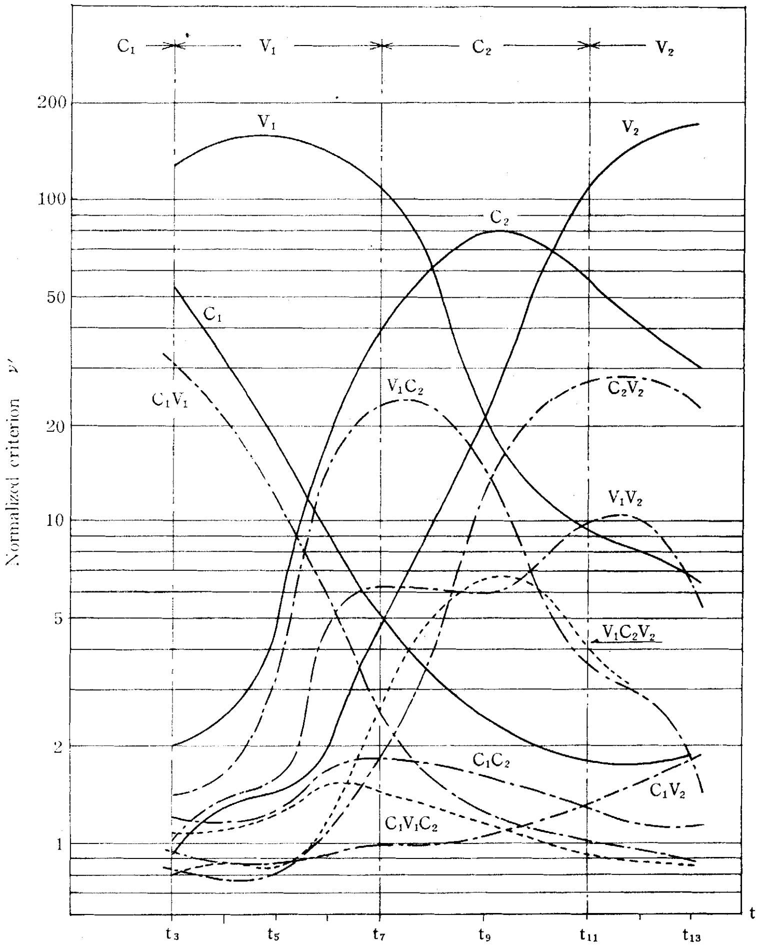

Fig. 2. Multivariate analysis of variance for four-factor design with repeated measurements.

Looking at Fig. 2, it is possible to persuade ourselves that the effect of C_2 is sufficiently significant as compared with the residual at $t=t_9$, t_{11}, etc., and that there exist definite physical characteristics which are independent of their phonemic environment.

As we see Fig. 2, however, a spectrum of any section in a utterance does not represent one phoneme only. Influence of other articulationally-combined phonemes always intervenes. Therefore, it is insufficient to conclude that spectra $\boldsymbol{x}_{ijklm}(t)$ measured within the regions of consonants represent spectra of consonants.

At $t=t_{11}$ when spectra of consonants /p, t, k, b, d, g, m, n, ŋ/ are defined as mentioned later, the effect of the vowel V_2 is so overpowering that the characteristics of consonant may be lost in the vowel if $\boldsymbol{x}_{ijklm}$ itself is used as spectrum of consonant.

We will adopt main effect vector $\boldsymbol{\gamma}_k$ $(k=1\sim c)$ of C_2 in the linear model of Eq. (2) as spectrum of consonant. $\boldsymbol{\gamma}_k$ is a vector which is determined only by the k-th consonant of C_2, and is a value independent of its phonemic environment. In other words, γ_k represents a spectrum which is regarded as a pure component of consonant C_2 after removing any component concerned with other phonemes except C_2 from x_{ijklm}. Although the problem is whether obtained $\gamma_1 \sim \gamma_c$ must be sufficiently and significantly different from each other as well as they are seemingly different, Fig. 2 clarifies this point.

We will now consider at what time point t we had better adopt $\gamma_k(t)$ as a consonant vector. Fig. 2 says that the effect of C_2 is maximum at the middle of the preceding vowel and the following one, that is $t=t_9$. It seems really maximum on the whole of the nine consonants, but what we can only classify in this time point are manner of articulation (M_1, M_2, M_3 of Table 1) and nasal sounds (/m, n, ŋ/); we can seldom do voiceless stops (/p, t, k/). (It will be clarified in the following section.)

Accordingly we want to select a time point at which these nine phonemes have equally balanced distribution even if the main effect becomes small. We presumed main effect vectors of C_2-factor at the boundary of C_2 and V_2 ($t=t_{11}$) to be spectra of the respective consonants.

Actually, maximum likelihood estimate of γ_k

$$\hat{\gamma}_k = x_{\cdot\cdot k\cdot\cdot} - x_{\cdot\cdot\cdot\cdot\cdot} \tag{9}$$

is used (See. Eq. (A.2) in Appendix A).

4. Classification from the Viewpoint of Manner and Place of Articulation

Consonants /p, t, k, b, d, g, m, n, ŋ/ are classified from the viewpoint of manner of articulation or place of articulation as described in Table 1. We examined whether this kind of classification was also reflected in the spectrum distribution. C_2 is interpreted in two ways—manner of articulation (M-factor) and place of articulation (P-factor)—as Table 1.

Assume $M_1 \sim M_3$ levels for M-factor and $P_1 \sim P_3$ levels for P-factor, respectively. These M-factor and P-factor are assigned to the first factor and the second one, respectively, of the analysis of variance of four-factor design of Eq. (2). And V_1 and V_2 are assigned to the third factor and the fourth, respectively, as before. We did an analysis of variance on trio-sequence $\sim V_1C_2V_2$ (disregarding C_1 from $C_1V_1C_2V_2$ sequence) which has repetitions corresponding to the number of consonants C_1 (nine in this time), regarding M, P, V_1, V_2 as four factors.

We consider, for example, utterances of /pame, tame, kame, bame, dame, game, mame, name, ŋame/ to be utterances /ame/ repeated nine times. In this case we used the first utterance out of three-repeated utterances of each combination in $C_1V_1C_2V_2$ words (2187/3=729 data).

Normalized criterion ν', in the same way as in Section 3, is shown in Fig. 3.

Table 1. Classification from the viewpoint of manner and place of articulation.

P_1	Labial
P_2	Alveolar
P_3	Palatal/Velar
M_1	Voiceless stop
M_2	Voiced stop
M_3	Nasal

			Place of articulation		
			Factor P		
			P_1	P_2	P_3
Manner of articulation	Factor M	M_1	/p/	/t/	/k/
		M_2	/b/	/d/	/g/
		M_3	/m/	/n/	/ŋ/

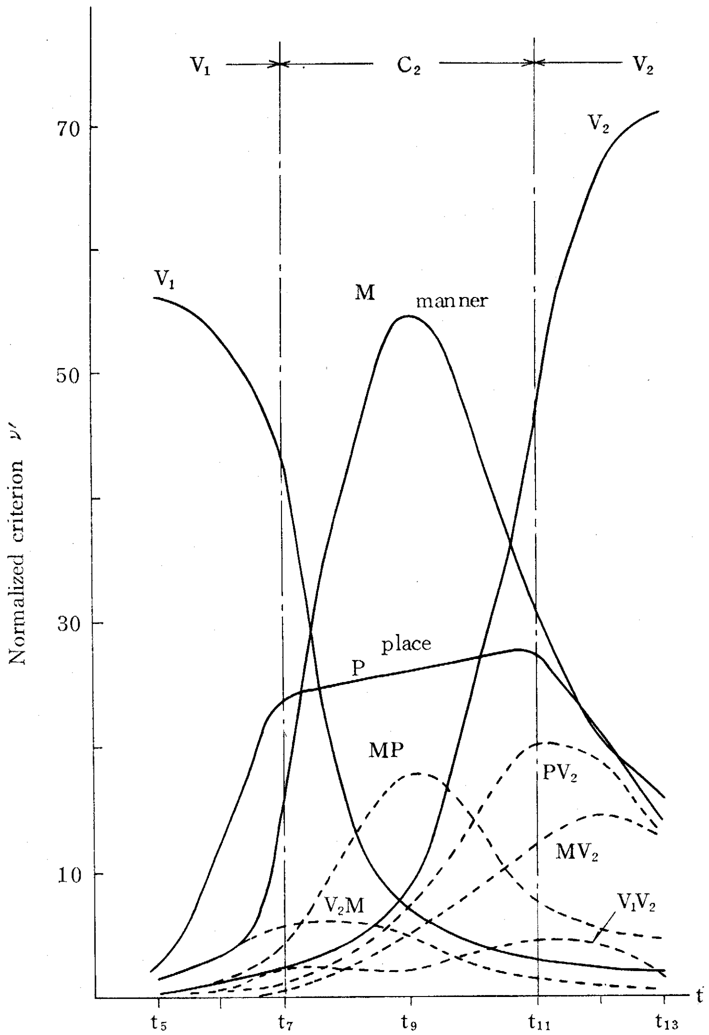

Fig. 3. Multivariate analysis of variance for four-factor design with repeated measurements.
V_1, V_2=/a,e,o/, M= (M_1, M_2, M_3) : manner, P= (P_1, P_2, P_3) : place.

The effects of manner and place of articulation are sufficiently separated. Especially amounts of the both effects at part V_2 are almost the same, and interactions between both are small at that part (More in detail, interactions at V_2 are larger in place of articulation than manner of one).

To verify the adequateness of the classification in Table 1, we perform analysis of variance for four-factor design of X, Y, V_1, V_2 based upon classification mixing elements of manner and place of articulation intentionally. The results are shown in Fig. 4.

Thus, we can see that the classification according to Table 2 is not good, because interaction of X, Y is large far beyond the main effects of them.

Furthermore, why the main effect of V_1 is smaller than V_2 is that we assumed $C_1V_1C_2V_2$ word to be $\sim V_1C_2V_2$. In despite of the fact that the influence of C_1

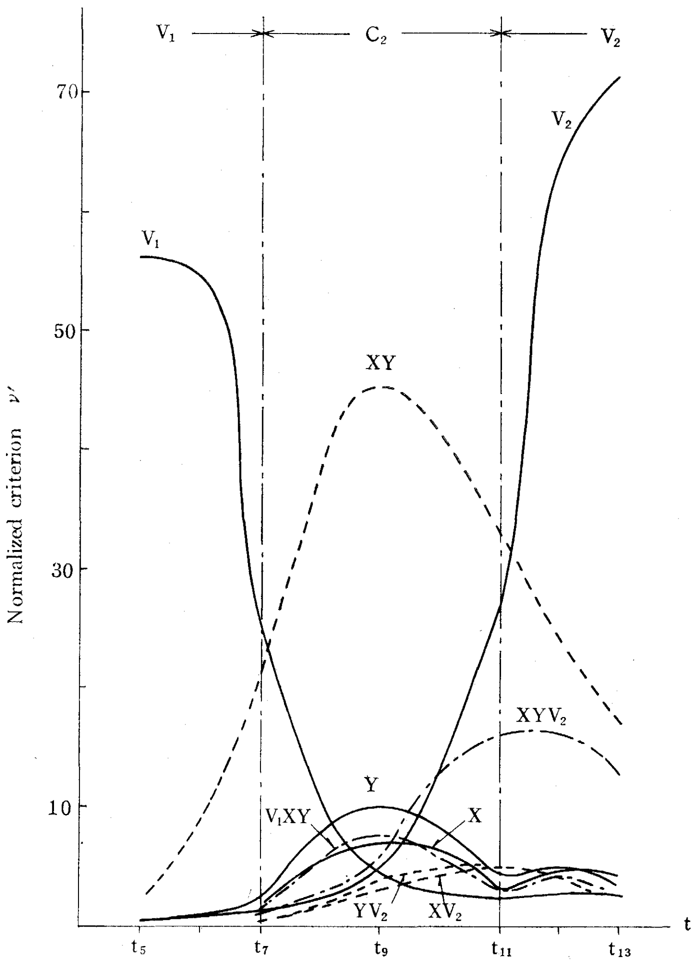

Fig. 4. Multivariate analysis of variance for four-factor design with repeated measurements.
$V_1, V_2 = /a, e, o/$, $X = (X_1, X_2, X_3)$, $Y = (Y_1, Y_2, Y_3)$.

Table 2. Some sort of classification.

		Factor Y		
		Y_1	Y_2	Y_3
Factor X	X_1	/d/	/p/	/ŋ/
	X_2	/k/	/n/	/b/
	X_3	/m/	/g/	/t/

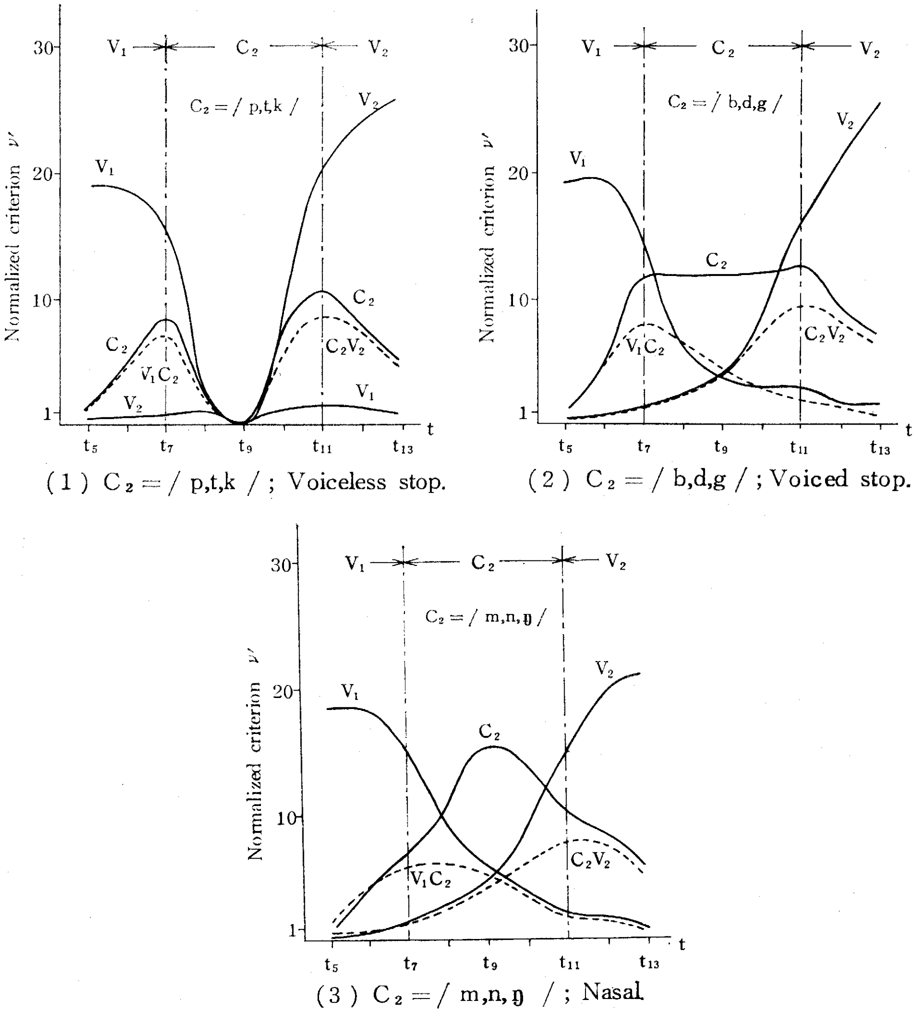

Fig. 5. Multivariate analysis of variance for three-factor design with repeated measurements. V_1, V_2=/a, e, o/.

=/p,t,k,b,d,g,m,n,ŋ/ on V_1 is considerably large, we disregarded it. This increased noise at part V_1, which accordingly made the ratio to the term of residual worse.

It would be relatively easy to classify in manner of articulation since the effect of it was enough large at time t_9, as clear in Fig. 3. So, we examined the effects of consonants belonging to each manner of articulation (M_1, M_2, M_3), respectively. Let us schme analysis of variance for three-factor design with repetition corresponding to the number of C_1 (nine), assigning the three factors to V_1, C_2, V_2 in trio-sequence $\sim V_1C_2V_2$ disregarding C_1 from $C_1V_1C_2V_2$. Models for three-factor design can be defined in the same means as Eq. (2)~Eq.(4).

The results of performing analysis of variance for three-factor design respectively on three cases of C_2—C_2=/p, t, k,/, C_2=/b, d, g/, and C_2=/m, n, ŋ/—are presented in Fig. 5.

It is understandable from Fig. 5 that we can distinguish nasals /m, n, ŋ/ at $t=t_9$, but absolutely not voiceless stops. Meanwhile, all the three cases are almost equally distinguishable at $t=t_{11}$.

Above is explained why we adopted γ_k at $t=t_{11}$ as consonant spectra.

5. EXPRESSION IN THREE-DIMENSIONAL SPACE

According to the preceding section, we will express the nine consonants /p, t, k, b, d, g, m, n, ŋ/ with vectors

$$\hat{\gamma}_1 \sim \hat{\gamma}_9 = \hat{\gamma}_p, \hat{\gamma}_t, \hat{\gamma}_k, \hat{\gamma}_b, \hat{\gamma}_d, \hat{\gamma}_g, \hat{\gamma}_m, \hat{\gamma}_n, \hat{\gamma}_\eta$$

(maximum likelihood estimates by 2189 words at $t=t_{11}$).

Now we would like to provide a less than three-dimensional space in which we can well understand the behavior of manner and place of artiuclation. If we can have the space like this, we have only to project orthogonally respective vectors in the 20-dimensional space on its subspace whose dimension is less than three.

Let $\boldsymbol{a}_{m_1}$, $\boldsymbol{a}_{m_2}$ represent eigenvectors (putting them in order of largeness of the corresponding eigenvalues) of covariance martix:

$$S_m = \frac{1}{3}(\gamma_{ptk}'\gamma_{ptk} + \gamma_{bdg}'\gamma_{bdg} + \gamma_{mn\eta}'\gamma_{mn\eta}) \tag{10}$$

which consists of three vectors of Eq. (11), that is,

$$\gamma_{ptk} = \frac{\gamma_p + \gamma_t + \gamma_k}{3}, \qquad \gamma_{bdg} = \frac{\gamma_b + \gamma_d + \gamma_g}{3}, \qquad \gamma_{mn\eta} = \frac{\gamma_m + \gamma_n + \gamma_\eta}{3}. \tag{11}$$

$$(\text{Note: } \gamma_{ptk} + \gamma_{bdg} + \gamma_{mn\eta} = \mathbf{0})$$

The extent of separation concerned with manner of articulation must be relatively favorable in these directions.

Likewise, suppose that $\boldsymbol{a}_{p_1}$, $\boldsymbol{a}_{p_2}$ are eigenvectors of covariance martix S_p which consists of three vectors of

$$\gamma_{pbm} = \frac{\gamma_p + \gamma_b + \gamma_m}{3}, \qquad \gamma_{tdn} = \frac{\gamma_t + \gamma_d + \gamma_n}{3}, \qquad \gamma_{kg\eta} = \frac{\gamma_k + \gamma_g + \gamma_\eta}{3},$$

$$(\text{Note: } \gamma_{pbm} + \gamma_{tdn} + \gamma_{kg\eta} = \mathbf{0}), \tag{12}$$

then the extent of separation concerned with place of articulation must be also favorable.

Let us project orthogonally the 20-dimensional vectors $\gamma_p \sim \gamma_\eta$ on its three-

Table 3. $\check{\gamma}=(\gamma_1, \gamma_2, \gamma_3)$ which is the orthogonal projection of γ $(1\times p)$ on W_1. $(p=20)$

$\check{\gamma}$	γ_1	γ_2	γ_3
p	4.53	−4.20	−3.27
t	4.83	3.48	−0.85
k	3.70	−2.75	1.07
b	1.16	−1.11	−1.92
d	1.00	6.75	1.68
g	0.78	−0.83	4.48
m	−5.69	−1.76	−1.55
n	−5.83	3.95	−0.81
ŋ	−4.48	−3.53	1.17

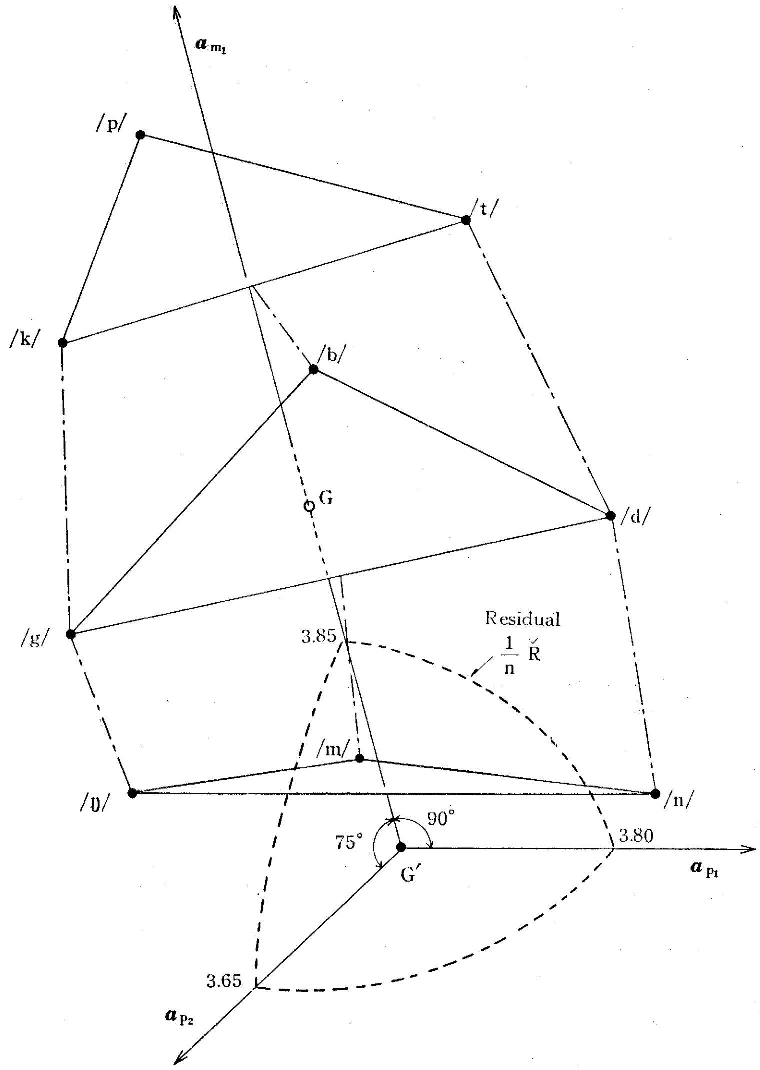

Fig. 6. Three-dimensional representation of Japanese consonants and 90%-probability ellipsoid of the residual $\check{R}/n$.

dimensional subspace which is generated by there vecotrs $\boldsymbol{a}_{m_1}$, $\boldsymbol{a}_{p_1}$ and $\boldsymbol{a}_{p_2}$. An arbitrary vector (through the origin) that belongs to the subspace W_1 generated by $\{\boldsymbol{a}_{m_1}, \boldsymbol{a}_{p_1}, \boldsymbol{a}_{p_2}\}$, is represented as

$$x_1\boldsymbol{a}_{m_1}+x_2\boldsymbol{a}_{p_1}+x_3\boldsymbol{a}_{p_2}=\check{x}A, \tag{13}$$

provided $A'(p\times 3)=(\boldsymbol{a}_{m_1}', \boldsymbol{a}_{p_1}', \boldsymbol{a}_{p_2}')$, $\check{x}=(x_1, x_2, x_3)$.
Let $\check{\gamma}_pA$ to be the orthogonal projection of γ_p on W_1. Since $\gamma-\check{\gamma}_pA$ is at right angles to a arbitrary vector $\check{x}A$ belonging to W_1,

$$0=(\gamma_p-\check{\gamma}_pA, \check{x}A)=(\gamma_pA'-\check{\gamma}_pAA', \check{x}). \tag{14}$$

$\check{x}$ is arbitrary, then

$$\gamma_pA'=\check{\gamma}_pAA', \text{ that is, } \check{\gamma}_p=\gamma_pA'(AA')^{-1}. \tag{15}$$

We made Table 3 by obtaining $\check{\gamma}_p\sim\check{\gamma}_ŋ$—orthogonal projections of $\gamma_p\sim\gamma_ŋ$ on W_1 through above process. The result of expressing $\check{\gamma}_p\sim\check{\gamma}_ŋ$ in three-dimensional space, whose coordinate axes are $\boldsymbol{a}_{m_1}$, $\boldsymbol{a}_{p_1}$ and $\boldsymbol{a}_{p_2}$ becomes as Fig. 6. All the vectors in Fig. 6 are moved parallel along the axis $\boldsymbol{a}_{m_1}$, in order to make it easier to understand. So that actual origin is represented by G.

This reveals that nine phonemes compose nearly a trianglular prism, clarifying relation between manner and place of articulation.

In Fig. 6 we also show the 90%-probability ellipsoid

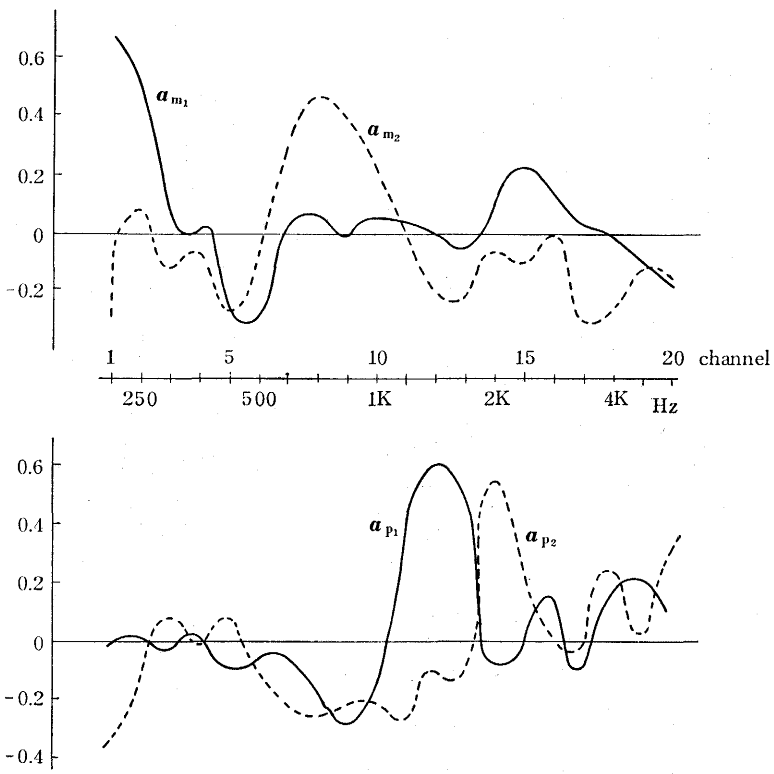

Fig. 7. Direction cosines of $\boldsymbol{a}_{m1}$, $\boldsymbol{a}_{m2}$, $\boldsymbol{a}_{p1}$ and $\boldsymbol{a}_{p2}$.

$$\check{x}(\check{R}/n)^{-1}\check{x}'=\chi^2_p(\alpha) \tag{16}$$

(where $\chi_p^2(\alpha)$ is the number such that Eq. (B.1) in Appendix B holds when $p=3$, $\alpha=0.1$) corresponding to the residual

$$\check{R}(3\times 3)=(A'(AA')^{-1})'R(A'(AA')^{-1}) \tag{17}$$

by $\check{x}_{ijklm}(1\times 3)$ which is orthogonal projection of 20-dimensional vector x_{ijklm} on the above three-dimensional subspace, where $R(20\times 20)$ is the residual variance as in Eq. (7) or Eq. (A.1) and $n=abcde$.

(See Appendix B and Appendix C)

We can see vertices of this prism separated significantly from each other as compared with the size of the residual ellipsoid $\check{R}/n$.

We showed the direction cosines of eigenvectors—a_{m_1}, a_{m_2}, a_{p_1}, a_{p_2}—in Fig. 7. The patterns are never contrary to the traditional spectral knowledge on manner and place of articulation.

6. Conclusion

We analyzed nine consonants, and investigated relations of their relative distribution in three-dimensional space. These consonants were placed at phoneme C_2 of $C_1V_1C_2V_2$ words (all of the total combinations of C_1, C_2=/p, t, k, b, d, g, m, n, ŋ/, and V_1, V_2=/a, e, o/). To begin with we extracted, from speech spactra (20-dimensional spectra), components which represent independently of co-articulation, then selected the boundary of C_2–V_2 as a section at which these consonants are evenly separated. When being projected orthogonally the nine consonants on three-dimensional subspace which is generated by three directions and promotes separation of consonants in viewpoint of manner and place of articulation, they took the shape nearly of a triangular prism in the space. It is guaranteed by multivariate statistical analysis that these vertices are significantly different from each other.

Spatial expression of consonants from physical and analytical points of view based on actual speech seems, for the first time, to be tried in this study, though spatial expression of vowels has been investigated from the early stage.

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(Aug. 31, 1974, received)

APPENDIX A

Supplement to the Linear Model of Multivariate Analysis of Variance for Four-Factor Design with Repeated Measurements.

The breakdown of total variance.

Let "′" denote the transposed matrix, then the breakdown of total variance Q (p×p) (matrix of sums of squares and cross products) is as follows: (See Table A.1.)

$$\begin{aligned} Q &= Q_1+Q_2+Q_3+Q_4+Q_5+Q_6+Q_7+Q_8+Q_9+Q_{10} \\ &\quad +Q_{11}+Q_{12}+Q_{13}+Q_{14}+Q_{15}+R, \end{aligned} \tag{A.1}$$

when

$$Q=\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{d}\sum_{m=1}^{e}(x_{ijklm}-x_{.....})'(x_{ijklm}-x_{.....}),$$

$$Q_1=bcde\sum_{i=1}^{a}(x_{i....}-x_{.....})'(x_{i....}-x_{.....}),$$

$$Q_2=acde\sum_{j=1}^{b}(x_{.j...}-x_{.....})'(x_{.j...}-x_{.....}),$$

$$Q_3=abde\sum_{k=1}^{c}(x_{..k..}-x_{.....})'(x_{..k..}-x_{.....}),$$

$$Q_4=abce\sum_{l=1}^{d}(x_{...l.}-x_{.....})'(x_{...l.}-x_{.....}),$$

$$\begin{aligned} Q_5=cde\sum_{i=1}^{a}\sum_{j=1}^{b}&(x_{ij...}-x_{i....}-x_{.j...}+x_{.....})' \\ &\cdot(x_{ij...}-x_{i....}-x_{.j...}+x_{.....}), \end{aligned}$$

$$\begin{aligned} Q_6=bde\sum_{i=1}^{a}\sum_{k=1}^{c}&(x_{i.k..}-x_{i....}-x_{..k..}+x_{.....})' \\ &\cdot(x_{i.k..}-x_{i....}-x_{..k..}+x_{.....}), \end{aligned}$$

$$\begin{aligned} Q_7=bce\sum_{i=1}^{a}\sum_{l=1}^{d}&(x_{i..l.}-x_{i....}-x_{...l.}+x_{.....})' \\ &\cdot(x_{i..l.}-x_{i....}-x_{...l.}+x_{.....}), \end{aligned}$$

$$\begin{aligned} Q_8=ade\sum_{j=1}^{b}\sum_{k=1}^{c}&(x_{.jk..}-x_{.j...}-x_{..k..}+x_{.....})' \\ &\cdot(x_{.jk..}-x_{.j...}-x_{..k..}+x_{.....}), \end{aligned}$$

$$\begin{aligned} Q_9=ace\sum_{j=1}^{b}\sum_{l=1}^{d}&(x_{.j.l.}-x_{.j...}-x_{...l.}+x_{.....})' \\ &\cdot(x_{.j.l.}-x_{.j...}-x_{...l.}+x_{.....}), \end{aligned}$$

$$\begin{aligned} Q_{10}=abe\sum_{k=1}^{c}\sum_{l=1}^{d}&(x_{..kl.}-x_{..k..}-x_{...l.}+x_{.....})' \\ &\cdot(x_{..kl.}-x_{..k..}-x_{...l.}+x_{.....}), \end{aligned}$$

$$\begin{aligned} Q_{11}=de\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}&(x_{ijk..}-x_{ij...}-x_{i.k..}-x_{.jk..}+x_{i....}+x_{.j...}+x_{..k..} \\ &-x_{.....})'\cdot(x_{ijk..}-x_{ij...}-x_{i.k..}-x_{.jk..}+x_{i....}+x_{.j...}+x_{..k..}-x_{.....}), \end{aligned}$$

$$Q_{12}=ce\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{l=1}^{d}(x_{ij.l.}-x_{ij...}-x_{i..l.}-x_{.j.l.}+x_{i....}+x_{.j...}+x_{...l.}$$

$$-x_{.....})'\cdot(x_{ij.l.}-x_{ij...}-x_{i..l.}-x_{.j.l.}+x_{i....}+x_{.j...}+x_{...l.}-x_{.....}),$$

$$Q_{13}=be\sum_{i=1}^{a}\sum_{k=1}^{c}\sum_{l=1}^{d}(x_{i.kl.}-x_{i.k..}-x_{i..l.}-x_{..kl.}+x_{i....}+x_{..k..}+x_{...l.}$$
$$-x_{.....})'\cdot(x_{i.kl.}-x_{i.k..}-x_{i..l.}-x_{..kl.}+x_{i....}+x_{..k..}+x_{...l.}-x_{.....}),$$

$$Q_{14}=ae\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{d}(x_{.jkl.}-x_{.jk..}-x_{.j.l.}-x_{..kl.}+x_{.j...}+x_{..k..}+x_{...l.}$$
$$-x_{.....})'\cdot(x_{.jkl.}-x_{.jk..}-x_{.j.l.}-x_{..kl.}+x_{.j...}+x_{..k..}+x_{...l.}-x_{.....}),$$

$$Q_{15}=e\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{d}(x_{ijkl.}-x_{ijk..}-x_{ij.l.}-x_{i.kl.}-x_{.jkl.}+x_{ij...}+x_{i.k..}$$
$$+x_{i..l.}+x_{.jk..}+x_{.j.l.}+x_{..kl.}-x_{i....}-x_{.j...}-x_{..k..}-x_{...l.}$$
$$+x_{.....})'\cdot(x_{ijkl.}-x_{ijk..}-x_{ij.l.}-x_{i.kl.}-x_{i.jkl.}+x_{ij...}+x_{i.k..}+x_{i..l.}$$
$$+x_{.jk..}+x_{.j.l.}+x_{..kl.}-x_{i....}-x_{.j...}-x_{..k..}-x_{...l.}+x_{.....}),$$

$$R=\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{d}\sum_{m=1}^{e}(x_{ijklm}-x_{ijkl.})'(x_{ijklm}-x_{ijkl.}),$$

and

$$x_{.....}=\frac{1}{abcde}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{d}\sum_{m=1}^{e}x_{ijklm},$$

$$x_{i....}=\frac{1}{bcde}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{d}\sum_{m=1}^{e}x_{ijklm},\ \dots,\ x_{...l.}=\frac{1}{abce}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{m=1}^{e}x_{ijklm},$$

Table A.1 Multivariate analysis of variance for four-factor design with repeated measurements.

	Factor	Effect vector	Q_l
Main effect	A	α_i	Q_1
	B	β_j	Q_2
	C	γ_k	Q_3
	D	δ_l	Q_4
Two-factor interaction	AB	ε_{ij}	Q_5
	AC	ζ_{ik}	Q_6
	AD	η_{il}	Q_7
	BC	θ_{jk}	Q_8
	BD	λ_{jl}	Q_9
	CD	μ_{kl}	Q_{10}
Three-factor interaction	ABC	ν_{ijk}	Q_{11}
	ABD	ρ_{ijl}	Q_{12}
	ACD	σ_{ikl}	Q_{13}
	BCD	τ_{jkl}	Q_{14}
Four-factor interaction	ABCD	φ_{ijkl}	Q_{15}

$$x_{ij\cdots}=\frac{1}{cde}\sum_{k=1}^{c}\sum_{l=1}^{d}\sum_{m=1}^{e}x_{ijklm},\ \ldots,\quad x_{\cdot j\cdot l\cdot}=\frac{1}{ace}\sum_{i=1}^{a}\sum_{k=1}^{c}\sum_{m=1}^{e}x_{ijklm},\ \ldots,$$

$$x_{ijk\cdot\cdot}=\frac{1}{de}\sum_{l=1}^{d}\sum_{m=1}^{e}x_{ijklm},\ \ldots,\qquad x_{i\cdot kl\cdot}=\frac{1}{be}\sum_{j=1}^{b}\sum_{m=1}^{e}x_{ijklm},\ \ldots,$$

$$x_{ijkl\cdot}=\frac{1}{e}\sum_{m=1}^{e}x_{ijklm}\ .$$

Table A.2 Degrees of freedom of Q_i ; $\ell_1+\ell_2=$abcd, and degrees of freedom of R is n−abcd. (n=abcde)

Factor	Q_i	ℓ_1
A	Q_1	a−1
B	Q_2	b−1
C	Q_3	c−1
D	Q_4	d−1
AB	Q_5	ab−a−b+1
AC	Q_6	ac−a−c+1
AD	Q_7	ad−a−d+1
BC	Q_8	bc−b−c+1
BD	Q_9	bd−b−d+1
CD	Q_{10}	cd−c−d+1
ABC	Q_{11}	abc−ab−ac−bc+a+b+c−1
ABD	Q_{12}	abd−ab−ad−bd+a+b+d−1
ACD	Q_{13}	acd−ac−ad−cd+a+c+d−1
BCD	Q_{14}	bcd−bc−bd−cd+b+c+d−1
ABCD	Q_{15}	abcd−abc−abd−acd−bcd+ab+ac+ad+bc+bd+cd−a−b−c−d+1

The maximum likelihood estimates of main effects and interactions.

$$\hat{\kappa}=x_{\cdots\cdots},$$
$$\hat{\alpha}_i=x_{i\cdots\cdot}-x_{\cdots\cdots},$$
$$\hat{\beta}_j=x_{\cdot j\cdots}-x_{\cdots\cdots},$$
$$\hat{\gamma}_k=x_{\cdot\cdot k\cdot\cdot}-x_{\cdots\cdots},$$
$$\hat{\delta}_l=x_{\cdots l\cdot}-x_{\cdots\cdots},$$
$$\hat{\varepsilon}_{ij}=x_{ij\cdots}-x_{i\cdots\cdot}-x_{\cdot j\cdots}+x_{\cdots\cdots},$$
$$\hat{\zeta}_{ik}=x_{i\cdot k\cdot\cdot}-x_{i\cdots\cdot}-x_{\cdot\cdot k\cdot\cdot}+x_{\cdots\cdots},$$
$$\hat{\eta}_{il}=x_{i\cdot\cdot l\cdot}-x_{i\cdots\cdot}-x_{\cdots l\cdot}+x_{\cdots\cdots},$$
$$\hat{\theta}_{jk}=x_{\cdot jk\cdot\cdot}-x_{\cdot j\cdots}-x_{\cdot\cdot k\cdot\cdot}+x_{\cdots\cdots},$$
$$\hat{\lambda}_{jl}=x_{\cdot j\cdot l\cdot}-x_{\cdot j\cdots}-x_{\cdots l\cdot}+x_{\cdots\cdots},$$
$$\hat{\mu}_{kl}=x_{\cdot\cdot kl\cdot}-x_{\cdot\cdot k\cdot\cdot}-x_{\cdots l\cdot}+x_{\cdots\cdots},$$
$$\hat{\nu}_{ijk}=x_{ijk\cdot\cdot}-x_{ij\cdots}-x_{i\cdot k\cdot\cdot}-x_{\cdot jk\cdot\cdot}+x_{i\cdots\cdot}+x_{\cdot j\cdots}+x_{\cdot\cdot k\cdot\cdot}-x_{\cdots\cdots},$$

$$
\begin{aligned}
\hat{\rho}_{ijl} &= \boldsymbol{x}_{ij\cdot l\cdot} - \boldsymbol{x}_{ij\cdot\cdot\cdot} - \boldsymbol{x}_{i\cdot\cdot l\cdot} - \boldsymbol{x}_{\cdot j\cdot l\cdot} + \boldsymbol{x}_{i\cdot\cdot\cdot\cdot} + \boldsymbol{x}_{\cdot j\cdot\cdot\cdot} + \boldsymbol{x}_{\cdot\cdot\cdot l\cdot} - \boldsymbol{x}_{\cdot\cdot\cdot\cdot\cdot}, \\
\hat{\sigma}_{ikl} &= \boldsymbol{x}_{i\cdot kl\cdot} - \boldsymbol{x}_{i\cdot k\cdot\cdot} - \boldsymbol{x}_{i\cdot\cdot l\cdot} - \boldsymbol{x}_{\cdot\cdot kl\cdot} + \boldsymbol{x}_{i\cdot\cdot\cdot\cdot} + \boldsymbol{x}_{\cdot\cdot k\cdot\cdot} + \boldsymbol{x}_{\cdot\cdot\cdot l\cdot} - \boldsymbol{x}_{\cdot\cdot\cdot\cdot\cdot}, \\
\hat{\tau}_{jkl} &= \boldsymbol{x}_{\cdot jkl\cdot} - \boldsymbol{x}_{\cdot jk\cdot\cdot} - \boldsymbol{x}_{\cdot j\cdot l\cdot} - \boldsymbol{x}_{\cdot\cdot kl\cdot} + \boldsymbol{x}_{\cdot j\cdot\cdot\cdot} + \boldsymbol{x}_{\cdot\cdot k\cdot\cdot} + \boldsymbol{x}_{\cdot\cdot\cdot l\cdot} - \boldsymbol{x}_{\cdot\cdot\cdot\cdot\cdot}, \\
\hat{\varphi}_{ijkl} &= \boldsymbol{x}_{ijkl\cdot} - \boldsymbol{x}_{ijk\cdot\cdot} - \boldsymbol{x}_{ij\cdot l\cdot} - \boldsymbol{x}_{i\cdot kl\cdot} - \boldsymbol{x}_{\cdot jkl\cdot} + \boldsymbol{x}_{ij\cdot\cdot\cdot} + \boldsymbol{x}_{i\cdot k\cdot\cdot} + \boldsymbol{x}_{i\cdot\cdot l\cdot} + \boldsymbol{x}_{\cdot jk\cdot\cdot} \\
&\quad + \boldsymbol{x}_{\cdot j\cdot l\cdot} + \boldsymbol{x}_{\cdot\cdot kl\cdot} - \boldsymbol{x}_{i\cdot\cdot\cdot\cdot} - \boldsymbol{x}_{\cdot j\cdot\cdot\cdot} - \boldsymbol{x}_{\cdot\cdot k\cdot\cdot} - \boldsymbol{x}_{\cdot\cdot\cdot l\cdot} + \boldsymbol{x}_{\cdot\cdot\cdot\cdot\cdot}
\end{aligned} \tag{A.2}
$$

Appendix B

Probability Ellipsoid

If $\boldsymbol{x}(1\times p)$ is distributeed according to the p-dimensional normal distribution $N(\boldsymbol{\mu}, \Lambda)$, $(\boldsymbol{x}-\boldsymbol{\mu})\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})'$ has a χ^2-distribution (Chi-squared distribution) with p degrees of freedom.

Let $\chi_p{}^2(\alpha)$ be the number such that

$$\Pr\{\chi_p{}^2 \geq \chi_p{}^2(\alpha)\} = \alpha, \tag{B.1}$$

where $\chi_p{}^2$ has a χ^2-distribution with p degrees of freedom.
Thus,

$$\Pr\{(\boldsymbol{x}-\boldsymbol{\mu})\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})' \leq \chi_p{}^2(\alpha)\} = 1-\alpha. \tag{B.2}$$

In the p-dimensional space of $\boldsymbol{x}$,

$$(\boldsymbol{x}-\boldsymbol{\mu})\Lambda^{-1}(\boldsymbol{x}-\boldsymbol{\mu})' \leq \chi_p{}^2(\alpha) \tag{B.3}$$

represents the surface and the interior of a ellipsoid whose center is $\boldsymbol{\mu}$. The shape of the ellipsoid depends on Λ^{-1}, and the size on $\chi_p{}^2(\alpha)$ for given Λ^{-1}. The interior of the ellipsoid (B.3) is considered to contain $(1-\alpha)\times 100\%$ of the population and is called the "Probability ellipsoid."

If $\Lambda^{-1} = I_p$, for example, (B.2) says that the probability is $1-\alpha$ that the distance between $\boldsymbol{x}$ and $\boldsymbol{\mu}$ is less than $\sqrt{\chi_p{}^2(\alpha)}$, where I_p is the p-dimensional unit matrix.

Appendix C

Matrix of Sums of Squares and Cross Products in Subspace.

Let B $(p\times q)$ be the orthogonal projection matrix of p-dimensional space on its q-dimensional subspace,

$$\boldsymbol{x}(1\times p) \rightarrow \check{\boldsymbol{x}}(1\times q) = \boldsymbol{x}\mathrm{B}. \tag{C.1}$$

And, let a matrix Q be such a matrix of sums of squares and cross products,

$$Q\,(p\times p) = \sum_{i=1}^{n}(\boldsymbol{x}_i - \boldsymbol{x}_{\cdot})'(\boldsymbol{x}_i - \boldsymbol{x}_{\cdot}), \quad \boldsymbol{x}_{\cdot} = \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i \quad . \tag{C.2}$$

Then Q is transformed by the above projection as follows.

$$
\begin{aligned}
\check{Q}\,(q\times q) &= \sum_{i=1}^{n}(\check{\boldsymbol{x}}_i - \check{\boldsymbol{x}}_{\cdot})'(\check{\boldsymbol{x}}_i - \check{\boldsymbol{x}}_{\cdot}) \\
&= \sum_{i=1}^{n}\mathrm{B}'(\boldsymbol{x}_i - \boldsymbol{x}_{\cdot})'(\boldsymbol{x}_i - \boldsymbol{x}_{\cdot})\mathrm{B}
\end{aligned}
$$

$$=B'(\sum_{i=1}^{n}(\boldsymbol{x}_i-\boldsymbol{x}.)'(\boldsymbol{x}_i-\boldsymbol{x}.))B$$
$$=B'QB, \tag{C.3}$$

where

$$\check{\boldsymbol{x}}.=\frac{1}{n}\sum_{i=1}^{n}\check{\boldsymbol{x}}_i \quad .$$

Now let us consider the Eq. (17).
Since R is a matrix of sums of squares and cross products, and $B=A'(AA')^{-1}$ from Eq. (15), then the residual R in the subspace is equal to

$$\check{R}=(A'(AA')^{-1})'R(A'(AA')^{-1}). \tag{C.4}$$