Nonlinear Adaptive Control and Its Applications

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Abstract

This thesis is concerned with the adaptive control theory and its applications to the autonomous flight control system of unmanned aerial vehicle.

First we propose a adaptive control system based on multiple module architecture and new learning algorithm based on Lyapunov design methods that is applicable in practical problems are proposed. We examine the performance of the proposed method both in simulations and experiments. It is shown that multiple modules are successfully trained and specialized for different domains in the state space in a cooperative way. Furthermore, the control system which consists of several online modules is applied to the autonomous flight control system of aero-robot, and we evaluate our method by flight experiments.

Second, the output regulation problem for linear time-invariant systems with unknown parameters is considered. Based on the Lyapunov stability theory, a stabilizing adaptive controller is derived. It is shown that an adaptive controller can be designed using the solution of the parameter dependent Riccati equation if the derivative of the solution is sufficiently small. Then sufficient conditions for the output regulation problem with full information to be solvable are established. Furthermore, the condition on the solution of the Riccati equation imposed above is relaxed.

Finally, adaptive output regulation for nonlinear systems described by multiple linear models with unknown parameters is considered. We design a local stabilizing controller for affine nonlinear system using the solution of the state dependent Riccati equation and local output regulation is established using a state dependent regulator equation. Then locally stabilizing adaptive state-feedback controllers for nonlinear systems described by multiple linear models with unknown parameters are designed based on the Lyapunov stability theory. Local adaptive output regulation is also established using a state dependent regulator equation. We extend our method to output feedback control. The adaptive laws are derived from Lyapunov stability analysis which guarantees that observer error and parameter estimation error are bounded provided that the state and the control are bounded. Simulation results are given to illustrate the theory.

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Chapter 1

Introduction

1.1 Background

1.1.1 Adaptive control: overview

In a broad sense, to adapt means "to change (oneself) so that one's behavior will conform to new or changed circumstances". Adaptive control is one of the ideas which can realize adaptive behavior of control systems. Specifically, adaptive control involves measuring elements and auto-tuning elements to modify the controller to cope with changes of plant or environment using online information. The most remarkable point is to introduce a new "mechanism" to achieve adaptive behavior, i.e., to maintain desirable performance of control system according to the changes of parameters of plant. On the other hand, even the most elementary feedback controllers can tolerate significant uncertainties. In fact it is also possible to realize the adaptive behavior of control systems using this property. Even though robust control can handle certain classes of parametric and dynamic uncertainties, it is not considered to be an adaptive system.

Adaptive control seems today a "natural" strategy to attack the stabilization and tracking of highly uncertain dynamical systems. In fact, since the late 50's, control theorists have struggled to develop adaptive control laws that guarantee closed loop stability in the presence of unmodeled dynamics and external disturbances. The Model Reference Adaptive Control (MRAC) architecture (Narendra and Annaswamy, 1989; Ioannou and Sun, 1995; Yoan, 1979) was first proposed for linear systems by Whitaker at the M.I.T. Instrumentation Laboratory in 1958 (Whitaker *et al.*, 1958; Osburn *et al.*, 1961). The MRAC approach was based on a heuristically constructed gradient or delta rule, also known as M.I.T. rule. Though this approach was originally a direct approach, in which parameters of controller is adjusted based on adaptive rule, the structure

of MRAC has changed to the combination of adaptive laws that can be guaranteed stability and controller which include parameters corresponding to the parameter of the controlled object to overcome the stability issue of the control system (Grayson, 1963; Parks, 1966). In 1970s, the theoretical framework of adaptive control called parametric adaptive control which include MRAC was established based on Lyapunov stability theories (Kalman and Bertram, 1959; Yoshizawa, 1966). An important feature of traditional adaptive control is its reliance on "certainly equivalence" controllers. This means that controller is first designed as if all the plant parameters were known. The controller parameters are determined as functions of the plant parameters. Given the true values of the plant parameters, the controller parameters are calculated by solving design equations for model-matching, optimality and so on. When the true plant parameters are unknown, the controller parameters are either estimated directly or computed by solving the same design equations with plant parameter estimates. The resulting controller is called a certainly equivalence controller. It is not obvious that certainly equivalence controller will work inside an adaptive feedback loop and achieve stabilization and tracking. Even when the plant is stable, bad parameter estimates may yield a destabilizing controller. The situation is more critical when the plant is unstable, because then the controller must achieve stabilization in addition to its tracking task. It is therefore significant that certainly equivalence controllers have been proven to be satisfactory for adaptive control of systems.

1.1.2 Reinforcement learning as a new approach of adaptive systems

In the field of modern artificial intelligence, "learning through interaction" is a key feature of new AI which is known as behavior-based approach (Brooks, 1991). Especially reinforcement learning (RL) (Sutton and Barto, 1998; Kaelbling *et al.*, 1996), which is one of the most active research areas in artificial intelligence, is a computational approach to learning whereby an agent tries to maximize the total amount of reward it receives when interacting with a complex, uncertain environment. From a historical perspective, Sutton and Barto identify two key trends that led to the reinforcement learning: the trial-and-error learning from psychology and the dynamic programming methods from mathematics (Sutton and Barto, 1998). The previously ignored areas lying between artificial intelligence and conventional engineering are now among the most active, including new fields such as neural networks, intelligent control, and our topic, reinforcement learning. In reinforcement learning we extend ideas from optimal control theory and stochastic approximation to address the broader and more ambitious goals of artificial intelligence.

While the main research effort of adaptive control theory was directed towards the theoretical stability assuming comparatively limited structure, in which parameter update rules are based on minimization of quadratic index function by gradient rule and so on, reinforcement learning deal with more complex problems which is far beyond adaptive control theory since it is only based on trial-and-error search and reward given by trying actions assuming the computational possibilities. Among the notable research of reinforcement learning is the recent success of reinforcement learning applications on difficult and diverse control problems (Anderson and Hong, 1994; Samuel, 1995; Tesauro, 1994; Crites and Barto, 1996; Singh and Bertsekas, 1997). Reinforcement learning is based on the idea that if an action is followed by a satisfactory state of affairs, or by an improvement in the state of affairs, then the tendency to produce that action is strengthened, i.e., reinforced. For a nonlinear, high-dimensional system, the conventional RL method necessitates a huge number of states, which makes learning very slow. Recently, many RL architecture to overcome this problem is proposed (Anderson and Hong, 1994; Haruno *et al.*, 2001; Morimoto and Doya, 1998, 2001).

1.1.3 Nonlinear adaptive control

Research in nonlinear control theory has been motivated by the inherently nonlinear characteristics of the dynamical systems we often try to control. Examples of such systems are Euler-Lagrange systems, limit and rate saturated control systems, dynamically coupled and interconnected systems, to list a few. If we add to the nonlinear nature of the dynamics, the fact that most systems are not well known and therefore not exactly modeled, it is clear that linear control techniques fall short in both their theoretical and practical aspects. Although linear systems are very well understood and controlled, linear control is not enough to guarantee stability and performance of nonlinear systems. A milestone in the extension of linear control techniques to nonlinear systems has been the development of nonlinear geometric control (Isidori, 1995; Marino and Tomei, 1996). Recent research involving differential geometric methods has rendered the design of controllers for a class of nonlinear systems somewhat systematic. This nonlinear control theory is based on coordinate transformations by which a class of nonlinear systems can be transformed into linear systems through feedback.

Adaptive control for nonlinear systems have developed together with the advent of feedback linearization (Sastry and Isidori, 1989; Krstic *et al.*, 1995). In the first stage of the research on adaptive nonlinear control, the strict conditions like matching conditions were supposed on the system. Later Kanellakopoulos *et al.* develop a systematic approach called backstepping to the design of adaptive controllers for linearizable systems. A backstepping control has become a very popular and powerful

tool in nonlinear adaptive control.However the applicable class of nonlinear systems is still small. The system must be affine with its input, and unknown parameters must appear linearly with regard to known functions (Kanellakopoulos *et al.*, 1991; Marino and Tomei, 1993).

1.1.4 Output regulation problem

The output regulation problem addresses design of a feedback controller to achieve asymptotic tracking and asymptotic disturbance rejection while maintaining closed-loop stability. This is a general mathematical formulation applicable to many control problems. The output regulation problem for linear systems was completely solved by the collective efforts of several researchers, including Davison, Francis, and Wonham, to name just a few (Saberi *et al.*, 2000; Francis and Wonham, 1975; Francis, 1977).

A simple extension of existing linear output regulation theory cannot dealt with output regulation for nonlinear systems. As a result of extensive work (Isidori and Byrnes, 1990), output regulation have now been successfully addressed to a certain degree. The idea to solve the nonlinear output regulation problem is similar to what has been used to solve the linear output regulation problem (Byrnes and Isidori, 1997; Huang, 2004). However, the control is much more difficult to find, since it is determined by a set of nonlinear partial differential and algebraic equations, which is a nonlinear analog of the regulator equations.

1.2 Motivation and Approach

There are two main objectives in this thesis: one is to apply adaptive controller motivated by reinforcement learning to real systems, i.e., to modify the method to match them and another one is to establish a new nonlinear adaptive control framework to ensure the stability based on the linear control theory, which is mainly a theoretical goal, but it is also aimed for practical application. The reason why the author does not focus on only one objective comes from the author's trails of researches. Reinforcement learning have been applied to many problems both in simulation and application and shown the possibilities to solve difficult and complex problem which existing control cannot deal with. However, as is often said, controller which is constructed by reinforcement learning method does not guarantee the stability of the closed-loop system. Moreover the stability problem of the learning process exists. When reinforcement learning controller is applied to real systems, the stability issue become more serious. Therefore, it is imperative that the controller be engineered with stable operation as a primary goal; performance is a secondary design consideration to be pursued after

stability is assured. For this point of view, it is necessary to restrict the objects and to examine the stability. If we restrict the system, it is easier to recognize possibilities, limits and expansibilities of the theory. For this reason, the author's interests have moved to control theoretical aspects of adaptive control of nonlinear systems.

To achieve the first objective, we employ multiple model-based approach to reinforcement learning. For real application, reinforcement learning controller may be still complicated enough from the control design viewpoint and further simplifications may be necessary. We assume that the plant is approximated by a linear model that is valid around a given operating point. Different operating points may lead to several different linear models that are used as plant models. To generate such plant model automatically, we employ the idea of softmax responsibility signal which is trained through experience.

The second objective is motivated by the fact that nonlinear systems is described by multiple local linear models, we can apply the linear control theory to each local linear model. This idea is similar to State Dependent Riccati Equations (SDRE). Consider the affine nonlinear systems:

$$\dot{x} = f(x) + g(x)u \tag{1.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control and $f(0) = 0$ and the goal will be to regulate the state x to the origin. A factorization is introduced such that it appears linear at any fixed state

$$\dot{x} = A(x)x + B(x)u \tag{1.2}$$

where $A(x)x = f(x)$ and $B(x) = g(x)$. Then the feedback

$$u = R(x)^{-1}B^T(x)X(x)x \tag{1.3}$$

is used where $X(x)$ is the solution of the SDRE:

$$A(x)^T X + XA(x) + Q(x) - XB(x)R^{-1}(x)B(x)^T X = 0. \tag{1.4}$$

and $Q(\cdot)$ and $R(\cdot)$ are the design parameters that satisfy pointwise positive definiteness condition $Q(x) > 0,\ R(x) > 0\ \forall x$. The greatest advantage of SDRE control is that physical intuition is always there and the designer can directly control the performance by tuning the weighting matrices $Q(x)$ and $R(x)$. Although SDRE approach has been demonstrated its effectiveness in many applications (Mracek and Cloutier, 1998; Erdem and Alleyne, 2004; Cloutier *et al.*, 1996), little is known about the stability properties associated with the SDRE controllers. In other to achieve the second objective of

establish the nonlinear adaptive control theory based on the linear control theory, we consider the system of the form:

$$\dot{x} = \sum_{i=1}^{r} \lambda_i(x)(A_i x + B_i u) \tag{1.5}$$

where $\lambda_i(x)$ are known functions of x such that

$$\sum_{i=1}^{r} \lambda_i(x) = 1, \lambda_i(x) \geq 0. \tag{1.6}$$

and employ an approach that combines SDRE and linear adaptive control. We also consider output regulation problem. In this case we also employ linear theory approach. We examine the solution of the regulator equations and design a controller to achieve adaptive output regulation. Then we examine the expansibilities to general nonlinear system.

1.3 Outline of the Thesis

This dissertation is organized as follows. Chapter 2 is concerned with adaptive control motivated by reinforcement learning method and its application to unmanned aerial vehicle. Chapters 3, 4 and 5 are concerned with adaptive control theory for linear and nonlinear systems.

Specifically, these chapters are organized as follows:

in Chapter 2, we focus on application of adaptive control to real system. Motivated by reinforcement learning approach, we derive the adaptive controller applicable to the practical control problems based on the optimal linear quadratic problem. We formulate learning algorithm based on Lyapunov design methods. We carried out numerical simulations to show that the dynamic relations between modules to achieve the task are learned and resulted controller is capable of nonlinear system. Moreover, it is applied to the autonomous flight control system of aero-robot and evaluated by a flight experiment. Experimental results revealed that softmax responsibility signal becomes binary signal, by which that the single model is used in each operating regime, and stability analysis holds even though the stability analysis doesn't hold for multiple modules with softmax responsibility signal.

In Chapter 3, we summarize the mathematical preliminaries used in Chapter 4 and 5. We review Riccati equation and regulator equation from linear theory and show the regularity properties of solution of the algebraic Riccati equation and the regulator equation with respect to system matrices. We also recall certain sufficient

conditions of asymptotic stability of a linear time-varying system and some estimates of the solution of the inhomogeneous system. Useful lemmas to analyze the stability of adaptive control system are enumerated.

In Chapter 4, we examine the stability properties of adaptive controller based on the linear quadratic problem. Then we propose a method of designing a full-information controller to achieve adaptive output regulation based on regulator equation for linear systems with parameter uncertainties.

In Chapter 5, we extend the method used in Chapter 4 to nonlinear systems. First we present a method to design controller for affine nonlinear systems with special form with known parameters. Secondly, We focus on the nonlinear system described by multiple local models. Then we describe extensibilities of the proposed method to general nonlinear systems.

In Chapter 6, we summarize the results of this thesis.

Chapter 2

Modular Learning and Its Application to Autonomous Aero-robot

2.1 Introduction

In 1990's, great progress was made in autonomous flight control of unmanned helicopter, for example, vision-based approach (Amidi *et al.*, 1998), linear control approach (Sato, 1999). These approaches were based on linear controller despite the fact that helicopter has strong nonlinearity inherent to the rotorcraft vehicle (Mettler, 2002). Then some approaches to deal with nonlinearity were proposed (Kim and Calise, 1997; Nakanishi *et al.*, 2003). Kim and Calise developed a direct adaptive control architecture using Neural Networks. In Nakanishi *et al.*, a robust controller using offline training of Neural Networks was applied to autonomous flight control of aero-robot. However most of these approaches were operated by single controller, regardless of whether the controller has the ability to adapt changes of the environment. To control the objects which vary considerably over the operating regime, it is impossible for a single controller to meet design specifications. It is generally difficult to design control system which meet various specifications even for linear systems. A nonlinear controller which is called gain scheduling controller is widely used to keep desirable performance over several operating regimes. Gain scheduling controllers consist of multiple controllers designed based on several partially linearized models. But designing gain scheduling controller is time-consuming and difficult because it depends on designers experience.

On the other hand, based on reinforcement learning framework, Doya *et al.* proposed a modular reinforcement learning architecture for nonlinear, non-stationary control tasks, called multiple model-based reinforcement learning (MMRL) (Doya *et al.*,

2002). The basic idea is to decompose a complex task into multiple domains in space and time based on the predictability of the environmental dynamics. This idea is similar to gain scheduling controller, however, how to decompose the system into several local models automatically determined. The resulting learning architecture, in which they extend the idea of a softmax combination of modules to the paradigm of reinforcement learning, decompose a nonlinear and/or nonstationary task through the competition and cooperation of multiple prediction models and reinforcement learning controllers. However, like most of RL algorithms, the learning agent uses the derivative of signal and is not appropriate to apply control systems of real application.

In this chapter, we propose a new learning architecture that is applicable to practical problems. The possibility of controller using multiple modules for the system which has nonlinear and time-varying dynamics is investigated. In the following sections, we first review the reinforcement learning method from control theoretical view and then introduce the basic MMRL architecture and point out its problem. We propose new learning algorithm based on Lyaounov design methods. We first test the performance of the proposed method by numerical simulation and also apply the method to the autonomous flight control system of an aero-robot.

2.2 Modular Learning

2.2.1 Reinforcement learning as adaptive optimal control

Reinforcement learning is based on the idea that if an action is followed by a satisfactory state of affairs, or by an improvement in the state of affairs, then the tendency to produce that action is strengthened, i.e., reinforced. Extending the original idea to allow action selections to depend on state information introduces aspects of feedback control. Control problems can be divided into two classed: 1) regulation and tracking problems, in which the objective is to follow reference trajectory, and 2) optimal control problems, in which the objective is to maximize a functional of the controlled system's behavior that is not necessarily defined in terms of a reference trajectory. For example, the majority of adaptive control methods including MRAC address regulation and tracking control because they have proven to be more tractable both analytically and computationally. However, adaptive methods for optimal control problems would be widely applicable if methods could be developed that were computationally feasible and that could be applied robustly to nonlinear systems. On the other hand, reinforcement learning can be viewed as an approach to the adaptive optimal control (Sutton *et al.*, 1992). Most formal results are for the control of Markov processes with unknown transition probabilities (Sutton, 1988; Watkins, 1989).

In recent years there have been many robotics and control applications that have used reinforcement learning. Here we will concentrate on the following example, although many other interesting ongoing robotics investigations are underway. Schaal and Atkeson constructed a two-armed robot, which learns to juggle a device known as a devil-stick. This is a complex nonlinear control task involving a six-dimensional state space and less than 200 msecs per control decision (Schaal and Atkeson, 1994). After about 40 initial attempts the robot learns to keep juggling for hundreds of hits. The juggling robot learned a world model from experience, which was generalized to unvisited states by a function approximation scheme known as locally weighted regression (Cleveland and Devlin, 1988; Moore and Atkeson, 1992). Between each trial, a form of dynamic programming specific to linear control policies and locally linear transitions was used to improve the policy. The form of dynamic programming is known as linear quadratic regulator design.

2.2.2 Multiple model-based reinforcement learning

Figure 2.1 shows the structure of the MMRL architecture. It is composed of n modules, each of which consists of a state prediction model and a reinforcement learning controller. The action output of reinforcement learning controllers as well as the learning rates of both the predictors and the controllers are weighted by the "responsibility signal", which is a Gaussian softmax function of errors in output of the prediction models. The advantage of this module selection mechanism is that the areas of specialization of the modules are determined in a bottom-up fashion based on the nature of the environment. In the following, we describe a specific algorithm of modular learning.

A nonlinear system is considered

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{2.1}$$

where $\boldsymbol{x}(t)$ is a state variable vector and $\boldsymbol{u}(t)$ is a control input vector. The goal of reinforcement learning is to find the policy that can maximize the cumulative future reward (Sutton and Barto, 1998). The task for the prediction model is to predict the state derivative $\dot{\boldsymbol{x}}(t)$. Output of the i-th module is denoted as $\hat{\dot{\boldsymbol{x}}}_i(t)$ and the prediction error as

$$E_i(t) = \| \dot{\boldsymbol{x}} - \hat{\dot{\boldsymbol{x}}}_i \|^2 . \tag{2.2}$$

The responsibility signal $\lambda_i(t)$ for the i-th module is then given by the softmax function of prediction errors

$$\lambda_i(t) = \frac{\exp(-E_i(t)/2\sigma^2)}{\sum_j \exp(-E_j(t)/2\sigma^2)}, \tag{2.3}$$

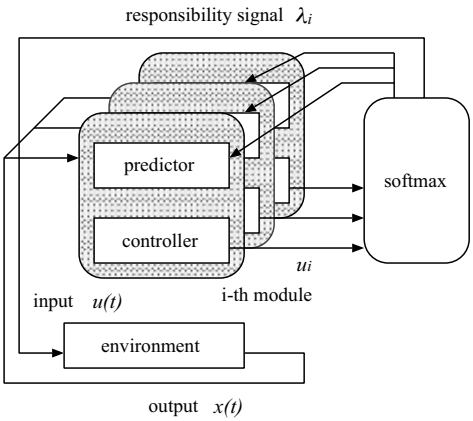


Figure 2.1: Multiple Model-based Reinforcement Learning (MMRL)

where σ is a parameter that controls the sharpness of module selection. Output of the i-th module

$$\hat{\dot{\boldsymbol{x}}}_i = f_i(\boldsymbol{x}(t), \boldsymbol{u}(t)), \tag{2.4}$$

is compared with the observed state dynamics $\dot{\boldsymbol{x}}$ to calculate the responsibility signal according to (2.3). Output of the prediction model of the i-th module is linearly weighted by the responsibility signal to make a prediction of the next state.

$$\hat{\dot{\boldsymbol{x}}} = \hat{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \sum_j \lambda_j f_j(\boldsymbol{x}(t), \boldsymbol{u}(t)). \tag{2.5}$$

Responsibility signal $\lambda_i(t)$ is also used for weighting the update of prediction models. Outputs of reinforcement learning controllers are also linearly weighted by $\lambda_i(t)$ to make the action output to the environment. The parameters of the local linear model of the i-th module A_i, B_i are updated by the weighted prediction errors $\lambda_i(t)(\hat{\dot{\boldsymbol{x}}}_i - \dot{\boldsymbol{x}}(t))$, respectively.

However, there are obstacles to apply MMRL to practical problems where it is difficult to measure derivatives of state variables $\dot{\boldsymbol{x}}$. This is because, like other RL techniques, gradient methods are used in MMRL. We must overcome this difficulties of MMRL for real applications, such as controlling an autonomous robot. In our approach, a Lyapunov function is used to formulate a learning rule so that we can guarantee the stability during operation of single module and in which we only use state variables not any derivatives so that we can use the proposed method for learning in a real environment.

2.2.3 Learning rule based on Lyapunov design method

In the following, we formulate a new learning rule based on Lyapunov design method. We assume that a locally linearized model of the controlled object (2.1) is described as

$$\dot{\boldsymbol{x}} = A(\boldsymbol{x}_k, \boldsymbol{u}_k)\boldsymbol{x} + B(\boldsymbol{x}_k, \boldsymbol{u}_k)\boldsymbol{u}, \tag{2.6}$$

where $\boldsymbol{x}_k, \boldsymbol{u}_k$ are a reference point and a prediction model

$$\hat{\dot{\boldsymbol{x}}} = \hat{A}\boldsymbol{x} + \hat{B}\boldsymbol{u} + \boldsymbol{v}, \tag{2.7}$$

for each module, where $\boldsymbol{v}$ is a pseudo control input which is introduced in order to guarantee stability. The error dynamics of the overall system can be expressed as

$$\dot{\boldsymbol{e}} = \tilde{A}\boldsymbol{x} + \tilde{B}\boldsymbol{u} - \boldsymbol{v}, \tag{2.8}$$

where $\tilde{A}$, $\tilde{B}$ are $\tilde{A} = A - \hat{A}$ and $\tilde{B} = B - \hat{B}$. To derive the stable learning rules, we consider a candidate of Lyapunov function

$$V = \frac{1}{2}\boldsymbol{e}^T P\boldsymbol{e} + \frac{1}{2}tr(\tilde{A}^T\Gamma_A\tilde{A} + \tilde{B}^T\Gamma_B\tilde{B}), \tag{2.9}$$

where P is a symmetric positive definite matrix, Γ_A, Γ_B are weight matrix and are positive defined, and trA denotes the trace of matrix A. The derivative of the Lyapunov function is given by

$$\dot{V} = \dot{\boldsymbol{e}}^T P\boldsymbol{e} + tr(\tilde{A}^T\Gamma_A\dot{\tilde{A}} + \tilde{B}^T\Gamma_B\dot{\tilde{B}}), \tag{2.10}$$

and we substitute (2.8) into (2.10) to obtain (2.11)

$$\begin{aligned}\dot{V} &= \boldsymbol{x}^T\tilde{A}^T P\boldsymbol{e} + \boldsymbol{u}^T\tilde{B}^T P\boldsymbol{e} \\ &\quad + tr(\tilde{A}^T\Gamma_A\dot{\tilde{A}} + \tilde{B}^T\Gamma_B\dot{\tilde{B}}) - \boldsymbol{v}^T P\boldsymbol{e} \\ &= tr[\tilde{A}^T(P\boldsymbol{e}\boldsymbol{x}^T + \Gamma_A\dot{\tilde{A}}) + \tilde{B}^T(P\boldsymbol{e}\boldsymbol{u}^T + \Gamma_B\dot{\tilde{B}})] \\ &\quad - \boldsymbol{v}^T P\boldsymbol{e}.\end{aligned} \tag{2.11}$$

Let $\dot{\hat{A}}, \dot{\hat{B}}$ be expressed as

$$\begin{aligned}\dot{\hat{A}} = -\dot{\tilde{A}} &= \Gamma_A^{-1}P\boldsymbol{e}\boldsymbol{x}^T, \\ \dot{\hat{B}} = -\dot{\tilde{B}} &= \Gamma_B^{-1}P\boldsymbol{e}\boldsymbol{u}^T, \\ \boldsymbol{v} &= \boldsymbol{e},\end{aligned} \tag{2.12}$$

then (2.11) will become

$$\dot{V} = -\boldsymbol{e}^T P\boldsymbol{e}. \tag{2.13}$$

From (2.13), it is guaranteed that V is a decreasing function while P is positive-definite matrix and the uniform ultimate boundedness of the error signal is achieved, then (2.8) and (2.12) is stable in the sense of Lyapunov stability. Actually, x_k, u_k in (2.6) dynamically change through learning process. If learning rules (2.12) are sufficiently fast, modules can track change of the locally linearized model (2.6). The proposed method can be applied to train linearly parameterized neural networks in order to use nonlinear prediction models. We can extend the proposed method to use modules in which a nonlinear prediction model is used. For examples, Tayler series model, linearly parameterized neural network model, and radial basis function model and so on are easily to use in our method. For simplicity, we focus on modules in which a linear prediction model is used.

2.3 Numerical Simulation

In this section, we show the result of numerical simulation. Consider the van der Pol equation

$$\ddot{x} = 0.2(1 - x^2)\dot{x} - x + u, \tag{2.14}$$

$$x(0) = 2 \ , \ \dot{x}(0) = 0, \tag{2.15}$$

and a desired trajectory is given by $2\sin(t)$. The origin is not stable but all trajectories asymptotically approach to a certain stable trajectory called a limit cycle. The local stability depends on whether the state is outside or inside of the cycle. We use two modules, each of which has a linear dynamic model and a quadratic reward model. E_i can be implemented by using a short-term average of the prediction error instead of the instantaneous prediction error to prevent chattering

$$\tau \dot{E}_i(t) = -E_i(t) + \| \boldsymbol{x} - \hat{\boldsymbol{x}}_i \|^2, \tag{2.16}$$

where we set $\tau = 0.01$, $\sigma = 0.5$ and initial prediction models of modules were set to have different property. Figure 2.2 shows transition of the state controlled by single module and Figures 2.3 and 2.4 show transition of the state and the responsibility signal controlled by two modules. As shown in Figure 2.2, about 60 seconds, the performance became good when it was controlled by single module. On the contrary, in Figure 2.3, after proper decomposition of the modules was learned around 25 seconds, the performance became good when it was controlled by multiple modules. This result shows that multiple modules learn faster than single module and our learning rules (2.12) work well in sufficient speed. Moreover the modules could keep the information about the controlled object. On the contrary, when one module is used in learning, the

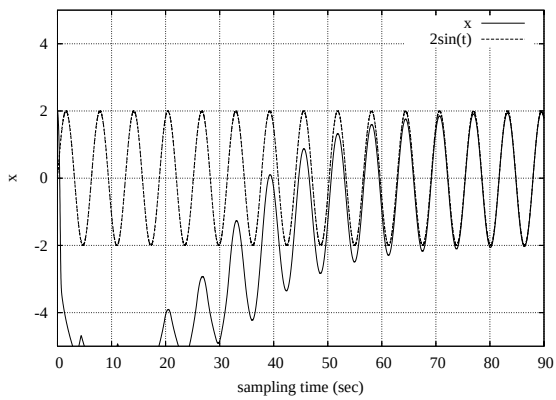


Figure 2.2: Transition of state controlled by single linear module

parameter used in the module kept varying, and knowledge about the controlled object didn't stored in the module, that is, the module kept forgetting collected information about the controlled object, which was necessary to keep control performance. Figure 2.5 show the relation between modules in the phase plane. After learning is finished, module A specialized in the internal region of the limit cycle and module B specialized in the external region of the limit cycle as shown in Figure 2.5. The result shows that multiple prediction models are successfully trained and specialized for different domains in the state space. In this example, modules which specialized in stable domain and unstable domain is formed through learning, so that they operate cooperatively, and the control task is efficiently achieved. It turns out that the learning rate of each module is successfully weighted by the responsibility signal. Consequently, modules that have different property and the dynamic relations between modules to achieve the task are learned. Thus, the relations between modules are not fixed but are decided in adaptive way depend on the behavior of each module. Therefore, the module configuration can be changed using prior and acquired information enough. Moreover, if it is possible to adapt to the change in the controlled object and the environment only by a part of module, information has already been acquired by learning is maintained in other modules.

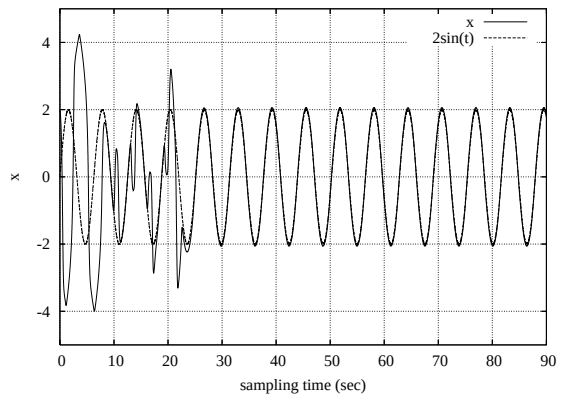


Figure 2.3: Transition of state controlled by two linear module

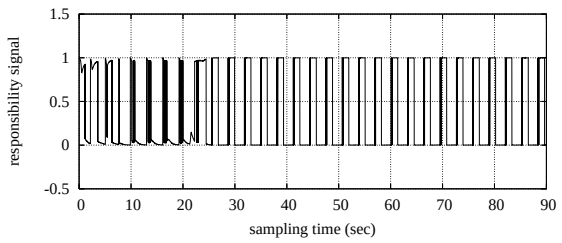


Figure 2.4: Transition of responsibility signal controlled by two linear module

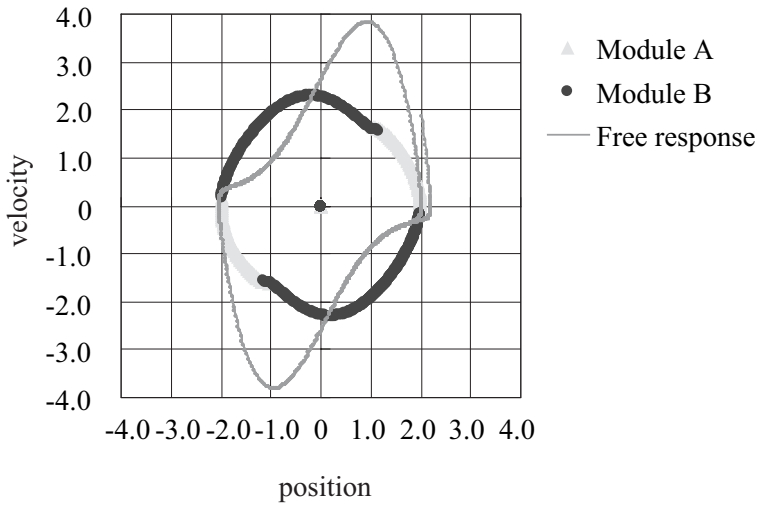


Figure 2.5: Relation between two modules in state space

2.4 Flight Experiment of Autonomous Aero-robot

2.4.1 Experimental settings

In this section, we show a result of a flight experiment of an autonomous aero-robot to confirm that the proposed method is applicable to practical problems. The autonomous aero-robot used in this experiment is based on an unmanned helicopter YAMAHA RMAX, and its photograph is shown in Figure 2.6. Table 2.1 shows the main specification of RMAX. RMAX is the best unmanned commercial aerial spray system which is widely used for agricultural purpose in Japan, and it is operated manually by use of a remote controller. Equipping various sensors for navigation, such as GPS (Global Positioning System) and a computer system for flight control, the state of the aero-robot can be measured with enough accuracy, and it can fly without manual control by developing an autonomous flight control system. Especially it can fly out of the operator's sight, therefore it can be applied for various purposes (Nakanishi *et al.*, 2003). Rotorcraft represent a challenging control problem with high-dimensional, complex, non-linear, dynamics, and are regarded as significantly more difficult to control than fixed-wing aircraft. Moreover, the atmospheric change, such as wind direction, the gust and so on, affects the behavior of the small sized helicopter significantly, so that adaptation to the environmental change is important property of the aero-robot. Designing an altitude controller is shown in this paper. In our previous report (Bando *et al.*, 2004), there existed steady-state error in the experiment. It becomes clear that

Figure 2.6: Autonomous Aero-robot

Table 2.1: Principal Specifications of YAMAHA RMAX

Overall Length	3,630(mm)
Main Rotor Diameter	3,115(mm)
Overall Height	1,080(mm)
Overall Width	720(mm)
Empty Weight	64(kg)
Payload	30(kg)
Flight Time	60(min)

the input offset, that is collective trim difference, which is not modeled in a simulation existed in an actual environment and it suggest that modules may have not learned in adequate way. In order to cancel steady-state error, we include the term of uncertainty into our prediction model to take uncertainties into account. We assume that

- The thrust coefficient of the main rotor doesn't depend on altitude and vertical velocity.
- The drag works in the movement of the vertical direction, and the magnitude is proportional to the vertical velocity.

and a linear prediction model of the aero-robot is described as

$$\dot{\hat{\boldsymbol{x}}} = \hat{A}\boldsymbol{x} + \hat{B}\boldsymbol{u} + \boldsymbol{v} + \boldsymbol{g}, \tag{2.17}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & \hat{a} \end{bmatrix}, \ \hat{B} = \begin{bmatrix} 0 \\ \tilde{b} \end{bmatrix}, \ \hat{\boldsymbol{g}} = \begin{bmatrix} 0 \\ \hat{g} \end{bmatrix},$$

for each module, where x_1, x_2, u, $\boldsymbol{v}$ is altitude, velocity, collective input and a pseudo control input respectively. Parameter $\hat{a}$ corresponds to the drag coefficient, $\hat{b}$ corresponds to the thrust coefficient and $\hat{g}$ corresponds to the uncertainties. In the flight experiment, desired altitude and vertical velocity are set to be varying and unit of desired altitude and velocity are m and m/s, respectively. Other states of the aero-robot is controlled to keep the initial value in this experiment. Initial parameters were obtained in advance by use of a flight simulator, which can emulate flight state of the aero-robot with sufficient accuracy. Reinforcement learning controller of each module

$$u = -Kx - \hat{g}, \tag{2.18}$$

is used. K is the optimal gain and computed continuously by solving Riccati equation so that it changes adaptively using prediction model. In our experiment, a performance index J described as

$$J = \int_{t=0}^{\infty} x_1^2(t) + 1.5u^2(t)dt, \tag{2.19}$$

was used.

2.4.2 Results and discussion

We first tested the performance of our proposed method with 3 modules. Figures 2.7 and 2.8 show transition of the state. As shown in these figures the aero-robot follows the desired trajectory and learning rules (2.12) works. Figure 2.9 summarize estimated value of parameters at flight experiment. This parameter corresponds to the thrust coefficient. This result shows the thrust coefficient of the module that takes charge of the hovering is bigger than that of the module when accelerating. It is also proved that when 3 modules are used in learning, the change in the parameter is small, and each module can maintain the information about the controlled object. To confirm how the number of modules affects on relation between modules, we also tested with 2 modules. Figures 2.10 and 2.11 shows the relation between modules. The control task was decomposed into negative acceleration and the other when two modules were used. When three module were used, the other task except for negative acceleration was decomposed into positive acceleration and hovering. The difference between 2 modules and 3 modules and of specialization is shown in Tables 2.2 and 2.3. The altitude did not influence the differentiation of the module according to the result. Because of small changes of altitude, it doesn't cause environmental changes such as air density and so on, and affect the dynamic characteristics little. If the environment and task become complicated or time-varying, it is impossible to determine the optimal number of modules in advance which are required to control the aero-robot with enough performance. However experiments show that modules can learn suitable differentiation and change their relation according to the situation by use of the proposed learning rule even if sufficient modules are not used.

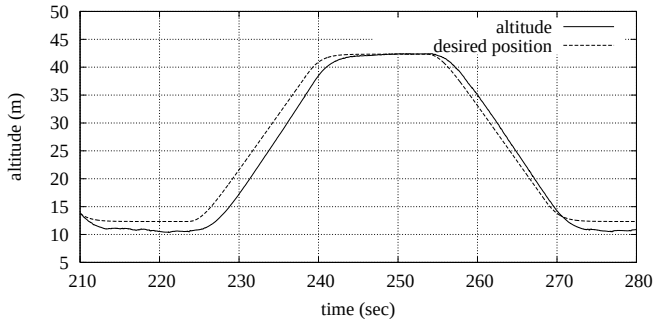


Figure 2.7: Response to desired position $z_d(t)$ in altitude control

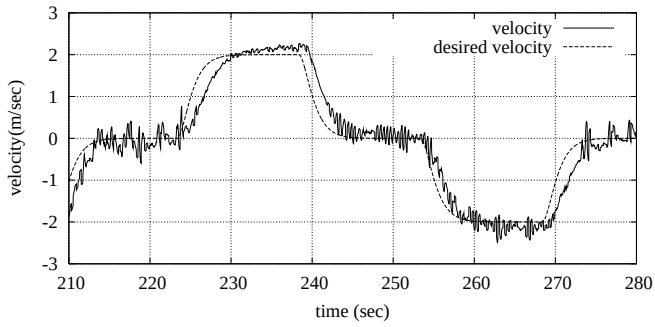


Figure 2.8: Response to desired velocity $v_d(t)$ in altitude control

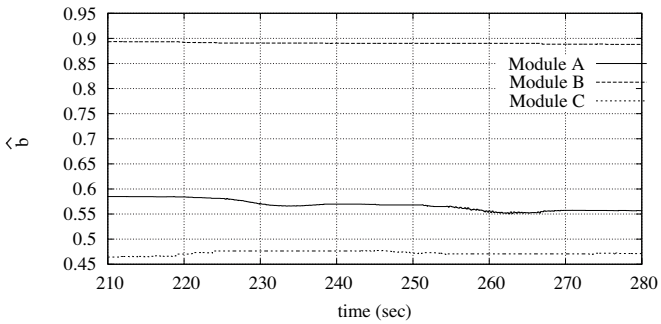


Figure 2.9: Estimated value of parameters at flighf experiment

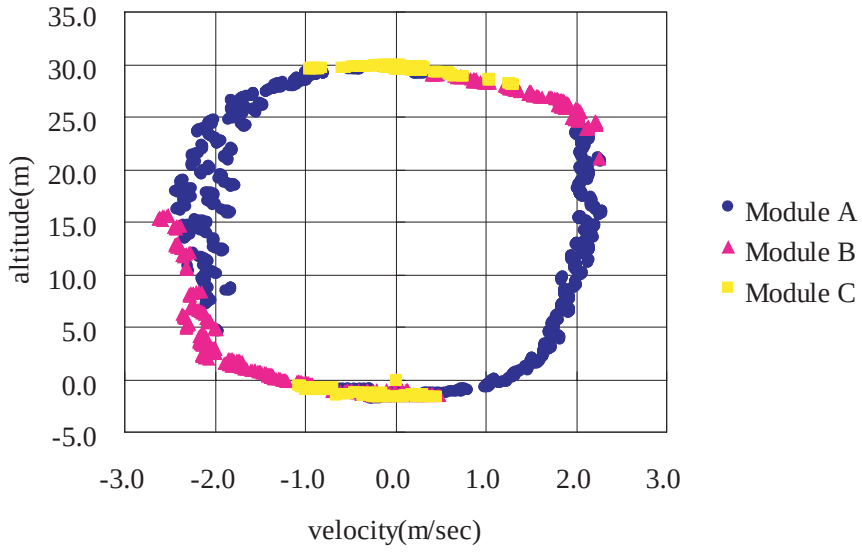


Figure 2.10: Relation between 3 modules

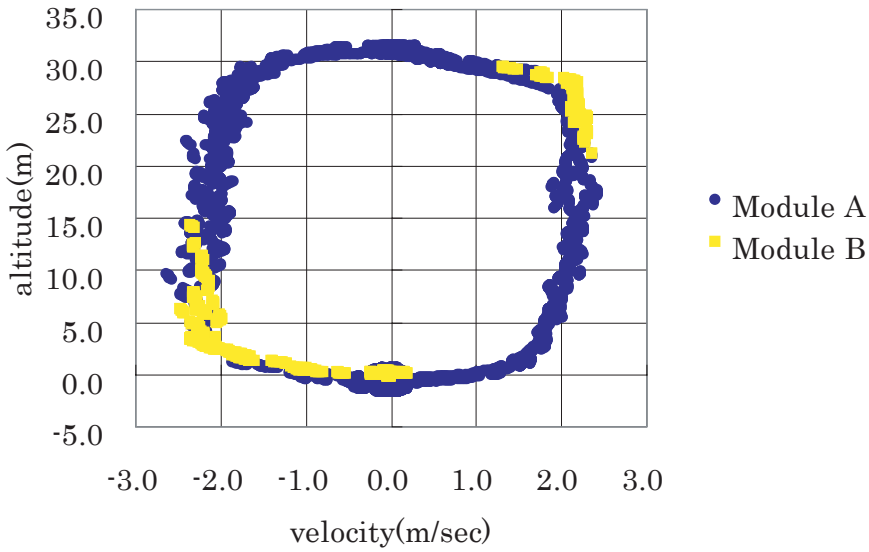


Figure 2.11: Relation between 2 modules

Table 2.2: specialization in 2 Modules

Module A	negative acceleration
Module B	the other

Table 2.3: specialization in 3 modules

Module A	negative acceleration
Module B	positive acceleration
Module C	hovering (the other)

2.5 Conclusion

In this chapter, we propose new learning algorithm based on Lyapunov design methods applicable in practical problems. We tested the performance of the proposed method both in simulations and experiments. It is shown that multiple modules are successfully trained and specialized for different domains in the state space in a cooperative way. Furthermore, the control system which consists of several online modules is applied to the autonomous flight control system of aero-robot, and we evaluated our method by a flight experiment. These results show that the proposed method can be applied to control various autonomous robots. Actually, the stability analysis doesn't hold for multiple modules with softmax responsibility signal. However, softmax responsibility signal becomes binary signal in our experiments and stability analysis holds.

Chapter 3

Mathematical Preliminaries

In this Chapter, the focus is on some elementary results and principles that are used in the design and analysis of adaptive systems.

3.1 Linear Control Theory

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \end{aligned} \tag{3.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, $z \in \mathbb{R}^q$ is the output to be regulated, and $w \in \mathbb{R}^s$ denotes bounded disturbances or reference signals generated by an anti-stable exosystem

$$\dot{w} = Sw. \tag{3.2}$$

All matrices A, B_1, B_2, C_1 and D_{11} are constant and of appropriate dimensions.

3.1.1 The algebraic Riccati equation

Let $C \in \mathbb{R}^{p \times n}$, and assume that (C, A) is detectable. If (A, B_2) is stabilizable, there exists a unique nonnegative stabilizing solution X of the algebraic Riccati equation

$$A^T X + XA + C^T C - XB_2 B_2^T X = 0. \tag{3.3}$$

Now we show that the solution X is a smooth function of the parameters of the matrices A, B_2 and C.

Lemma 3.1.1. *Let (A, B_2, C)=(A_0, B_{20}, C_0) be a stabilizable and detectable triple. Then the following holds:*

1. *There exists a neighborhood N_1 of $(A_0,\ B_0,\ C_0)$ such that each $(A,\ B,\ C) \in N_1$ is stabilizable and detectable. Moreover for each $(A,\ B,\ C) \in N_1$, there exists a nonnegative stabilizing solution $X = X(A, B, C)$ of (3.3).*

2. *There exists a neighborhood N_2 of $(A_0,\ B_{20},\ C_0)$ such that for each $(A,\ B_2,\ C) \in N_2$, there exists a unique solution $X = X(A, B_2, C)$ of (3.3) which is continuous and continuously differentiable in each element of the matrices A, B_2, and C.*

Proof. 1. Let $M \in \mathbb{R}^{n\times m}$ with $n \leq m$. Since the rank of a matrix is lower semi-continuous (Beltrami, 1970) the set $\{M : rank\ M \leq n-1\}$ is closed. First we show by contradiction that the stabilizable pair $(A,\ B)$ forms an open set. Suppose each neighborhood of $(A_0,\ B_0)$ contains a pair $(A,\ B)$ which is not stabilizable. Then we can construct a sequence $(A_k,\ B_k)$ which is not stabilizable. Moreover, there exists an eigenvalue λ_k of A_k such that $Re\ \lambda_k \geq 0$ and $rank[\lambda_k I - A_k \quad B_k] \leq n-1$. Let p_k be the normalized eigenvector corresponding to λ_k so that $A_k p_k = \lambda_k p_k$. Taking a subsequence if necessary, we can find limits λ_0 and p_0 such that $Re\ \lambda_0 \geq 0$, $\|p_0\| = 1$, and $A_0 p_0 = \lambda_0 p_0$. By the lower semicontinuity of the rank we obtain $rank[\lambda_0 I - A_0 \quad B_0] \leq n-1$. This is a contradiction to the stabilizability of $(A_0,\ B_0)$. Hence if $(A_0,\ B_0)$ is stabilizable, then there exists a neighborhood M_1 of $(A_0,\ B_0)$ such that each $(A,\ B) \in M_1$ is stabilizable. Similarly, if $(C_0,\ A_0)$ is detectable, there exists a neighborhood M_2 of $(C_0,\ A_0)$ such that each $(C,\ A) \in M_2$ is detectable. Combining these observations we obtain the first assertion. The second part is well-known.

2. We set

$$\boldsymbol{\theta} = [vec(A)^T \quad vec(B)^T], \quad \boldsymbol{x} = vec(X),$$

where $vec(\cdot)$ denotes the vector formed by the column vectors of the matrix, taking them from left and placing them from top. The Riccati equation (3.3) is equivalently transformed to

$$F(\boldsymbol{\theta}, \boldsymbol{x}) = vec(A^T X + XA + C^T C - XBB^T X) = 0. \tag{3.4}$$

We shall apply the implicit function theorem (Khalil, 2002) to (3.4), and show that X is a continuously differentiable function of A and B. Let X_0 be a nonnegative stabilizing solution of (3.3) associated with $(A_0,\ B_0,\ C_0)$ and $\boldsymbol{\theta}_0$, $\boldsymbol{x}_0$ the corresponding vectors. Then $F(\boldsymbol{\theta}_0, \boldsymbol{x}_0) = 0$. Its Jacobian at $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ and $\boldsymbol{x} = \boldsymbol{x}_0$ is given by

$$\begin{aligned} J &= \frac{\partial F(\boldsymbol{\theta}, \boldsymbol{x})}{\partial \boldsymbol{x}^T}\Big|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0, \boldsymbol{x}=\boldsymbol{x}_0} \\ &= I_n \otimes (A_0 - B_0 B_0^T X_0)^T + (A_0 - B_0 B_0^T X_0)^T \otimes I_n \end{aligned}$$

where $\otimes$ denotes the Kronecker product of matrices (Brewer, 1978). For example if A is an m-by-n matrix and B is a p-by-q matrix, then the Kronecker product $A \otimes B$ is the mp-by-nq block matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

By the assumption, the matrix $(A_0 - B_0 B_0^T X_0)$ is stable. Hence $\det J \neq 0$, i.e., J is nonsigular at $(\boldsymbol{\theta}_0, \boldsymbol{x}_0)$. Note that $F(\boldsymbol{\theta}, \boldsymbol{x})$ is a continuous function of $(\boldsymbol{\theta}, \boldsymbol{x})$ and C^1. Then, by the implicit function theorem, there exists a neighborhood V of $\boldsymbol{\theta}_0$ in which a unique continuous solution $\boldsymbol{x} = f(\boldsymbol{\theta})$ exists and $\dfrac{\partial f}{\partial \boldsymbol{\theta}}$ is continuous.

□

3.1.2 The regulator equation

The output regulation problem of (3.1) and (3.2) with full information is to find a stabilizing feedback control

$$u = Fx + Gw$$

such that $z(t) \to 0$ as $t \to \infty$ for any $x(0) = x_0$ and $w(0) = w_0$.

It is known that the output regulation problem is solvable if and only if the regulator equation

$$\begin{aligned} &A\Pi - \Pi S + B_1 + B_2\Gamma = 0, \\ &C_1\Pi + D_{11} + D_{12}\Gamma = 0, \end{aligned} \tag{3.5}$$

has a solution (Π, Γ) (Saberi *et al.*, 2000). In this case, $u = Fx + (\Gamma - F\Pi)w$, for any stabilizing F, is a solution.

We recall that the regulator equation (3.5) is solvable if and only if the matrix

$$A_1(\lambda) = \begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full row-rank for each eigenvalue λ of S (Saberi *et al.*, 2000). For fixed C_1, D_{11} and D_{12} we show that the solution of the regulator equation (3.5) is a continuously differentiable function of the elements of A, B_1 and B_2. Suppose

$$rank \begin{bmatrix} A_0 - \lambda I & B_{20} \\ C_1 & D_{12} \end{bmatrix} = n + q \qquad \forall \lambda \in \sigma(S).$$

Then there exists a neighborhood M_1 of $(A_0,\, B_0)$ such that for any $(A,\, B_2) \in M_1$,

$$rank \begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + q \qquad \forall \lambda \in \sigma(S). \tag{3.6}$$

Lemma 3.1.2. *Suppose A_1 is a square matrix, i.e., the input signal u and the output signal z have the same dimension. Then the solution $(\Pi,\, \Gamma)$ of the regulator equation (3.5) is continuously differentiable function of B_1 and $(A,\, B_2)$ in M_1.*

Proof. In M_1, the solvability of the regulator equation (4.3) is guaranteed. We set

$$\boldsymbol{\theta} = [vec(A)^T \; vec(B_2)^T],\; \boldsymbol{x} = [vec(\Pi)^T \; vec(\Gamma)^T].$$

We can rewrite (4.3) as

$$F_1(\boldsymbol{\theta}, \boldsymbol{x}) = A_c \boldsymbol{x} - \boldsymbol{b} = 0, \tag{3.7}$$

where

$$A_c = \begin{bmatrix} (I_s \otimes A) - (S^T \otimes I_n) & -(I_s \otimes B_2) \\ (I_s \otimes C_1) & vecD_{12} \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -vecB_1 \\ -vecD_{11} \end{bmatrix}.$$

From (3.6), it follows that $A_c \in \mathbb{R}^{(n+q)s \times (n+q)s}$ has full row-rank Zhou and Duan (2006). Hence $\boldsymbol{x}$ is a continuously differentiable function of $\boldsymbol{\theta}$. □

If A_1 is a wide matrix, then the solution of the regulator equation is not unique. The solutions consist of two parts, those corresponding to a nonsingular submatrix of order $(n+q)s$ and those which can be set arbitrary. But the nonsingular submatrix may change when $(A,\, B_2)$ changes and the continuity of the solution is not always ensured. However, we can show the regularity of the solution for systems of special structure. In fact, we assume that $(A,\, B_2)$ is in the controllable canonical form and that the matrix C_1 picks up first elements of all subsystems so that

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B^1 \\ B^2 \\ \vdots \\ B^m \end{bmatrix}, C_1 = \begin{bmatrix} C^1 & C^2 & \cdots & C^m \end{bmatrix}, \tag{3.8}$$

with

$$A_{ii} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \cdots & * \end{bmatrix} \in \mathbb{R}^{n_i \times n_i}, \tag{3.9}$$

$$A_{ij} = \begin{bmatrix} & & 0 & & \\ & & & & \\ * & * & * & \cdots & * \end{bmatrix} \in \mathbb{R}^{n_i \times n_j}, \tag{3.10}$$

$$B^i = \begin{bmatrix} & & 0 & & & & \\ & & & & & & \\ 0 & 0 & \cdots & 1 & * & \cdots & * \end{bmatrix} \in \mathbb{R}^{n_i \times m}, \tag{3.11}$$

$$C^i = \begin{bmatrix} 0 & \\ \vdots & \\ 1 & 0 \\ \vdots & \\ 0 & \end{bmatrix} \in \mathbb{R}^{q \times n_i}, \tag{3.12}$$

where the i-th elements of the last row of B^i and of the first column of C^i are one, and the symbol $*$ stand for possible nonzero elements and are called the parameters of A and B_2. We assume $q \leq m$.

Lemma 3.1.3. *Suppose $D_{12} = 0$ and that $(A,\ B_2,\ C_1)$ has the structure (3.8)-(3.12). Then there exists a solution $(\Pi,\ \Gamma)$ of the regulator equation (3.5) which is continuously differentiable function of the parameters of A and B_2. Moreover Π is independent of the parameters of A and B_2.*

Proof. We set

$$\boldsymbol{\theta} = [vec(A)^T\ vec(B_2)^T],\ \boldsymbol{x} = [vec(\Pi)^T\ vec(\Gamma)^T].$$

We can rewrite (4.3) as

$$F_1(\boldsymbol{\theta}, \boldsymbol{x}) = A_c \boldsymbol{x} - \boldsymbol{b} = 0, \tag{3.13}$$

where

$$A_c = \begin{bmatrix} (I_s \otimes A) - (S^T \otimes I_n) & -(I_s \otimes B_2) \\ (I_s \otimes C_1) & (I_s \otimes vecD_{12}) \end{bmatrix},\ \ \boldsymbol{b} = \begin{bmatrix} -vecB_1 \\ -vecD_{11} \end{bmatrix}.$$

First we show the solvability of the regulator equation (4.3). We recall that if A_c has full row rank, the solution of (3.13) exists. Let Λ be the diagonal matrix consisting of eigenvalues of S^T and $S^T = P\Lambda P^{-1}$ where P is the eigenvector matrix of S^T. Using

the property of the Kronecker product $V_1V_2 \otimes W_1W_2 = (V_1 \otimes W_1)(V_2 \otimes W_2)$ (Brewer, 1978), we obtain (Zhou and Duan, 2006)

$$\begin{aligned} A_c &= (P \otimes I_{n+m})(I_s \otimes \begin{bmatrix} A & B_2 \\ C_1 & D_{12} \end{bmatrix} - \Lambda \otimes \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix})(P^{-1} \otimes I_{n+m}) \\ &= (P \otimes I_{n+m})\tilde{A}_c(P^{-1} \otimes I_{n+m}) \end{aligned}$$

Hence the regulator equation (3.13) is reduced to the equation of the form

$$\tilde{A}_c\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{b}} = 0,$$

where $\tilde{\boldsymbol{x}} = (P^{-1} \otimes I_{n+m})\boldsymbol{x}$. By direct calculation we obtain

$$\tilde{A}_c = \begin{bmatrix} A_1(\lambda_1) & & & 0 \\ & A_1(\lambda_2) & & \\ & & \ddots & \\ 0 & & & A_1(\lambda_s) \end{bmatrix}, \tag{3.14}$$

where $\lambda_i \in \sigma(S)$.For simplicity we assume $q = 1$ and $m = 2$. Then $A_1(\lambda_1)$ becomes

$$A_1(\lambda_1) = \left[\begin{array}{cccc|cccc|cc} -\lambda_1 & 1 & & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & & \vdots & & & \vdots & \vdots & \vdots \\ 0 & \cdots & -\lambda_1 & 1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ * & \cdots & \cdots & * & * & \cdots & \cdots & * & 1 & * \\ \hline 0 & \cdots & \cdots & 0 & -\lambda_1 & 1 & & 0 & 0 & 0 \\ \vdots & & & \vdots & \vdots & \ddots & \ddots & & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & -\lambda_1 & 1 & 0 & 0 \\ * & \cdots & \cdots & * & * & \cdots & \cdots & * & 0 & 1 \\ \hline 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \end{array}\right].$$

$\tilde{A}_c$ has full row rank. In fact, we can see $rank\ A_1(\lambda_1) = n+1$ and $rank\ \tilde{A}_c = (n+1)s$ since $\tilde{A}_c$ has a block structure (3.14). Now A_c is of full row rank and the solvability of the regulator equation (4.3) is guaranteed.

Next we show that there exists Π which does not depend on the parameters of A and B_2. Suppose A_c is a square matrix. For simplicity we assume $q = 2$, $m = 2$ and $s = 2$. Then A_c becomes

$$A_c = \begin{bmatrix} (I_s \otimes A) - (S^T \otimes I_n) & -(I_s \otimes B_2) \\ (I_s \otimes C_1) & (I_s \otimes D_{12}) \end{bmatrix} = \begin{bmatrix} A - s_{11}I_n & -s_{12}I_n & -B_2 & 0 \\ s_{21}I_n & A - s_{22}I_n & 0 & -B_2 \\ C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \end{bmatrix}$$

$$
= \left[\begin{array}{ccc|ccc|cc|cc}
-s_{11} & 1 & 0 & -s_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -s_{11} & 1 & 0 & -s_{12} & 0 & -1 & * & 0 & 0 \\
* & * & * & 0 & 0 & -s_{12} & 0 & -1 & 0 & 0 \\
\hline
-s_{21} & 0 & 0 & -s_{22} & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -s_{21} & 0 & 0 & -s_{22} & 1 & 0 & 0 & -1 & * \\
0 & 0 & -s_{21} & * & * & * & 0 & 0 & 0 & -1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

where $S = (s_{ij})$. Note that nth- and $2n$th rows contain the parameters of A and B_2. Without nth- and $2n$th rows, we have

$$\bar{A}_c vec(\Pi) - vecB_1 = 0. \tag{3.15}$$

Since $\bar{A}_c$ is nonsingular, we can uniquely determine Π. Then Π is independent of the parameters of A and B_2. If $q < m$, A_c become a wide matrix. In this case there are $m - q$ parameters of Π which are free. Choosing these elements independent of the parameters of A and B_2, the rest of the solutions are uniquely determined by the same discussion as in the square case, and Π becomes independent of the parameters of A and B_2. □

3.2 Stability of Linear Time-Varying System

Consider the linear system

$$\dot{x} = A(t)x, \tag{3.16}$$

where $x \in \mathbb{R}^n$ and the elements of $A(t)$ are piecewise continuous, bounded and differentiable for all $t \geq t_0 \geq 0$. We recall sufficient conditions for asymptotic stability (Ioannou and Sun, 1995).

Lemma 3.2.1. *Assume the condition*

$$Re\{\lambda_i(A(t))\} \leq -\sigma_s \quad \forall t \geq 0$$

for any eigenvalue $\lambda_i(A(t))$ of $A(t)$, where σ_s is a positive constant. If $\|\dot{A}\| \in \mathcal{L}_2[0, \infty)$, then the equilibrium state $x_e = 0$ of (3.16) is uniformly asymptotically stable in the large.

By Theorem 4.11 in (Khalil, 2002), under the conditions of Lemma 3.2.1 the state transition matrix $\Phi(t,\tau)$ satisfies

$$\|\Phi(t,\tau)\| \leq \lambda_0 e^{-\alpha_0(t-\tau)} \tag{3.17}$$

for some λ_0, $\alpha_0 > 0$.

Let x be locally square integrable and define

$$\|x_t\|_{2\delta} \triangleq \left(\int_0^t e^{-\delta(t-\tau)} x^T(\tau)x(\tau)d\tau)\right)^{\frac{1}{2}},$$

and $\delta \geq 0$ is a constant. This is called the $\mathcal{L}_{2\delta}$-norm (Ioannou and Sun, 1995). Consider the system

$$\dot{x} = A(t)x + B(t)u, \quad x(0) = x_0 \tag{3.18}$$

where $u \in \mathbb{R}^m$ and the elements of $B(t)$ are bounded continuous functions. Suppose the inequality (3.17) holds. Then for each $\delta \in [0, 2\alpha_0)$ and a locally square intergrable function u, we have the following useful estimates (Ioannou and Sun, 1995).

Lemma 3.2.2.

(i) $|x(t)| \leq \dfrac{c_0\lambda_0}{\sqrt{2\alpha_0 - \delta}}\|u_t\|_{2_\delta} + \epsilon_t$

(ii) $\|x_t\|_{2\delta} \leq \dfrac{c_0\lambda_0}{(\delta_1 - \delta)(\sqrt{2\alpha_0 - \delta_1})}\|u_t\|_{2_\delta} + \epsilon_t$

where $c_0 = \sup_t \|B(t)\|$, $\epsilon_t = \lambda_0 e^{-\alpha_0 t}|x_0|$ *and* δ_1 *satisfies* $\delta < \delta_1 < 2\alpha_0$.

A key lemma for analysis of adaptive control schemes is the following.

Lemma 3.2.3 (Barbalat's lemma). *If* $\lim_{t\to\infty}\int_0^t \phi(\tau)d\tau$ *exists and is finite, and* $f(t)$ *is a uniformly continuous function, then* $\lim_{t\to\infty} f(t) = 0$.

Lemma 3.2.4 (Bellman-Gronwall Inequality). *Let* $\lambda : [a,b] \to \mathbb{R}$ *be continuous and* $\mu : [a,b] \to \mathbb{R}$ *be continuous and nonnegative. If a continuous function* $y : [a,b] \to \mathbb{R}$ *satisfies*

$$y(t) \leq \lambda(t) + \int_a^t \mu(s)y(s)ds$$

for $a \leq t \leq b$*, then on the same integral*

$$y(t) \leq \lambda(t) + \int_a^t \lambda(s)\mu(s)\Big[\int_s^t \mu(\tau)d\tau\Big]ds$$

Chapter 4

Adaptive Output Regulation for Linear Systems

4.1 Introduction

The output regulation problem addresses design of a feedback controller to achieve asymptotic tracking and asymptotic disturbance rejection while maintaining closed-loop stability. This is a general mathematical formulation applicable to many control problems. Consider a linear plant of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control, and $z \in \mathbb{R}^q$ is the output to be regulated. The signal $w \in \mathbb{R}^s$ denotes disturbances or reference signals generated by an anti-stable exosystem

$$\dot{w} = Sw. \tag{4.2}$$

The output regulation problem is to find a control law such that $z(t)$ converges to zero as $t \to \infty$ for any initial conditions of the plant and the exosystem. When the state and the signal are available, it is called the output regulation with full information. In this case a control of the form $u = Fx + Gw$ is sought. The output regulation problem for linear systems was completely solved by the collective efforts of several researchers, including Davison, Francis, and Wonham, to name just a few (Saberi *et al.*, 2000; Francis and Wonham, 1975; Francis, 1977). The following results are known.

Theorem 4.1.1 (Saberi *et al.* (2000)). *Suppose (A, B_2) is stabilizable. Then the output regulation problem with full information is solvable if and only if there exist two*

matrices Π *and* Γ *which satisfy the regulator equation*

$$\begin{aligned} A\Pi - \Pi S + B_1 + B_2\Gamma &= 0, \\ C_1\Pi + D_{11} + D_{12}\Gamma &= 0. \end{aligned} \tag{4.3}$$

Under this condition admissible controllers are given by

$$u(t) = Fx(t) + (\Gamma - F\Pi)w(t),$$

where F *is any matrix such that* $A + BF$ *is exponentially stable.*

However if the system parameters contain uncertainties, the above controller is no more feasible. Adaptive control is one of the ideas which can deal with uncertainties. Specifically, adaptive control involves measuring elements and auto-tuning elements to modify the controller to cope with changes of plant or environment using online information. On the other hand, even the most elementary feedback controllers can tolerate significant uncertainties. In (Huang, 2004) it is shown robust output regulation can handle parametric uncertainties. In this chapter, we consider the adaptive output regulation problem for linear time-invariant systems with unknown parameters.

In 1970s, the theoretical framework of adaptive control called parametric adaptive control which include MRAC was established based on Lyapunov stability theories (Kalman and Bertram, 1959; Yoshizawa, 1966). The main contribution was certainly equivalence principle by which traditional adaptive controllers are designed regarding estimated parameters as true parameters. The greatest advantage of the "certainly equivalence" controller is that we can use linear control theory for known systems. Given the true values of the plant parameters, the controller parameters are calculated by solving design equations for model-matching, optimality and so on. When the true plant parameters are unknown, the controller parameters are either estimated directly or computed by solving the same design equations with plant parameter estimates. The resulting controller is called a certainly equivalence controller. Even when the plant is stable, bad parameter estimates may yield a destabilizing controller. The situation is more critical when the plant is unstable, because then the controller must achieve stabilization in addition to its tracking task. The stability of the system can be analyzed by considering estimated parameter as a time-varying parameter. Hence the stability problem of the adaptive system can be stated in terms of the stability of a linear time-varying system depending on parameters and the stability property in terms of parameters are discussed in (Narendra and Annaswamy, 1989). In (Ioannou and Sun, 1995) stability results of linear time-varying systems based on a $\mathcal{L}_{2\delta}$ norm that are particularly useful in the analysis of adaptive systems are shown. However It is not obvious that certainly equivalence controller based on the algebraic Riccati

equation and the regulator equation will work inside an adaptive feedback loop and achieve stabilization and output regulation. When we design certainly equivalence controllers based on the algebraic Riccati equation and the regulator equation from linear theory, the analysis of adaptive feedback loop become more complex. Moreover it is not obvious that under what conditions the regularity properties of solution of the algebraic Riccati equation, which is often used in real applications, with respect to system matrices hold. The solutions of the regulator equation become more complex. It is therefore significant that such controllers are proven to be satisfactory for adaptive control of systems.

This chapter is organized as follows. Section 1 is introduction. Section 2 is concerned with adaptive stabilization and adaptive output regulation. We introduce an estimator of the system and adaptive laws which decrease the estimation error of the state and parameters. We then introduce a feedback law based on the algebraic Riccati equation. Based on this result, we consider the output regulation problem with full information and give sufficient conditions for solvability in terms of the regulator equation. In Section 3 to relax the condition on the solution of the Riccati equation imposed above, we introduce normalized adaptive laws. In Section 4 two examples are worked out numerically and simulation results of the tracking of a step function and a sine function are presented. Section 5 is conclusion.

4.2 Adaptive Output Regulation

Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \end{aligned} \tag{4.4}$$

where A, B_1 and B_2 contain unknown parameters and the other matrices are assumed to be known. We introduce the signal w generated by (4.2) and assume that it is bounded. We assume that the state x and the signal w are accessible and consider the output regulation problem for (4.4) under the following conditions.

Assumption 4.2.1.

(i) $(A,\ B_2)$ is stabilizable for any unknown parameters of $(A,\ B_2)$.

(ii) The matrix

$$A_1 = \begin{bmatrix} A - \lambda I & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full row-rank for each eigenvalue λ of S for any unknown parameters of (A, B_2).

Assumption 4.2.2. *One of the following conditions is satisfied.*

1. *A_1 is a square matrix, i.e. the input signal u and the output signal z have the same dimension.*

2. *$D_{12} = 0$ and (A, B_2, C_1) has the structure (3.8)-(3.12).*

First we consider the stabilization problem.

4.2.1 Adaptive stabilization

Introduce an estimator and adaptive laws of the form (Narendra and Annaswamy, 1989):

$$\dot{\hat{x}} = A_m\hat{x} + (\hat{A} - A_m)x + \hat{B}_1 w + \hat{B}_2 u, \tag{4.5}$$

$$\begin{aligned} \dot{\hat{A}} &= \dot{\Phi} = -Pex^T, \\ \dot{\hat{B}}_1 &= \dot{\Psi}_1 = -Pew^T, \\ \dot{\hat{B}}_2 &= \dot{\Psi}_2 = -Peu^T, \end{aligned} \tag{4.6}$$

where A_m is an $n \times n$ stable matrix, P is the solution of the following matrix equation

$$A_m^T P + PA_m = -Q_0$$

for some positive-definite matrix Q_0 and

$$e = \hat{x} - x, \quad \Phi = \hat{A} - A, \quad \Psi_1 = \hat{B}_1 - B_1, \quad \Psi_2 = \hat{B}_2 - B_2.$$

Then the error equation is given by

$$\dot{e} = A_m e + \Phi x + \Psi_1 w + \Psi_2 u. \tag{4.7}$$

If some elements of A, B_1 and B_2 are known, we can omit their adaptive laws in (4.6), but for notational convenience we use (4.6).

Lemma 4.2.1. *The system (4.6) and (4.7) is globally stable.*

Proof. Consider the function

$$V(e, \Phi, \Psi_1, \Psi_2) = e^T P e + tr(\Phi^T \Phi + \Psi_1^T \Psi_1 + \Psi_2^T \Psi_2), \tag{4.8}$$

where trA denotes the trace of the matrix A. The time derivative of (4.8) along the solutions of (4.6) and (4.7) is given by

$$\begin{aligned}\dot{V} &= e^T(A_m^T P + A_m P)e + 2e^T P\Phi x + 2e^T P\Psi_1 w + 2e^T P\Psi_2 u + 2tr(\Phi^T\dot{\Phi} + \Psi_1^T\dot{\Psi}_1 + \Psi_2^T\dot{\Psi}_2) \\ &= -e^T Q_0 e \leq 0.\end{aligned}$$

Hence the origin of (4.6) and (4.7) is globally stable. It follows that e, Φ, Ψ_1 and Ψ_2 are bounded for all $t \geq 0$ and $e \in \mathcal{L}_2$. □

Let Q be positive-definite. Then $(C, \hat{A})$ is observable where $C = \sqrt{Q}$. Since $(\hat{A}(t), \hat{B}_2(t))$ is stabilizable for each t by Assumtion 4.2.1, there exists a positive stabilizing solution $X(t)$ of the algebraic Riccati equation

$$\hat{A}^T(t)X + X\hat{A}(t) + Q - X\hat{B}_2(t)\hat{B}_2^T(t)X = 0. \tag{4.9}$$

Lemma 4.2.2. *$X(t)$ is continuously differentiable with respect to t and is uniformly bounded.*

Proof. We set

$$\hat{\boldsymbol{\theta}} = [vec(\hat{A})^T \quad vec(\hat{B}_2)^T].$$

By the proof of Lemma 3.1.1, $\boldsymbol{x} = vec(X)$ is given by $f(\hat{\boldsymbol{\theta}})$ and $\dot{\boldsymbol{x}} = \dfrac{\partial f}{\partial \boldsymbol{\theta}}\dot{\hat{\boldsymbol{\theta}}}$. From Lemma 4.2.1 $\hat{\boldsymbol{\theta}}$ is bounded and hence $\boldsymbol{x} = f(\hat{\boldsymbol{\theta}})$ is bounded. □

Now we introduce the control law

$$u = -\hat{B}_2^T(t)X(t)\hat{x}. \tag{4.10}$$

Before stability property is established, the existance of the solutions must be shown. The overall adaptive system is represented by

$$\dot{x} = (A - B_2\hat{B}_2^T X)x + B_1 w, \tag{4.11}$$

$$\dot{e} = A_m e + (\hat{A} - A)x + (\hat{B}_1 - B_1)w - (\hat{B}_2 - B_2)\hat{B}_2^T X x, \tag{4.12}$$

$$\dot{\hat{A}} = -Pex^T, \tag{4.13}$$

$$\dot{\hat{B}}_1 = -Pew^T, \tag{4.14}$$

$$\dot{\hat{B}}_2 = Pe(\hat{B}_2^T X x)^T. \tag{4.15}$$

Since the right-hand side of (4.11) -(4.15) is continuous with respect to t, the existence of a solution is assured for all $t \in [t_0,\ t_0+\alpha]$ for some $\alpha > 0$. Since the Lyapuov function defined in (4.8) ensures that $e, \hat{A}, \hat{B}_1, \hat{B}_2$ are bounded for all $t \in [t_0,\ t_0+\alpha]$. Then (4.11) can be considered as a linear time-varying differential equation with bounded coefficients. It follows that $x(t)$ grows at most exponentially and the existence of solutions of (4.11) -(4.15) on $t \in [t_0,\ \infty)$ is assured.

Now we consider the stability of the adaptive control system.

Theorem 4.2.1. *If $\dot{\hat{A}}(t)$, $\dot{\hat{B}}_2(t) \to 0$ as $t \to \infty$, then x and $\hat{x}$ are bounded and $\lim_{t\to\infty} e(t) = 0$. Moreover, if $w = 0$ then $\lim_{t\to\infty} x(t) = 0$.*

Proof. Substituting (4.10) into (4.5) , we have

$$\dot{\hat{x}} = (\hat{A} - \hat{B}_2\hat{B}_2^T X)\hat{x} + (A_m - \hat{A})e + \hat{B}_1 w. \tag{4.16}$$

From Lemma 4.2.2, $\dot{\boldsymbol{x}} = \dfrac{\partial f}{\partial \boldsymbol{\theta}}\dot{\hat{\boldsymbol{\theta}}}$ and $\dot{\boldsymbol{x}} \to 0$. Hence $\dot{X} \to 0$ as $\dot{\hat{A}}$, $\dot{\hat{B}}_2 \to 0$. Consider the homogeneous part of (4.16)

$$\dot{\xi} = (\hat{A} - \hat{B}_2\hat{B}_2^T X)\xi. \tag{4.17}$$

Since $\dot{X} \to 0$, there exists a $\delta > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi^T X\xi = -\xi^T(Q + X\hat{B}_2\hat{B}_2^T X - \dot{X})\xi \le -\delta\|\xi\|^2 \tag{4.18}$$

for large t. Since X stays in a compact set, $X(t) \ge \alpha I$ for some $\alpha > 0$ and $(\hat{A} - \hat{B}_2\hat{B}_2^T X)$ is exponentially stable. Since $\hat{A}e$ and $\hat{B}_1 w$ are bounded in (4.16), $\hat{x}$ is also bounded, which in turn implies x is bounded. Now, we conclude from (4.7) that $\dot{e}$ is bounded. Therefore by Barbalat's lemma we obtain $\lim_{t\to\infty} e(t) = 0$. If in particular $w = 0$ then $\lim_{t\to\infty} \hat{x}(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$. □

4.2.2 Adaptive output regulation

Now we consider the adaptive output regulation problem associated with (4.4), (4.5) and (4.6). In this case regulator equation is given by

$$\begin{aligned} &\hat{A}(t)\Pi - \Pi S + \hat{B}_1(t) + \hat{B}_2(t)\Gamma = 0, \\ &C_1\Pi + D_{11} + D_{12}\Gamma = 0. \end{aligned} \tag{4.19}$$

Lemma 4.2.3. *Under Assumptions 4.2.1 and 4.2.2, then there exists a solution (Π, Γ) of (4.19) which is continuously differentiable function of $\hat{A}$, $\hat{B}_1$ and $\hat{B}_2$ and is uniformly bounded.*

Proof. The first part is shown by the same arguments as Lemma 4.2.2. Since $\hat{A}$, $\hat{B}_1$ and $\hat{B}_2$ are uniformly bounded, $(\Pi(t), \Gamma(t))$ is also bounded. □

Corollary 4.2.1. *If $\dot{\hat{A}}(t)$, $\dot{\hat{B}}_2(t) \to 0$ as $t \to \infty$, then $(\dot{\Pi}(t), \dot{\Gamma}(t)) \to 0$ as $t \to \infty$.*

We choose the controller

$$u = -\hat{B}_2^T(t)X(t)\hat{x} + (\Gamma(t) + \hat{B}_2^T(t)X(t)\Pi(t))w, \tag{4.20}$$

where $X(t)$ is the solution of the Riccati equation (4.9) corresponding to (4.4), (4.5) and (4.6). The following result is obtained.

Theorem 4.2.2. *Suppose Assumuptions 4.2.1 and 4.2.2 hold. If $\dot{\hat{A}}(t)$, $\dot{\hat{B}}_2(t) \to 0$ as $t \to \infty$, then the adaptive output regulation is fulfilled i.e., $\lim_{t\to\infty} z(t) = 0$.*

Proof. Since $\dot{\hat{A}}$, $\dot{\hat{B}}_2 \to 0$, we can use the arguments in the proof of Theorem 4.2.1 to conclude that $\hat{x}$ and x are bounded. It also follows from (4.7) that $\dot{e}$ is bounded. Therefore we conclude by Barbalat's lemma that $\lim_{t\to\infty} e = 0$. Now consider

$$\tilde{x} = \hat{x} - \Pi w$$

then

$$\begin{aligned} \dot{\tilde{x}} &= \dot{\hat{x}} - \Pi\dot{w} - \dot{\Pi}w \\ &= (\hat{A} - \hat{B}_2\hat{B}_2^T X)\tilde{x} + (A_m - \hat{A})e - \dot{\Pi}w. \end{aligned}$$

Since $\dot{\hat{A}}$, $\dot{\hat{B}}_2 \to 0$, $\lim_{t\to\infty} \dot{\Pi} = 0$ by Corollary 4.2.1. Since $\lim_{t\to\infty} e = 0$, $\lim_{t\to\infty} \tilde{x} = 0$. Now

$$\begin{aligned} z &= C_1 x + D_{11} w + D_{12} u \\ &= (C_1 - D_{12}\hat{B}_2^T X)\tilde{x} + (C_1\Pi + D_{11} + D_{12}\Gamma)w + C_1 e \\ &= (C_1 - D_{12}\hat{B}_2^T X)\tilde{x} + C_1 e \to 0. \end{aligned}$$

Hence output regulation is achieved. □

4.3 Modification of Adaptive Laws

In this section, we design a controller with normalized adaptive laws (Ioannou and Sun, 1995) so that all signals in the closed-loop plant are bounded and regulation is fulfilled. We use adaptive laws driven by the normalized estimation error which are not directly related to the regulation error x. As a result, the stability analysis of the closed-loop adaptive system is more complicated. We shall introduce new adaptive laws following (Ioannou and Sun, 1995).

4.3.1 Adaptive output regulation with normalized adaptive laws

We consider (4.4) and (4.5). Define

$$\theta \triangleq [\Phi \quad \Psi_1 \quad \Psi_2], \quad \phi \triangleq [x^T \quad w^T \quad u^T]^T.$$

Then the error equation (4.7) is written as

$$\dot{e} = A_m e + \theta\phi. \tag{4.21}$$

We modify this system and define

$$\dot{\epsilon} = A_m \epsilon + \theta\phi - \epsilon n_s^2, \tag{4.22}$$

where $n_s^2 = \phi^T\phi$. This is called the normalized estimation error (Ioannou and Sun, 1995). Note that $\dfrac{\phi}{\sqrt{1+n_s^2}} \in \mathcal{L}_\infty$.

We modify adaptive laws (4.6) and define

$$\begin{aligned} \dot{\hat{A}} &= \dot{\Phi} = -P\epsilon x^T, \\ \dot{\hat{B}}_1 &= \dot{\Psi}_1 = -P\epsilon w^T, \\ \dot{\hat{B}}_2 &= \dot{\Psi}_2 = -P\epsilon u^T. \end{aligned} \tag{4.23}$$

Lemma 4.3.1. *The adaptive law (4.23) guarantees that*

(i) $\theta, \epsilon \in \mathcal{L}_\infty$, *and*

(ii) $\epsilon, \epsilon n_s, \dot{\theta} \in \mathcal{L}_2$

independently of the boundedness properties of ϕ.

Proof. Let us now consider the following quadratic function for the system (4.22) and (4.23)

$$V(\epsilon, \Phi, \Psi_1, \Psi_2) = \epsilon^T P \epsilon + tr(\Phi^T\Phi + \Psi_1^T\Psi_1 + \Psi_2^T\Psi_2). \tag{4.24}$$

The time derivative of (4.24) along the solutions of (4.22) is given by

$$\begin{aligned} \dot{V} = \epsilon^T(A_m^T P + PA_m)\epsilon + 2\epsilon^T P\Phi x + 2\epsilon^T P\Psi_1 w + 2\epsilon^T P\Psi_2 u \\ + 2tr(\Phi^T\dot{\Phi} + \Psi_1^T\dot{\Psi}_1 + \Psi_2^T\dot{\Psi}_2) - 2\epsilon^T P\epsilon n_s^2. \end{aligned}$$

Then by (4.23) we have

$$\dot{V}(\epsilon, \Phi, \Psi) = -\epsilon^T Q\epsilon - 2\epsilon^T P\epsilon n_s^2 \leq 0,$$

which together with (4.24) implies that V, ϵ, Φ, $\Psi \in \mathcal{L}_\infty$. Moreover $\epsilon, \epsilon n_s \in \mathcal{L}_2$. From the adaptive laws (4.23), we have

$$|\dot{\theta}| \leq \|P\| \, |\epsilon| \sqrt{1+n_s^2} \frac{|\phi|}{\sqrt{1+n_s^2}}. \tag{4.25}$$

Since $\epsilon^2(1+n_s^2) = \epsilon^2 + \epsilon^2 n_s^2$ and ϵ, $\epsilon n_s \in \mathcal{L}_2$, we have $\epsilon\sqrt{1+n_s^2} \in \mathcal{L}_2$, which together with $\dfrac{|\phi|}{\sqrt{1+n_s^2}} \in \mathcal{L}_\infty$ implies $\dot{\theta} \in \mathcal{L}_2$. □

We choose the control law

$$u = -\hat{B}_2^T X x + [\Gamma + \hat{B}_2^T X \Pi)]w \tag{4.26}$$

where X is the solution of the algebraic Riccati equation (4.9). The system (4.4) can be written as

$$\dot{x} = \hat{A}x + \hat{B}_1 w + \hat{B}_2 u - \theta\phi. \tag{4.27}$$

Substituting (4.26) into (4.27), we have

$$\dot{x} = (\hat{A} - \hat{B}_2\hat{B}_2^T X)x - \theta\phi + (\hat{B}_1 + \hat{B}_2\Gamma + \hat{B}_2\hat{B}_2^T X\Pi)w. \tag{4.28}$$

Define $\hat{A}_c \triangleq \hat{A} - \hat{B}_2\hat{B}_2^T X$ and we consider the homogeneous part of (4.28).

Lemma 4.3.2. *The homogeneous part of (4.28) is uniformly asymptotically stable.*

Proof. By taking the derivative of (4.9), we obtain

$$\hat{A}_c^T \dot{X} + \dot{X}\hat{A}_c = -Q(t),$$

where

$$Q(t) = \dot{\hat{A}}^T X + X\dot{\hat{A}} - X\dot{\hat{B}}_2\hat{B}_2^T X - X\hat{B}_2\dot{\hat{B}}_2^T X. \tag{4.29}$$

First we show $Re\{\lambda_i(\hat{A}_c(t))\} \leq -\sigma_s$, $\forall t \geq 0$ and for $i = 1, 2, \ldots, n$ where $\sigma_s > 0$ is some constant. Suppose there exists a i and sequence $t_i \to \infty$ such that $Re\{\lambda_i(\hat{A}_c(t_i))\} \to 0$. Along a subsequence, again denoted by t_i there exists limits $\hat{A}_\infty \triangleq \lim_{t_i\to\infty} \hat{A}(t_i)$, $\hat{B}_\infty \triangleq \lim_{t_i\to\infty} \hat{B}_2(t_i)$ and $X_\infty \triangleq \lim_{t_i\to\infty} X(t_i)$. And $\hat{A}_\infty$, $\hat{B}_\infty$ and X_∞ satisfy

$$\hat{A}_\infty^T X_\infty + X_\infty \hat{A}_\infty + Q - X_\infty \hat{B}_\infty \hat{B}_\infty^T X_\infty = 0.$$

Moreover there exists an eigenvalue of $(\hat{A}_\infty - B_\infty \hat{B}_\infty^T X_\infty)$ on the imaginary axis. This is a contradiction since $(\hat{A}_\infty - B_\infty \hat{B}_\infty^T X_\infty)$ is exponentially stable.

Since $\hat{A}_c$ is exponentially stable, $\dot{X}$ can be expressed as

$$\dot{X}(t) = \int_0^\infty e^{\hat{A}_c^T(t)\tau} Q(t) e^{\hat{A}_c(t)\tau} d\tau$$

for each $t \geq 0$. Therefore

$$\|\dot{X}\| \leq \|Q(t)\| \int_0^\infty \|e^{\hat{A}_c^T(t)\tau}\| \|e^{\hat{A}_c(t)\tau}\| d\tau.$$

Since $\|e^{\hat{A}_c(t)\tau}\| \leq k_1 e^{-\sigma_s \tau}$ for some $k_1 \geq 0$, it follows that

$$\|\dot{X}(t)\| \leq c\|Q(t)\|$$

for some $c \geq 0$. In view of $\dot{\hat{A}}$, $\dot{\hat{B}}_2 \in \mathcal{L}_2$ and $\hat{A}$, $\hat{B}_2$, $X \in \mathcal{L}_\infty$, we obtain $Q(t) \in \mathcal{L}_2$ from (4.29) and hence $\|\dot{X}(t)\| \in \mathcal{L}_2$. From the estimate

$$\|\dot{A}_c\| \leq \|\dot{\hat{A}}\| + 2\|\dot{\hat{B}}_2\|\|\hat{B}_2\|\|X\| + \|\hat{B}_2\|^2\|\dot{X}\|,$$

we obtain $\dot{A}_c \in \mathcal{L}_2$. Since $\hat{A}_c$ satisfies the condition of Lemma 3.2.1, $\hat{A}_c$ is uniformly asymptotically stable. □

Next, we show the boundedness of x, $\dot{x}$ and u by using the properties of the $\mathcal{L}_{2\delta}$ norm. We define $\bar{e} \triangleq x + \epsilon$, then (4.22) and (4.28) yield

$$\dot{\bar{e}} = \hat{A}_c\bar{e} + (A_m - \hat{A}_c)\epsilon - \epsilon n_s^2 + (\hat{B}_1 + \hat{B}_2\Gamma + \hat{B}_2\hat{B}_2^T X\Pi)w. \tag{4.30}$$

From Lemma 4.3.2 the state transition matrix of $\dot{x} = \hat{A}_c x$ satisfies $\|\Phi(t,\tau)\| \leq \lambda_0 e^{-\alpha_0(t-\tau)}$ for some constant λ_0 and α_0. We choose $\delta \in [0, \alpha_0)$ and define m_f as

$$m_f^2(t) \triangleq 1 + \|x_t\|_{2\delta}^2 + \|w_t\|_{2\delta}^2 + \|u_t\|_{2\delta}^2. \tag{4.31}$$

Now, we verify that the signal m_f bounds x, $\dot{x}$, and u from above.

Lemma 4.3.3. *Consider the system (4.26) and (4.27). For any given $\delta \in [0, 2k_1)$ $\frac{|x(t)|}{m_f}, \frac{|\dot{x}(t)|}{m_f}, \frac{|u(t)|}{m_f} \in \mathcal{L}_\infty$.*

Proof. Applying Lemma 3.2.2 to (4.28), using Φ, Ψ_1, $\Psi_2 \in \mathcal{L}_\infty$, we obtain

$$\begin{aligned} |x(t)| &\leq \frac{c_0\lambda_0}{\sqrt{2\alpha_0 - \delta}}(\|x_t\|_{2_\delta} + \|w_t\|_{2\delta} + \|u_t\|_{2_\delta}) + c \\ &\leq cm_f(t) + c, \end{aligned}$$

for some generic constant $c \geq 0$. From (4.26) and the facts $\hat{B}_2$, X, Π, $\Gamma \in \mathcal{L}_\infty$, we obtain

$$|u(t)| \leq c|x(t)| + c \leq cm_f(t) + c.$$

From (4.28) we also have

$$\begin{aligned} |\dot{x}(t)| &\leq |\hat{A}_c x - \theta\phi + (\hat{B}_1 + \hat{B}_2\Gamma + \hat{B}_2\hat{B}_2^T X\Pi)w| \\ &\leq cm_f(t) + c. \end{aligned}$$

□

Now we are ready to show the boundedness of x, $\dot{x}$ and u and $\lim_{t\to\infty} \epsilon = 0$.

Lemma 4.3.4. *For the adaptive control system (4.4), (4.22), (4.23) and (4.26) the following is true.*
$x(t), \dot{x}(t), u(t) \in \mathcal{L}_\infty$ *and* $\lim_{t\to\infty} \epsilon = 0$.

Proof. (4.31) and the $\mathcal{L}_{2\delta}$ norm of the control (4.26) yields

$$m_f^2(t) \leq c + c\|x_t\|_{2\delta}^2.$$

Since $x = \bar{e} - \epsilon$, we have $\|x_t\|_{2\delta} \leq \|\bar{e}_t\|_{2\delta} + \|\epsilon_t\|_{2\delta}$. Hence from the inequality above we obtain

$$m_f^2(t) \leq c + c\|\bar{e}_t\|_{2\delta}^2 + c\|\epsilon_t\|_{2\delta}^2. \tag{4.32}$$

On the other hand applying Lemma 3.2.2 to (4.30) we have

$$\|\bar{e}_t\|_{2\delta}^2 \leq c + c\|\epsilon_t\|_{2\delta}^2 + c\|(\epsilon n_s^2)_t\|_{2\delta}^2. \tag{4.33}$$

Since Lemma 4.3.3 implies $\dfrac{|\phi|}{m_f} \in \mathcal{L}_\infty$ and since $n_s^2 = \phi^T\phi$

$$|\epsilon n_s^2| \leq |\epsilon n_s| \frac{|\phi|}{m_f} m_f \leq c|\epsilon n_s m_f|.$$

Combining (4.32), (4.33) and then using the fact $\epsilon \in \mathcal{L}_2$ we have

$$m_f^2(t) \leq c + c\|\epsilon_t\|_{2\delta}^2 + c\|(\epsilon n_s^2)_t\|_{2\delta}^2 \leq c + c\|(\epsilon n_s m_f)_t\|_{2\delta}^2. \tag{4.34}$$

By the definition of $\mathcal{L}_{2\delta}$ norm, (4.34) can be rewritten as

$$m_f^2(t) \leq c + c\int_0^t e^{-\delta(t-\tau)}\epsilon^2 n_s^2 m_f^2(\tau)d\tau.$$

Applying Bellman-Gronwall lemma and using the fact $\epsilon n_s \in \mathcal{L}_2$, we obtain

$$m_f^2(t) \leq c + c\int_{t_0}^t e^{-\delta(t-\tau)} e^{\int_\tau^t \epsilon^2 n_s^2 ds} d\tau < c.$$

By Lemma 4.3.3 x, $\dot{x}$, $u \in \mathcal{L}_\infty$. Since ϵ, x, u, $\epsilon n_s^2 \in \mathcal{L}_\infty$, it follows from (4.22) that $\dot{\epsilon}$ is bounded. But $\epsilon \in \mathcal{L}_\infty \bigcap \mathcal{L}_2$ and hence $\lim_{t\to\infty} \epsilon = 0$ by Barbalat's lemma. □

The following theorem is the main result of this section.

Theorem 4.3.1. *Under Assumuptions 4.2.1 and 4.2.2, the adaptive output regulation is fulfilled i.e.,* $\lim_{t\to\infty} z(t) = 0$.

Proof. Define

$$\tilde{e} = \bar{e} - \Pi w.$$

Then (4.30) gives

$$\begin{aligned}\dot{\tilde{e}} &= \dot{\bar{e}} - \Pi\dot{w} - \dot{\Pi}w \\ &= \hat{A}_c\tilde{e} + (A_m - \hat{A}_c)\epsilon - \epsilon n_s^2 - \dot{\Pi}w. \end{aligned} \tag{4.35}$$

Since this is a linear equation in $\tilde{e}$, we consider responses to the input $(A_m - \hat{A}_c)\epsilon - \epsilon n_s^2$ and $\dot{\Pi}w$ separately. Since $\epsilon, \epsilon n_s^2 \in \mathcal{L}_2$, the former converges to zero as $t \to \infty$. Since $\lim_{t\to\infty} \epsilon = 0$, $\lim_{t\to\infty} \dot{\hat{A}} = 0$ and $\lim_{t\to\infty} \dot{\hat{B}}_2 = 0$ by (4.23). Hence $\lim_{t\to\infty} \dot{\Pi} = 0$ by Corollary 4.2.1. Then the response to $\dot{\Pi}w$ also converges to zero. Hence the response to the sum of two inputs also converges to zero. Since both $\tilde{e} = x + \epsilon - \Pi w$ and ϵ converges to zero $\lim_{t\to\infty}(x - \Pi w) = 0$. Now

$$\begin{aligned} z &= C_1 x + D_{11} w + D_{12} u \\ &= (C_1 - D_{12}\hat{B}_2^T X)(x - \Pi w) + (C_1\Pi + D_{11} + D_{12}\Gamma)w \\ &= (C_1 - D_{12}\hat{B}_2^T X)(x - \Pi w) \to 0. \end{aligned}$$

Hence output regulation is achieved. □

4.4 Examples

Example 4.4.1

Consider the system

$$\dot{x} = Ax + B_1 w + B_2 u \tag{4.36}$$

with

$$A = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ a_1 & a_2 & a_3 & a_4 & a_5 \\ \hline 0 & 0 & 0 & 0 & 1 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{array}\right], \; B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; B_2 = \left[\begin{array}{c|c} 0 & 0 \\ 0 & 0 \\ 1 & b_1 \\ \hline 0 & 0 \\ 0 & 1 \end{array}\right],$$

where $a_1 = 1, a_2 = 2, a_3 = -1, a_4 = 1.5, a_5 = 1, a_6 = -2, a_7 = 1, a_8 = 1, a_9 = -1, a_{10} = -2, b_1 = 1$ are assumed to be unknown. For this system we design a state feedback

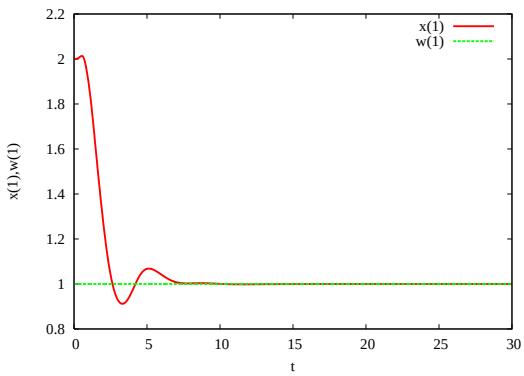


Figure 4.1: Step tracking

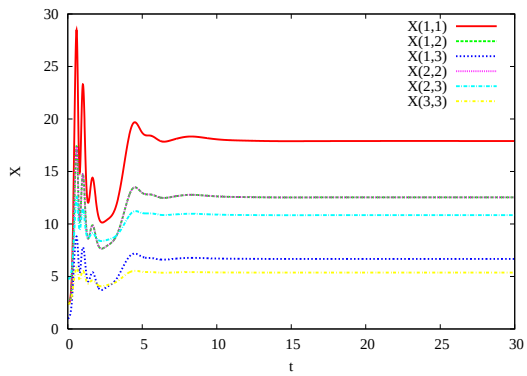


Figure 4.2: Solution of the Riccati equation

controller such that $x_1(t) \to 1$ as $t \to \infty$ under the sinusoidal disturbance. Then we can set $C_1 = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$, $D_{11} = [0 \;\; 0 \;\; -1]$ and take the following exosystem

$$S = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]. \tag{4.37}$$

The exosystem (4.37) represents both sinusoidal disturbance and reference step signal. The solution of the regulator equation exists since A_1 has full row-rank. The solution is

$$\Pi = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_1 & p_2 & p_3 \\ -p_2 & p_1 & 0 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} -1 + (-\hat{a}_4 + \hat{b}_1 + \hat{a}_9\hat{b}_1)p_1 + (\hat{a}_5 - \hat{a}_{10}\hat{b}_1)p_2 & -(1+\hat{a}_9)p_1 + \hat{a}_{10}p_2 \\ (-\hat{a}_5 + \hat{a}_{10}\hat{b}_1)p_1 - (\hat{a}_4 - \hat{b}_1 - \hat{a}_9\hat{b}_1)p_2 & -\hat{a}_{10}p_1(1+\hat{a}_9)p_2 \\ -\hat{a}_1 + \hat{a}_6\hat{b}_1 + (-\hat{a}_4 + \hat{a}_9\hat{b}_1)p_3 & -\hat{a}_6 - \hat{a}_9 p_3 \end{bmatrix}^T,$$

where p_1, p_2 and p_3 are free parameters. Now we choose $p_1 = p_2 = p_3 = 0$. Adaptive output regulation by (4.20) with $Q = I_2$ is considered. The simulation result with $x(0) = [2 \;\; 0 \;\; 0 \;\; 0 \;\; 0]^T$, $w(0) = [0 \;\; 1 \;\; 1]^T$, $\hat{a}_i(0) = 0$ $(i = 1, \cdots, 10)$ and $\hat{b}_1(0) = 0$ is shown in Figure 4.1. The solution of the Riccati equation (4.9) is also shown in Figure 4.2. We can see that $\dot{X} \to 0$ and assumption $Q - \dot{X} \geqslant \delta I$ holds for $t > 10$.

Example 4.4.2

Consider the system (4.36) with

$$A = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ a_1 & a_2 & a_3 \\ \hline 0 & 0 & 1 \end{array}\right], \; B_2 = \left[\begin{array}{c|c} 0 & 0 \\ 1 & b_1 \\ \hline 0 & 1 \end{array}\right], C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, D_{11} = \begin{bmatrix} -1 & 0 \end{bmatrix},$$

where $a_1 = 1, a_2 = -2, a_3 = 3, b_1 = 1$ are assumed to be unknown. For this system we design a state feedback controller such that $x_1(t) \to \sin(t)$ as $t \to \infty$. Then we take the following exosystem

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4.38}$$

which generates a reference signal. In this case A_1 is not square, but Assumption 4.2.2 holds. The solution of the regulator equation (4.19) is

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ p_1 & p_2 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} -\hat{a}_1 - \hat{a}_3 p_1 + \hat{b}_1(p_1 + p_2) - 1 & -\hat{a}_2 - \hat{a}_3 p_2 - \hat{b}_1(p_1 - p_2) \\ -p_1 - p_2 & p_1 - p_2 \end{bmatrix},$$

where p_1, and p_2 are free parameters and we choose $p_1 = p_2 = 0$. The simulation result with $x(0) = [2 \;\; 0 \;\; 0]^T$, $w(0) = [0 \;\; 1]^T$, $\hat{a}_i(0) = 0$ and $\hat{b}_1(0) = 0$ is shown in Figure 4.3. The solution of the Riccati equation (4.9) is also shown in Figure 4.4. We can see that $\dot{X} \to 0$ and assumption $Q - \dot{X} \geqslant \delta I$ holds for $t > 15$. Figures 4.5 and 4.6 show the results using normalized adaptive laws. We can see that the adaptive controller with normalized adaptive laws also fulfills output regulation.

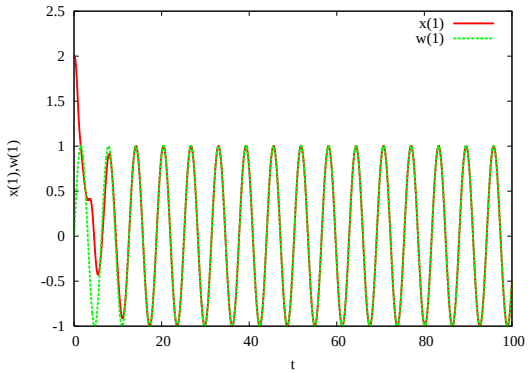


Figure 4.3: Sine tracking

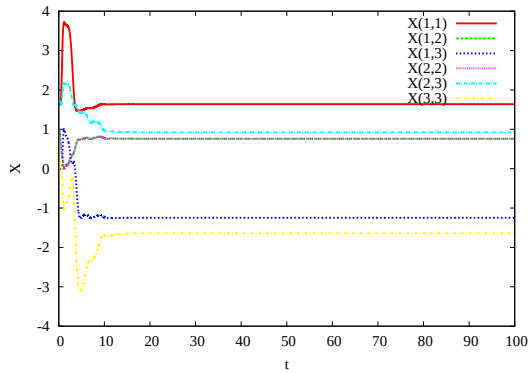


Figure 4.4: Solution of the Riccati equation

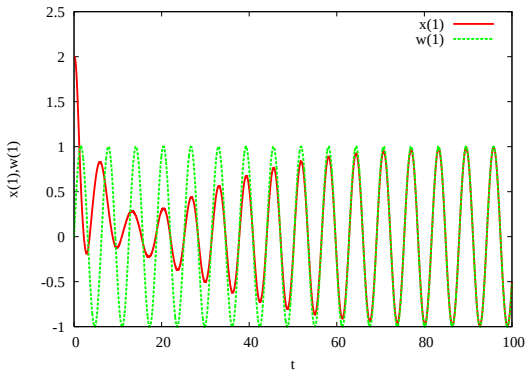


Figure 4.5: Sine tracking with normalized adaptive laws

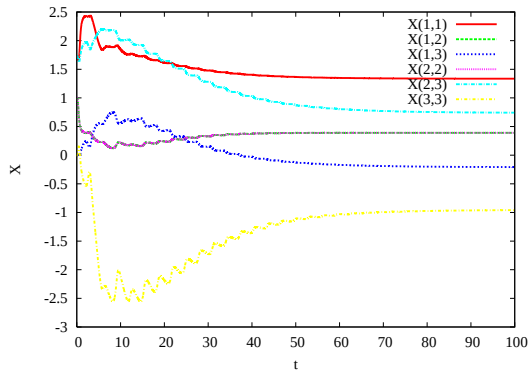


Figure 4.6: Solution of the Riccati equation

4.5 Conclusion

In this chapter, the output regulation problem for linear time-invariant systems with unknown parameters was considered. Based on the Lyapunov stability theory, an adaptive controller which stabilize the system was derived. It was shown that an adaptive controller can be designed using the solution of the Riccati equation if the derivative of the solution is sufficiently small. Then sufficient conditions for the output regulation problem with full information to be solvable are established. Furthermore, the condition on the solution of the Riccati equation imposed above was relaxed introducing normalized adaptive laws. Simulation results were given to illustrate the theory.

Chapter 5

Adaptive Output Regulation of Nonlinear Systems described by Multiple Linear Models

5.1 Introduction

Modelling and control of complex nonlinear dynamical systems is a difficult task, and a natural approach is to use multiple local models and controllers. It is referred to as a multiple model approach, and is used in many areas (Murray-Smith and Johansen, 1997). For example, if local models are linear and underlying systems are described by their convex combination, they form an important subclass which contains Takagi-Sugeno fuzzy systems (Murray-Smith and Johansen, 1997). Weights in the convex combination are referred to as model validity functions. This subclass has theoretical and computational advantages that ample design methods of controllers from linear systems theory and many efficient algorithms already developed can be used. In Takagi-Sugeno fuzzy systems, model validity functions are fuzzy weights from the firing strengths of IF-THEN rules in the systems. For such systems much work has been done on stability, stabilization and H_∞ control (Tanaka and Sugeno, 1992; Tanaka *et al.*, 1996; Yoneyama *et al.*, 2000, 2001*a*,*b*; Nishikawa *et al.*, 2000; Katayama and Ichikawa, 2002, 2004; Feng, 2006, (and references therein)). An application of multiple model approach can be found in (Bando and Nakanishi, 2006). There multiple linear models are used to design a controller of an unmanned helicopter. It is based on an algebraic Riccati equation, and a soft-max function of estimation error was used as model validity functions.

In (Bando and Ichikawa, 2007), a nonlinear system described by multiple linear models, where model validity functions are smooth functions of the state variable, was

considered. It is assumed that linear systems are of controllable canonical form with the same structure and contain unknown parameters. Introducing an estimator of the system, adaptive laws, and feedback controllers based on algebraic Riccati equations, local adaptive stabilization was established.

As pointed out in (Chen *et al.*, 2000; Feng and Harris, 2001; Lam *et al.*, 2000), for most of the nonlinear systems, state variables are not available in practice. Then the above requirement may be too restrictive or does not hold. In this situation, observer-based adaptive controllers are more appealing. In this chapter we first consider an adaptive state feecback controller for nonlinear systems described by multiple linear models and then consider an adaptive observer and an adaptive output feedback controller for a class of SISO uncertain nonlinear systems.

Our approach is similar to the state dependent Riccati equation (SDRE) method (Cloutier *et al.*, 1996) for the affine nonlinear systems. In the SDRE approach, the system is described in the linear form with matrices depending on the state, and the nonlinear system described by multiple linear models is a special case. While SDRE approach has been shown to be effective in many applications (Mracek and Cloutier, 1998; Erdem and Alleyne, 2004; Cloutier *et al.*, 1996), very little is known about the stability properties associated with the SDRE controllers. In (Langson and Alleyne, 1999) and (Curtis and Beard, 2002) the stability of SDRE approach is investigated. In (Curtis and Beard, 2002) the SDRE approach is combined with satisficing (Curtis and Beard, 2004), and the stability of the system is shown by a control Lyapunov function. A sufficient condition for global asymptotic stability is derived in (Langson and Alleyne, 1999). These works are concerned with known dynamics with state-feedback control. Our system is uncertain, and an adaptive output feedback is employed to establish the local stability. Moreover, our controller can take physical intuition into account, since the controller is designed based on state variables. It is desirable that there be room for physical intuition in controller design. This gives the designer the opportunity to tune the performance of the controller by adjusting the physically significant parameters. while feedback linearization, for example, which is often used in nonlinear control, is a method which allows very little room for physical intuition, for the physical significance of the parameters are easily lost during coordinate transformation.

The chapter is organized as follows. Section 1 is introduction, and Section 2 is concerned with output regulation of affine nonlinear systems. Section 3 gives adaptive regulation and adaptive output regulation by state feedback. Section 4 is concerned with adaptive regulation by output feedback . As examples, adaptive regulation of a nonlinear mass-spring system and the van der Pol equation is considered, and simulation results are given.

5.2 Output Regulation of Affine Nonlinear Systems

Here we assume that A and B_2 are continuously differentiable functions of the state x. Then the system (4.4) becomes an affine nonlinear system

$$\begin{aligned} \dot{x} &= A(x)x + B_1 w + B_2(x)u, \\ z &= C_1 x + D_{11} w + D_{12} u. \end{aligned} \tag{5.1}$$

We shall consider the output regulation problem for (5.1) under the following condition:

Assumption 5.2.1. *Suppose $D_{12} = 0$ and that $(A(x),\ B_2(x),\ C_1)$ has the structure (3.8)-(3.12) for each x.*

5.2.1 Stabilization

Consider first the regulation problem of (5.1) with $w = 0$. Let Q be positive-definite. Then $(C,\ A(x))$ is observable, where $C = \sqrt{Q}$. Since $(A(x),\ B_2(x))$ is by Assumption 5.2.1 stabilizable for all x, there exists a positive stabilizing solution $X = X(x)$ of the algebraic Riccati equation

$$A(x)^T X + XA(x) + Q - XB_2(x)B_2(x)^T X = 0. \tag{5.2}$$

Now we show the regularity of the solution X in $x = (x_i)$.

Lemma 5.2.1. *For all x, there exists a unique positive stabilizing solution $X = X(x)$ of (5.2) which is continuous and continuously differentiable with respect to x. Moreover, if x stays in a compact domain, $\dfrac{\partial X}{\partial x_i}$ is bounded for any i.*

Proof. If $(A,\ B_2,\ C)$ of (5.2) is of the form (3.8)-(3.12), then by Lemma 3.1.1 X is a continuously differentiable function of parameters of A and B_2. Now the parameters of $A(x)$ and $B_2(x)$ are continuously differentiable functions of x, and hence $X(x)$ is a continuously differentiable with respect to x. □

Now we introduce the control law

$$u = -B_2^T X(x)x, \tag{5.3}$$

and we shall examine the stability of the closed-loop system

$$\dot{x} = [A(x) - B_2(x)B_2(x)^T X(x)]x. \tag{5.4}$$

Theorem 5.2.1. *The equilibrium $x_e = 0$ of the system (5.1) is locally asymptotically stable.*

Proof. Consider the Lyapunov function candidate

$$V(x) = x^T X(x) x. \tag{5.5}$$

The time derivative of (5.5) along the solutions of (5.4) is given by

$$\begin{aligned}
\dot{V}(x) &= \frac{\partial}{\partial x}(x^T X(x) x)\dot{x} \\
&= [(Xx)^T + x^T X + x^T \frac{\partial}{\partial x}(X(x)x)](A - B_2 B_2^T X)x, \\
&= [(Xx)^T + x^T X + x^T (\nabla_x \otimes X)(I_n \otimes x)](A - B_2 B_2^T X)x, \\
&\leq -x^T [Q - (\nabla_x \otimes X)(I_n \otimes x)(A - B_2 B_2^T X)]x
\end{aligned}$$

where $\nabla_x = \left[\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right]$, $\otimes$ denotes Kronecker product and $M \otimes N$ is a block matrix with (i,j) block $m_{ij}N$, and for simplicity we have omitted x in $A(x)$, $B_2(x)$ and $X(x)$. Define $\Omega = \{x \in \mathbb{R}^n |\ Q - (\nabla_x \otimes X)(I_n \otimes x)(A - B_2 B_2^T X) > 0\}$. Then Ω is an open set containing the origin. If $x \in \Omega$, then $\dot{V} \leq 0$. Hence the origin is locally asymptotically stable. □

Corollary 5.2.1. *If $Q - (\nabla_x \otimes X)(I_n \otimes x)(A - B_2 B_2^T X) \geqslant \delta I$ for some $\delta > 0$, then the equilibrium $x_e = 0$ of the system (5.4) is globally asymptotically stable.*

Let $x(t; x_0)$ be the solution of (5.4) with $x(0) = x_0$.

Corollary 5.2.2. *Suppose $x(t; x_0)$ exists for all $t \geq 0$. If $Q - \frac{d}{dt}X(x(t; x_0)) \geqslant \delta I$ for some $\delta > 0$ and for $t > T > 0$, then $x(t; x_0) \to 0$ as $t \to \infty$.*

By Theorem 5.2.1 the existence of local controller is assured. In this section we construct the controller which can assure the stability in larger domain. To achieve this we consider following:

1. For small ϵ_0, construct the controller such that if $|x(t)| \leq \epsilon_0$ then $x(t) \to 0$.
2. Choose $\epsilon < \epsilon_0$ and constrct the controller by which state converges to $\boldsymbol{B}_\epsilon = \{x \in \mathbb{R}^n; |x| \leq \epsilon\}$.
3. We use the controller constructed in first and second steps.

We consider the domain

$$\boldsymbol{B}_\epsilon^r = \{x \in \mathbb{R}^n; \epsilon \leq |x| < r\} \tag{5.6}$$

and set

$$\alpha \triangleq \min_{\epsilon \leq |x| < r} \frac{|A(x)x|}{|x|}, \quad \beta \triangleq \min_{\epsilon \leq |x| < r} \frac{|B_2^T(x)x|}{|x|}.$$

To achieve 2, we consider the following controller:

$$u = -\rho B_2^T x. \tag{5.7}$$

Lemma 5.2.2. *Assume $\beta \neq 0$. Choose ρ such that*

$$\rho > \frac{\alpha}{\beta^2} \tag{5.8}$$

Then the solution of (5.9) starting from $x_0 \in \boldsymbol{B}_\epsilon^r$ asymptotically converges to $\boldsymbol{B}_\epsilon$.

Proof. Closed-loop sytem becomes

$$\dot{x} = [A(x) - \rho B_2(x) B_2^T(x)]x. \tag{5.9}$$

Consider the Lyapunov function candidate

$$V(x) = x^T x. \tag{5.10}$$

The time derivative of (5.10) along the solutions of (5.4) is given by

$$\begin{aligned} \dot{V}(x) &= 2x^T[A(x) - \rho B_2 B_2^T]x, \\ &\leq (\alpha - \rho\beta^2)|x|^2 < 0 \end{aligned}$$

Hence the solution of (5.9) starting from $x_0 \in \boldsymbol{B}_\epsilon^r$ asymptotically converges to $\boldsymbol{B}_\epsilon$. □

5.2.2 Output regulation

Consider the output regulation problem associated with (5.1). By Assumption 5.2.1 and Lemma 3.1.3, there exists a solution (Π, Γ) of the regulator equation

$$\begin{aligned} &A(x)\Pi - \Pi S + B_1 + B_2(x)\Gamma = 0, \\ &C_1\Pi + D_{11} + D_{12}\Gamma = 0, \end{aligned} \tag{5.11}$$

such that Π is independent of x. We choose the controller

$$u = -B_2^T(x)X(x)x + (\Gamma(x) + B_2^T(x)X(x)\Pi)w, \tag{5.12}$$

where $X(x)$ is the solution of the Riccati equation (5.2).

Theorem 5.2.2. *Under Assumption 5.2.1, the local output regulation is fulfilled, i.e., $\lim_{t\to\infty} z(t) = 0$ for any small x_0 and w_0.*

Proof. Substituting (5.12) into (5.1), we have

$$\begin{aligned} \dot{x} &= (A - B_2^T B_2^T X)x + (B_1 + \Gamma + B_2^T X \Pi)w \\ &\triangleq A_f x + Bw. \end{aligned} \tag{5.13}$$

The time derivative of $V(x)$ along the solution of (5.13) is given by

$$\begin{aligned} \dot{V} &\leq -x^T[Q - (\nabla_x \otimes X)(I_n \otimes x)(A_f x + Bw)]x + 2x^T XBw \\ &\leq -x^T[Q - M(x,w)]x + \frac{1}{\beta}|w|^2, \quad \beta > 0 \end{aligned}$$

where $M(x,w) \triangleq (\nabla_x \otimes X)(I_n \otimes x)(A_f x + Bw) + \beta XBB^T X$. For sufficiently small $\alpha > 0$, $\epsilon > 0$ and $\beta > 0$, there exists a δ such that $V(x) \leq \alpha$ and $|w| \leq \epsilon$ imply

$$\dot{V} \leq -\delta V + \frac{1}{\beta}|w|^2, \tag{5.14}$$

and $\dfrac{\epsilon^2}{\beta\delta} < \alpha$. In fact there exist positive numbers δ' and δ'' such that $Q - \delta'' I \geq \delta' I$. Choose small positive numbers $\alpha > 0$, $\epsilon_0 > 0$ and $\beta > 0$ such that $V(x) \leq \alpha$ and $|w| \leq \epsilon_0$ imply $\max \|M(x,w)\| \leq \delta''$. Then

$$Q - M(x,w) \geq Q - \|M(x,w)\| I \geq Q - \delta'' I \geq \delta' I.$$

Now for x satisfying $V(x) \leq \alpha$, there exists a $\delta > 0$ such that $\delta'|x|^2 \geq \delta V(x)$ and hence $\dot{V} \leq -\delta V + \dfrac{1}{\beta}|w|^2$. Choose $0 < \epsilon \leq \epsilon_0$ such that $\dfrac{\epsilon^2}{\delta\beta} < \alpha$. Then $V(x) \leq \alpha$ and $|w| \leq \epsilon$ imply $\dot{V} \leq -\delta V + \dfrac{1}{\beta}|w|^2$.

By integrating (5.14) we have

$$V(x(t)) \leq e^{-\delta t} V(x_0) + \frac{\beta}{\delta}|w|^2.$$

Finally choose α_0 such that $\alpha_0 + \dfrac{\beta\epsilon^2}{\delta} < \alpha$ and x_0 such that $V(x_0) \leq \alpha_0$. Then the solution of (5.13) starting from x_0 stays in $\Omega_\alpha = \{x \in \mathbb{R}^n |\ V(x) \leq \alpha\}$ for all $t \geq 0$. Consider

$$\tilde{x} = x - \Pi w$$

then

$$\dot{\tilde{x}} = (A - B_2 B_2^T X)\tilde{x} = A_f \tilde{x}. \tag{5.15}$$

Consider $\tilde{V}(x,\tilde{x}) = \tilde{x}^T X(x)\tilde{x}$. The time derivative of $\tilde{V}(x)$ along the solutions of (5.13) and (5.15) is given by

$$\begin{aligned} \dot{\tilde{V}} &\leq -\tilde{x}^T[Q - (\nabla_x \otimes X)(I_n \otimes x)(A_f x + Bw)]\tilde{x} \\ &\leq -\delta'|\tilde{x}|^2 \end{aligned}$$

for some $\delta' > 0$, since $x \in \Omega_\alpha$ and $|w| \leq \epsilon$ for all $t \geq 0$. Then the origin of (5.15) is locally asymptoticaly stable. Now

$$\begin{aligned} z &= C_1 x + D_{11} w + D_{12} u \\ &= (C_1 - D_{12} B_2^T X)\tilde{x} + (C_1 \Pi + D_{11} + D_{12}\Gamma) w \\ &= (C_1 - D_{12} B_2^T X)\tilde{x} \to 0. \end{aligned}$$

Hence local output regulation is achieved. □

Corollary 5.2.3. *If $Q - (\nabla_x \otimes X)(I_n \otimes x)(A_f x + Bw) \geqslant \delta I$ for $w = w(t; w_0)$, $t \geq 0$ and x, then the global output regulation is fulfilled i.e., $\lim_{t\to\infty} z(t) = 0$ for any x_0 and w_0.*

Corollary 5.2.4. *Suppose $x(t; x_0)$ exists for all $t \geq 0$. If $Q - \frac{d}{dt} X(x(t; x_0)) \geqslant \delta I$ for some $\delta > 0$ and for $t > T > 0$, then $z(t) \to 0$ as $t \to \infty$.*

We shall apply Theorem 5.2.2 to the van der Pol equation.

5.2.3 Example

Example 5.2.1

Consider the van der Pol equation

$$\ddot{\xi} = 0.2(1 - \xi^2)\dot{\xi} - \xi + u. \tag{5.16}$$

This system can be represented by the following affine nonlinear system:

$$\dot{x} = A(x)x + B_2 u, \tag{5.17}$$

where

$$A(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0.2(1 - x_1^2) \end{bmatrix}, \; B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix}.$$

We design a state feedback controller such that $x_1(t) \to 0.3sin(t)$. In this case we set $C_1 = [1 \;\; 0]$, $D_{11} = [-1 \;\; 0]$ and take the following exosystem

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then the solution of the regulation equation (5.11) is $\Pi = I_2$ and $\Gamma = [-1, -0.2(1-x_1^2)]$. We consider the controller

$$u = F(x)x + (\Gamma(x) + B_2^T(x)X(x)\Pi)w,$$

and choose three different types of feedback gain $F(x)$.

- SDRE control

  We choose $F(x) = -B_2^T(x)X(x)$ where $X(x)$ is the solution of the Riccati equation (5.2).The first controller tried in simulation is SDRE with Q=diag([1, 1]), R=1. Another SDRE controller with state-dependent Q(x) and R(x) matrices was also simulated on the model.

- Pole placement

  Consider $F(x) = -K_p(x)$ and $K_c(x)$ are calculated from

$$\det(sI - A + B_2K_p) = (s+1)^2, \tag{5.18}$$

  for each x where A_c^* are given monic Hurwitz polynomials of degree n. The roots of $A_c^*(s) = 0$ represent the desired pole locations of the transfer function of the closed-loop system. We select the closed-loop polynomial $A_c^* = (s+1)^2$ and solve (5.18) for $K_p(x)$ to obtain

$$K_p(x) = [0 \quad -0.2(1 - x_1^2)].$$

  Since $A - B_2K_p$ is assymptotically stable, there exists a slolution P of a Lyapunov equation

$$(A - B_2K_p)^T P + P(A - B_2K_p) = -Q_p \tag{5.19}$$

  where $Q_p = Q_p^T$ is positive definite matrix for all x. P is a continuously differentiable function of parameters of A and B_2 and hence $P(x)$ is a continuously differentiable with respect to x. Consider the Lyapunov function candidate

$$V(x) = x^T P(x)x. \tag{5.20}$$

  and local asymptotic stability is shown by the same discussion as in Theorem 5.2.1.

- H_∞ control

  We choose $F(x) = -B_2^T(x)X(x)$ where $X(x)$ is the solution of the Riccati equation

$$A(x)^T X + XA(x) + Q - X(B_2(x)B_2(x)^T - \gamma^{-2}B_1B_1^T)X = 0. \tag{5.21}$$

  where $Q = I_2$ and $B_1 = [0 \;\; 1]^T$. For $x = [2 \;\; 0]^T$, the solution of (5.21) is shown in Figure 5.1. and we select $\gamma = 1.2$.

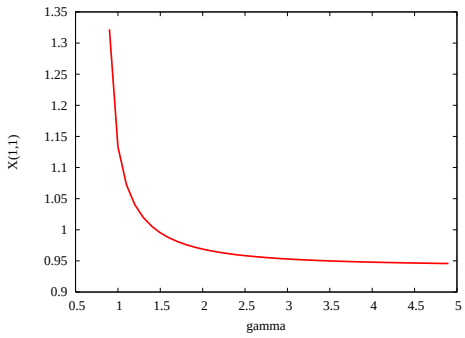
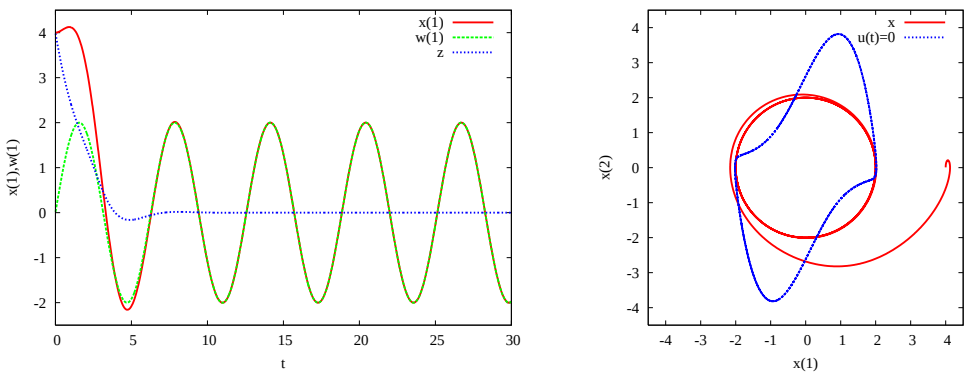
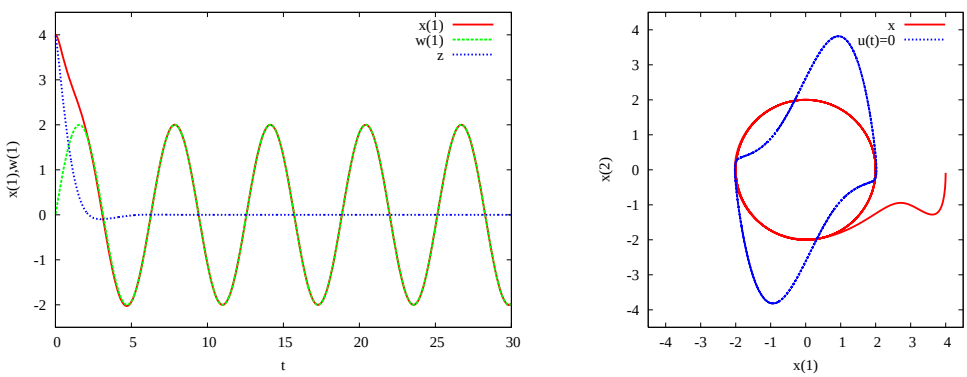


Figure 5.1: γ.

The free system (5.16) has a periodic solution. In Figure 5.2, the responses to the controllers mentioned above are shown with initial conditions $x(0) = [4 \quad 0]^T$, $w(0) = [0 \;\; 2]^T$. Figure 5.2(a) corresponds to the controller with Q=diag([1, 1]). Figure 5.2(b) corresponds to the controller with diag($[x_1^2+1,\ x_2^2+1]$). This example illustrates how in SDRE control the weighting matrices Q and R can be chosen as functions of the states so as to obtain the desired system response. The closed-loop poles are shown in Figures 5.3. It is seen that pole placement controller respond faster than other controllers which have poles smaller than -1 initially.

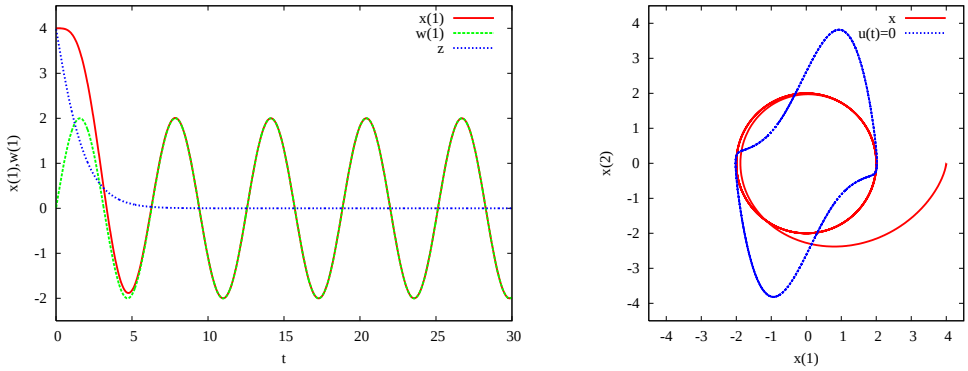


(a) SDRE control (Q(x) = diag([1, 1]))

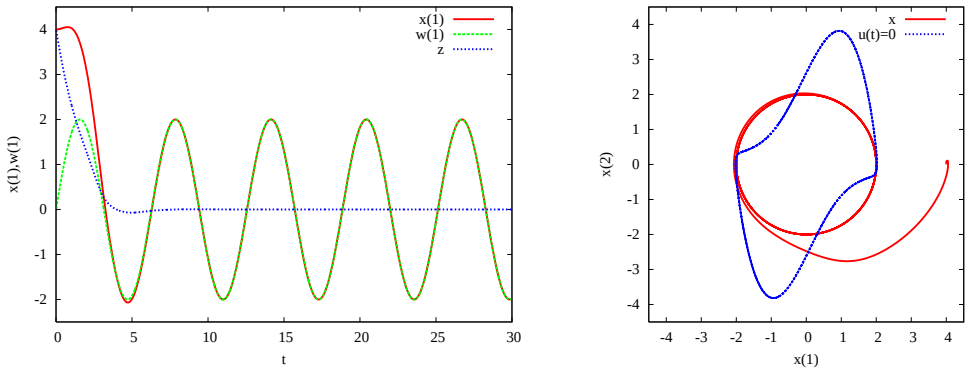


(b) SDRE control (Q(x) = diag([$x_1^2 + 1,\ x_2^2 + 1$]))

Figure 5.2: The trajectories of the state (left) and the trajectories of the state in the phase plane (right).



(b) Pole placement



(b) H_∞ control

Figure 5.2: Continue

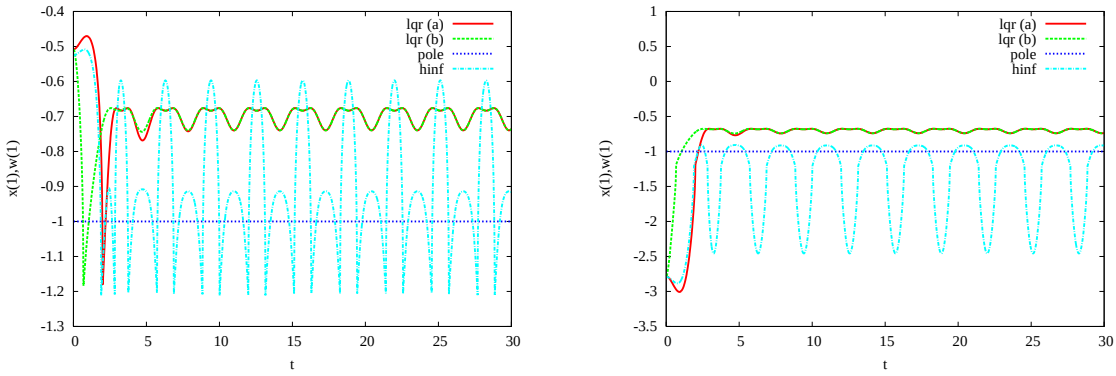


Figure 5.3: The closed loop poles.

5.3 Adaptive Output Regulation of Nonlinear Systems Described by Multiple Linear Models

Consider the nonlinear system described by multiple linear models

$$\begin{aligned} \dot{x} &= \sum_{i=1}^{r} \lambda_i(x) A_i x + B_1 w + \sum_{i=1}^{r} \lambda_i(x) B_{2i} u, \\ z &= C_1 x + D_{11} w + D_{12} u, \end{aligned} \tag{5.22}$$

where the constant matrices A_i and B_{2i} contain unknown parameters, and the other matrices are assumed to be known. We assume A_i, B_{2i} and C_1 satisfy Assumption 5.2.1. Note that all pairs $(A_i,\ B_{2i})$ are in the same controllable canonical form and hence $(\sum_{i=1}^{r} \lambda_i(x) A_i,\ \sum_{i=1}^{r} \lambda_i(x) B_{2i})$ is also in the controllable canonical form for any unknown parameters of $(A_i,\ B_{2i})$ and for any x. $\lambda_i(x)$ are locally Lipshitz given functions of x such that

$$\sum_{i=1}^{r} \lambda_i(x) = 1, \lambda_i(x) \geq 0 \quad i = 1, \cdots, r. \tag{5.23}$$

Note that (5.22) is a special case of (5.1) if A_i and B_{2i} are known. Hence Theorems 5.2.1 and 5.2.2 can be applied.

5.3.1 Adaptive regulation

First we consider the stabilization problem. Introduce an estimator and adaptive laws of the form:

$$\dot{\hat{x}} = A_m \hat{x} + \sum_{i=1}^{r} \lambda_i(x)(\hat{A}_i - A_m) x + B_1 w + \sum_{i=1}^{r} \lambda_i(x) \hat{B}_{2i} u, \tag{5.24}$$

$$\begin{aligned} \dot{\hat{A}}_i &= \dot{\Phi}_i = -\lambda_i(x) P e x^T, \\ \dot{\hat{B}}_{2i} &= \dot{\Psi}_{2i} = -\lambda_i(x) P e u^T, \end{aligned} \tag{5.25}$$

where A_m is an $n \times n$ stable matrix, P is the solution of the following matrix equation

$$A_m^T P + P A_m = -Q_0$$

for some positive-definite matrix Q_0 and

$$e = \hat{x} - x,\ \Phi_i = \hat{A}_i - A,\ \Psi_{2i} = \hat{B}_{2i} - B_2.$$

Then the error equation is given by

$$\dot{e} = A_m e + \sum_{i=1}^{r} \lambda_i(x)(\Phi_i x + \Psi_{2i} u). \tag{5.26}$$

If some elements of A_i and B_{2i} are known, we can omit their adaptive laws in (5.25), but for notational convenience we use (5.25).

Lemma 5.3.1. *If x(t) and u(t) are bounded for all $t \geq 0$, then (5.25) and (5.26) are globally stable.*

Proof. Consider the Lyapunov function candidate

$$V(e, \Phi_i, \Psi_{2i}) = e^T P e + \sum_{i=1}^{r} tr(\Phi_i^T \Phi_i + \Psi_{2i}^T \Psi_{2i}). \tag{5.27}$$

The time derivative of (5.27) along the solutions of (5.26) is given by

$$\dot{V}(e, \Phi_i, \Psi_{2i}) = -e^T Q_0 e \leq 0.$$

Hence the origin of (5.25) and (5.26) is globally stable. It follows that e, Φ_i and Ψ_{2i} are bounded for all $t \geq 0$ and $e \in \mathcal{L}_2$. □

Consider (5.2) with $(A(x),\ B_2(x))$ replaced by $(\hat{A}(x,t),\ \hat{B}_2(x,t))$, where $\hat{A}(x,t) = \sum_{i=1}^{r} \lambda_i(x)\hat{A}_i(t)$, $\hat{B}_2(x,t) = \sum_{i=1}^{r} \lambda_i(x)\hat{B}_{2i}(t)$. Let Q be positive-definite. Then for each t fixed, (5.2) becomes

$$\begin{aligned}(\sum_{i=1}^{r} \lambda_i(x)\hat{A}_i)^T X + X(\sum_{i=1}^{r} \lambda_i(x)\hat{A}_i) + Q \\ - X(\sum_{i=1}^{r} \lambda_i(x)\hat{B}_{2i})(\sum_{i=1}^{r} \lambda_i(x)\hat{B}_{2i})^T X = 0.\end{aligned} \tag{5.28}$$

By Lemma 3.1.1 and 5.2.1, $X = X(x, \theta(t))$ is continuously differentiable function of x and $\theta(t)$, where

$$\theta = [vec\ \hat{A}_1, \cdots, vec\ \hat{A}_r, vec\ \hat{B}_{21}, \cdots, vec\ \hat{B}_{2r}]^T.$$

Now introduce the control law

$$u = -\hat{B}_2^T X(x, \theta(t))\hat{x} \tag{5.29}$$

and consider the stability of the adaptive control system.

Theorem 5.3.1. *For sufficiently small $x(0)$, $\hat{x}(0)$, $e(0)$ and $\theta(0)$, x, $\hat{x}$, e and θ are bounded and $\lim_{t\to\infty} e(t) = 0$. Moreover, if $w = 0$, then $\lim_{t\to\infty} x(t) = 0$.*

Proof. Substituting (5.29) into (5.22) and (5.24) , we have

$$\begin{aligned}
\dot{x} &= (\hat{A} - \hat{B}_2\hat{B}_2^T X)x + \sum_{i=1}^{r} \lambda_i(x)(\Phi_i x - \Psi_i \hat{B}_2^T X\hat{x}) + B_1 w. \\
&\triangleq \hat{A}_f x + \Phi x + \Psi\hat{x} + B_1 w \qquad (5.30)\\
\dot{\hat{x}} &= (\hat{A} - \hat{B}_2\hat{B}_2^T X)\hat{x} + (A_m - \hat{A})e + B_1 w \\
&\triangleq \hat{A}_f\hat{x} + Ee + B_1 w, \qquad (5.31)
\end{aligned}$$

where we have suppressed x and t in $\hat{A}$ and $\hat{B}_2$ and X and so on. Consider $V_1 = x^T X(x,\theta)x$. The time derivative of V_1 along the solution of (5.30) is given by

$$\begin{aligned}
\dot{V}_1 \quad &\le -x^T[Q - (\nabla_x \otimes X)(I_n \otimes x)(\hat{A}_f x + \Phi x + \Psi\hat{x} + Bw) - (\nabla_\theta \otimes X)(I_{2r} \otimes \theta)\dot{\theta}]x \\
&+2x^T X\Phi x + 2x^T X\Psi\hat{x} + 2x^T XBw.
\end{aligned}$$

There exist ϵ_e, ϵ_w, α, β_1, $\beta_2 > 0$ such that $|e|, |\theta| \le \epsilon_e$, $|w| \le \epsilon_w$ and $V_1(x) \le \alpha$ imply

$$\dot{V}_1 \le -\delta V_1 + \frac{1}{\beta_1}|w|^2 + \frac{1}{\beta_2}\hat{x}^T\hat{x} \le -\delta V_1 + \frac{1}{\beta_1}|w|^2 + \frac{\gamma}{\beta_2}\hat{x}^T X\hat{x} \qquad (5.32)$$

and $\frac{\epsilon_w^2}{\beta_1\delta} < \alpha$, where $\gamma > 0$ is a positive constant. Then there exists a $\rho > 0$ such that $\frac{\epsilon_w^2}{\beta_1\delta} + \frac{\gamma\rho}{\beta_2\delta} < \alpha$. Now choose α_0 such that $\alpha_0 + \frac{\epsilon_w^2}{\beta_1\delta} + \frac{\gamma\rho}{\beta_2\delta} < \alpha$ and x_0 such that $V_1(x_0) \le \alpha_0$. Now consider also $V_2 = \hat{x}^T X(x,\theta)\hat{x}$. The time derivative of V_2 along the solution of (5.31) is given by

$$\begin{aligned}
\dot{V}_2 \quad &\le -\hat{x}^T[Q - (\nabla_x \otimes X)(I_n \otimes x)(\hat{A}_f x + \Phi x + \Psi\hat{x} + \hat{B}_1 w) - (\nabla_\theta \otimes X)(I_{2r} \otimes \theta)\dot{\theta}]\hat{x} \\
&+2x^T X\Phi x + 2\hat{x}^T XEe + 2\hat{x}^T XBw. \\
&\le -\delta V_2 + \frac{1}{\beta_3}|e|^2 + \frac{1}{\beta_4}|w|^2, \qquad (5.33)
\end{aligned}$$

where β_3 and β_4 are small positive numbers and conditions for (5.32) are assumed. Choose ρ_0 such that $\rho_0 + \frac{\epsilon_w^2}{\beta_3\delta} + \frac{\epsilon_e^2}{\beta_4\delta} < \rho$ and $\hat{x}_0$ such that $V_2(\hat{x}_0) \le \rho_0$. Then by (5.33), $V_2(\hat{x}) \le \rho$. Combining this and (5.32), we obtain $V_1(x) \le \alpha$. Thus the solution starting from x_0, $\hat{x}_0$ stays in $\Omega = \{x, \hat{x} \in \mathbb{R}^n |\ V_1(x) \le \alpha, V_2(\hat{x}) \le \rho\}$.

Now, we conclude from (5.26) that $\dot{e}$ is bounded. Therefore by Barbalat's lemma (Narendra and Annaswamy, 1989) we obtain $\lim_{t\to\infty} e(t) = 0$. If in particular $w = 0$, then by (5.33) $\lim_{t\to\infty} \hat{x}(t) = 0$, and hence $\lim_{t\to\infty} x(t) = 0$. □

5.3.2 Adaptive output regulation

Now we consider the adaptive output regulation problem associated with (5.22), (5.24) and (5.25). In this case regulator equation is given by

$$\begin{aligned} &\hat{A}(x,\theta)\Pi - \Pi S + B_1 + \hat{B}_2(x,\theta)\Gamma = 0, \\ &C_1\Pi + D_{11} + D_{12}\Gamma = 0. \end{aligned} \tag{5.34}$$

By Assumption 5.2.1 and Lemma 3.1.3, there exists a solution $(\Pi,\ \Gamma)$ of (5.34) such that Π is a constant matrix. Using this solution, we choose the controller

$$u = -\hat{B}_2^T X(x,\theta)\hat{x} + (\Gamma(x,\theta) + \hat{B}_2^T X(x,\theta)\Pi)w, \tag{5.35}$$

where $X(x,\theta)$ is the solution of the Riccati equation (5.28) corresponding to (5.22), (5.24) and (5.25). Then the following result is obtained.

Theorem 5.3.2. *For sufficiently small $x(0)$, $\hat{x}(0)$, $e(0)$ and $\theta(0)$, the adaptive local output regulation is fulfilled i.e., $\lim_{t\to\infty} z(t) = 0$.*

Proof. Choose $x(0)$, $\hat{x}(0)$, $e(0)$ and $\theta(0)$ as in Theorem 4.2.1. Consider

$$\tilde{x} = \hat{x} - \Pi w$$

then

$$\dot{\tilde{x}} \;=\; (\hat{A} - \hat{B}_2\hat{B}_2^T X)\tilde{x} + (A_m - \hat{A})e.$$

Proceeding as Theorem 4.2.1 with $V_3 = \tilde{x}^T X(x,\theta)\tilde{x}$, we obtain $\dot{V}_3 \le -\tilde{\delta}|\tilde{x}|^2 + \dfrac{1}{\beta'}|e|^2$ for some $\tilde{\delta} > 0$ and $\beta' > 0$. Since $\lim_{t\to\infty} e(t) = 0$, $\lim\limits_{t\to\infty} \tilde{x}(t) = 0$. Now

$$\begin{aligned} z &= C_1 x + D_{11} w + D_{12} u \\ &= (C_1 - D_{12}\hat{B}_2^T X)\tilde{x} + C_1 e \to 0. \end{aligned}$$

Hence local output regulation is achieved. □

Corollary 5.3.1. *Suppose $x(t;x_0)$ exists for all $t \ge 0$. If $Q - \dfrac{d}{dt}X(x(t;x_0)) \geqslant \delta I$ for some $\delta > 0$ and for $t > T > 0$, then $z(t) \to 0$ as $t \to \infty$.*

Remark We can extend Theorem 4.2.1 and 5.3.2 to the system (5.22) where $\lambda_i(x)$ are continuously differentiable functions of x but does not satisfy (5.23). For example, van der Pol equation (5.16) can be written as the three linear models:

$$\dot{x} = \Big(A_0 + \sum_{i=1}^{r} \lambda_i(x) A_i\Big)x + Bu, \tag{5.36}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix}, \ \lambda_1(x) = 1, \ \lambda_2(x) = 1 - x_1^2, a_1 = -1, \ a_2 = 0.2.$$

By replacing $\hat{A} = \sum_{i=1}^{r} \lambda_i(x)\hat{A}_i$ in (5.24), (5.28) and (5.34) by $\left(A_0 + \sum_{i=1}^{r} \lambda_i(x)\hat{A}_i\right)$, Theorem 4.2.1 and 5.3.2 hold for this system.

5.3.3 Example

Example 5.3.1

Consider the nonlinear mass-spring system

$$\ddot{\xi} = -0.01\xi - 0.67\xi^3 + u, \tag{5.37}$$

The nonlinear term satisfies the following conditions for $\xi \in [-1 \ \ 1]$:

$$\begin{aligned} -0.67\xi \leqslant -0.67\xi^3 \leqslant 0\xi, \qquad \xi \geqslant 0, \\ 0\xi \leqslant -0.67\xi^3 \leqslant -0.67\xi, \qquad \xi \leqslant 0. \end{aligned}$$

Hence it can be represented by the following two linear models:

$$\dot{x} = \sum_{i=1}^{2} \lambda_i(x)(A_i x + B_i u), \tag{5.38}$$

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 1 \\ a_3 & a_4 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix},$$
$$\lambda_1(x) = 1 - x_1^2, \ \lambda_2(x) = x_1^2,$$
$$a_1 = -0.01, \ a_2 = 0, \ a_3 = -0.68, \ a_4 = 0.$$

Here a_1 and a_3 are regarded unknown and $\lambda_i(x)$ are given functions. Adaptive stabilization by (5.29) with $Q = I_2$ is considered. The simulation result with $x(0) = [0.8 \ \ 0]^T$, $[\hat{a}_1(0), \hat{a}_3(0)] = [0, 0]$ is shown in Figure 5.4.

Example 5.3.2

For the system (5.37) we design a state feedback controller such that $x_1(t) \to 0.5$. For this purpose we set $C_1 = 1$, $D_{11} = -1$ and take the following exosystem

$$S = 0, \ w(0) = 0.5.$$

In this case $\Pi = [1 \quad 0]^T$ and $\Gamma = -(1 - x_1^2)\hat{a}_1 - x_1^2\hat{a}_3$. The simulation result with $x(0) = [0.8 \quad 0]^T$, $[\hat{a}_1(0), \hat{a}_3(0)] = [0, 0]$ is shown in Figure 5.5.

Example 5.3.3

We design a state feedback controller for (5.37) such that $x_1(t) \to 0.3sin(t)$. In this case we set $C_1 = [1 \quad 0]$, $D_{11} = [-1 \quad 0]$ and take the following exosystem

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then $\Pi = I_2$ and $\Gamma = [-1 + (x_1^2 - 1)\hat{a}_1 - x_1^2\hat{a}_3, (x_1^2 - 1)\hat{a}_2 - x_1^2\hat{a}_4]$. The simulation result with $x(0) = [0.8 \quad 0]^T$, $w(0) = [0 \quad 0.3]^T$, $[\hat{a}_1(0), \hat{a}_3(0)] = [0, 0]$ is shown in Figures 5.6 and 5.7. In this example $\dot{X}$ does not converge to zero but the assumption in the Corollary 5.3.1 hold as shown in Figure 5.7.

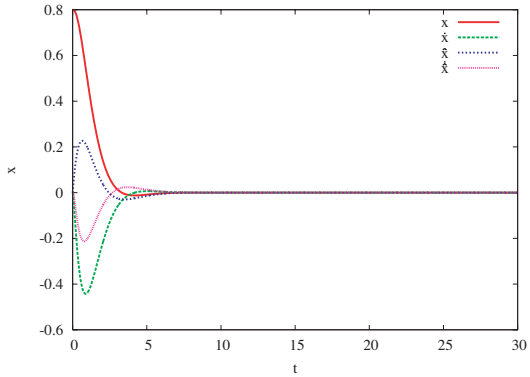


Figure 5.4: The trajectories of the state.

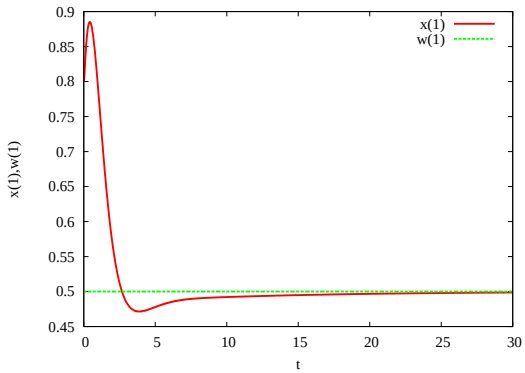


Figure 5.5: Step tracking.

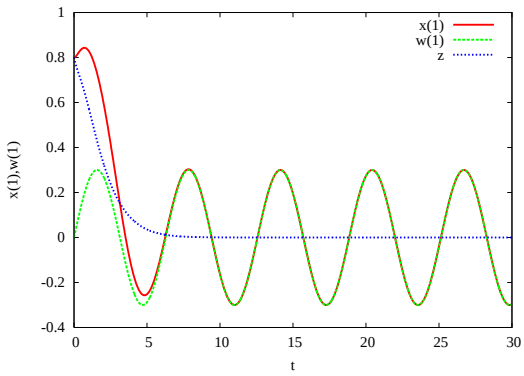


Figure 5.6: Sine tracking.

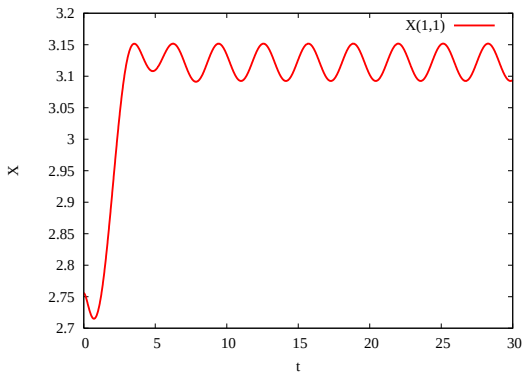


Figure 5.7: Solution of the Riccati equation.

Example 5.3.4

For the van der Pol equation (5.36), we design a state feedback controller such that $x_1(t) \to 2sin(t)$. Here a_1 and a_2 are regarded unknown and λ_i are given functions. In this case we set $C_1 = [1 \;\; 0]$, $D_{11} = [-1 \;\; 0]$ and take the following exosystem

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then the solution of the regulation equation (5.34) is

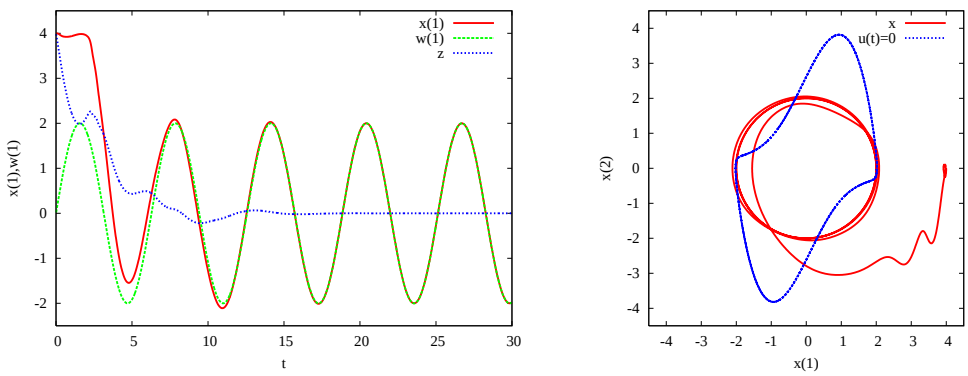
$$\Pi = I_2,$$

$$\begin{aligned} \Gamma &= \begin{bmatrix} -(\hat{a}_1 + 1)\lambda_1(x) & -\hat{a}_2\lambda_2(x) \end{bmatrix} \\ &= \begin{bmatrix} -(\hat{a}_1 + 1) & -\hat{a}_2(1 - x_1^2) \end{bmatrix}. \end{aligned}$$

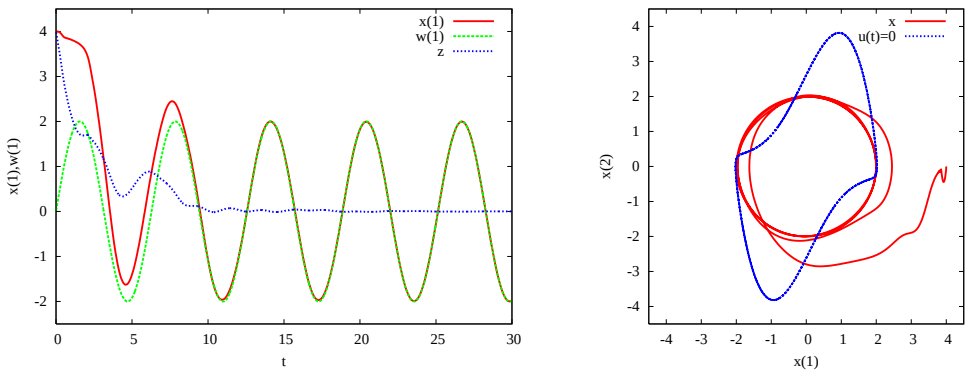
As in Example 5.2.1, we consider the controller

$$u = F(x)x + (\Gamma(x) + B_2^T(x)X(x)\Pi)w,$$

and choose three different types of feedback gain $F(x)$ as in Example 5.2.1. The simulation results with $x(0) = [4 \;\; 0]^T$, $w(0) = [0 \;\; 2]^T$ and $[\hat{a}_1(0), \hat{a}_2(0)] = [0, 0]$ are shown in Figure 5.8. The pole of the closed-loop system are shown in Figure 5.9 .

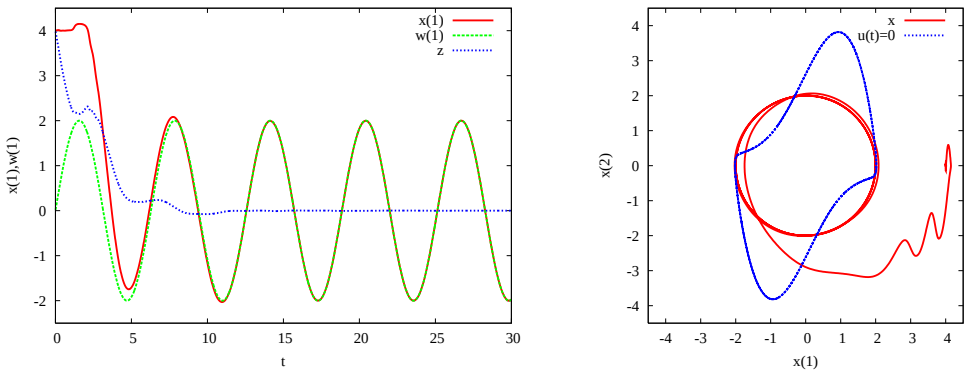


(a) SDRE control (Q(x) = diag([1, 1]))



(b) Pole placement

Figure 5.8: The trajectories of the state (left) and the trajectories of the state in the phase plane (right).



(b) H_∞ control

Figure 5.8: Continue

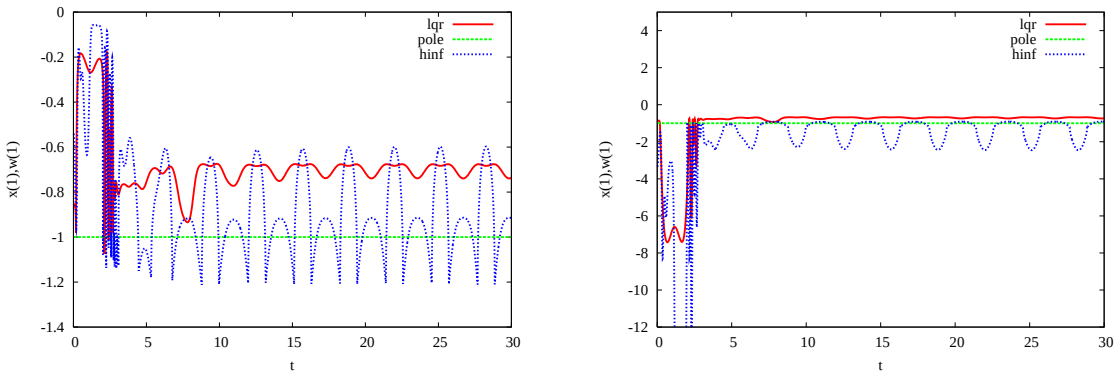


Figure 5.9: The closed loop poles.

5.4 Adaptive Regulation of Nonlinear Systems by Output Feedback

In this section we consider the observer-based adaptive regulation of (5.22). Consider the single-input/single-output system in the controllable canonical form,

$$\begin{aligned}
\dot{x} &= \sum_{i=1}^{r} \lambda_i(x) \left(\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & a_{in} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_i \end{bmatrix} \right) u, \\
y &= \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} x, \\
\sum_{i=1}^{r} \lambda_i(x) &= 1, \quad \lambda_i(x) \geq 0, \quad i = 1, \cdots, r.
\end{aligned} \tag{5.39}$$

(5.39) becomes

$$\begin{aligned}
\dot{x} &= Ax + B \sum_{i=1}^{r} \lambda_i(x)[a_i x + b_i u], \\
y &= Cx.
\end{aligned} \tag{5.40}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, a_i = \begin{bmatrix} a_{i1} & \cdots & a_{in} \end{bmatrix}. \tag{5.41}$$

a_i and $b_i \geq b_0 > 0$ are unknown, $\lambda_i(x)$ are locally Lipshitz given functions of x and y is the observation.

We consider the observer

$$\begin{aligned}
\dot{\hat{x}} &= A\hat{x} + B \sum_{i=1}^{r} \lambda_i(\hat{x})[\hat{a}_i \hat{x} + \hat{b}_i u] + J(y - \hat{y}), \\
\hat{y} &= C\hat{x},
\end{aligned} \tag{5.42}$$

where J is the observer gain, chosen such that $A - JC$ is Hurwitz because (C, A) is observable. Now we set

$$e \triangleq \hat{x} - x, \ \phi_i \triangleq \hat{a}_i - a_i, \ \psi_i \triangleq \hat{b}_i - b_i.$$

Then the error equation is given by

$$\begin{aligned}\dot{e} &= (A - JC)e + B\sum_{i=1}^{r}[\lambda_i(\hat{x})(\hat{a}_i\hat{x} + \hat{b}_i u) - \lambda_i(x)(a_i x + b_i u)] \\ &= (A - JC)e + B\big[\sum_{i=1}^{r}\lambda_i(\hat{x})(\phi_i\hat{x} + \psi_i u) + w_0\big], \\ e_1 &= Ce,\end{aligned} \tag{5.43}$$

where $w_0 = \sum_{i=1}^{r}\lambda_i(\hat{x})(a_i\hat{x} + b_i u) - \lambda_i(x)(a_i x + b_i u)$. (5.43) can be written as

$$e_1 = H(s)\big[\sum_{i=1}^{r}\lambda_i(x)(\phi_i\hat{x} + \psi_i u) + w_0\big], \tag{5.44}$$

where

$$H(s) = C[sI - (A - JC)]^{-1}B.$$

Note that $H(s)$ is a known stable transfer function. If $n > 1$, $H(s)$ cannot be SPR (Tao and Ioannou, 1988, 1990). To achieve SPR in the $n > 1$ case, following (Tong *et al.*, 2004, 2005; Leu *et al.*, 1999), a stable low pass filter $L^{-1}(s)$ is introduced into (5.44) as

$$e_1 = H(s)L(s)\big[\sum_{i=1}^{r}(\phi_i\hat{x}_{fi} + \psi_i u_{fi} + \epsilon_i) + w_f\big], \tag{5.45}$$

where

$$\begin{bmatrix}\hat{x}_{fi} \\ u_{fi}\end{bmatrix} = L^{-1}(s)\lambda_i(\hat{x})\begin{bmatrix}\hat{x} \\ u\end{bmatrix}, \tag{5.46}$$

$$\epsilon_i = L^{-1}(s)\lambda_i(\hat{x})(\phi_i\hat{x} + \psi_i u) - \phi_i\hat{x}_{fi} + \psi_i u_{fi} \tag{5.47}$$

$$w_f = L^{-1}(s)w_0, \tag{5.48}$$

and $L(s)$ is chosen so that $L^{-1}(s)$ is a proper stable transfer function and $H(s)L(s)$ is a proper SPR transfer function. Suppose that $L(s) = s^m + \beta_1 s^{m-1} + \cdots \beta_m$, where $m = n$ or $m = n - 1$. Then the state-space realization of (5.45) can be written as

$$\begin{aligned}\dot{e_c} &= A_c e_c + B_c\big[\sum_{i=1}^{r}(\phi_i\hat{x}_{fi} + \psi_i u_{fi} + \epsilon_i) + w_f\big], \\ e_1 &= C_c e_c.\end{aligned} \tag{5.49}$$

where

$$A_c = A - JC, B_c = \begin{bmatrix} 0 & \cdots & \beta_1 & \cdots & \beta_m \end{bmatrix}^T, C_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

For the given positive-definite matrix Q_c, there exists positive-definite solution P_c for the matrix equations

$$\begin{aligned} A_c^T P_c + P_c A_c &= -Q_c < 0, \\ P_c B_c &= C_c^T. \end{aligned} \tag{5.50}$$

We choose adaptive laws for adjusting $\hat{a}_i(t)$ and $\hat{b}_i(t)$

$$\begin{aligned} \dot{\hat{a}}_i &= \dot{\phi}_i = -e_1 \hat{x}_{fi}^T - k_1 \hat{a}_i, \\ \dot{\hat{b}}_i &= \dot{\phi}_i = \begin{cases} -e_1 u_{fi}^T - k_2 \hat{b}_i & \text{if } \hat{b}_i > b_0, \\ -e_1 u_{fi} - k_2 \hat{b}_i & \text{if } \hat{b}_i = b_0 \text{ and } e_1 u_f > 0, \\ 0 & \text{if } \hat{b}_i = b_0 \text{ and } e_1 u_f \leq 0, \end{cases} \end{aligned} \tag{5.51}$$

where $k_1 > 0$ and $k_2 > 0$.

Lemma 5.4.1. *If $|\hat{x}(t)| < c_1$ and $|u(t)| < c_2$, $|x(t)| < c_3$ for all $t \geq 0$, then (5.43), (5.49) and (5.51) is globally stable, provided the following conditions hold:*

$$\lambda_{min} Q_c > 2c^2, \quad k_1 > 2c_1^2, \quad k_2 > 2c_2^2. \tag{5.52}$$

Proof. Since $A - JC$ is stable matrix, for the given positive-definite Q_1 there exists positive-definite solution P for the matrix equation

$$(A - JC)^T P + P(A - JC) = -Q_1 < 0. \tag{5.53}$$

Consider the Lyapunov function candidate

$$V(e, e_c, \phi_i, \psi_i) = e^T P e + e_c^T P_c e_c + \sum_{i=1}^{r} (\phi_i^T \phi_i + \psi_i^T \psi_i), \tag{5.54}$$

where P and P_c is a solution for the matrix equations (5.53) and (5.50), respectively. The time derivative of (5.54) along the solutions of (5.49) is given by

$$\begin{aligned} \dot{V}(e, e_c, \phi_i, \psi_i) = e^T[(A - JC)^T P + P(A - JC)]e + 2e^T PB \sum_{i=1}^{r} \lambda(\hat{x})(\phi_i \hat{x} + \psi_i u) \\ +2e^T PBw_0 + e_c^T (A_c^T P_c + P_c A_c) e_c + 2e_c^T P_c B_c \sum_{i=1}^{r} (\phi_i \hat{x}_{fi} + \psi_i u_{fi} + \epsilon_i) \\ +2e_c^T P_c B_c w_f + 2 \sum_{i=1}^{r} (\dot{\phi}_i^T \phi_i + \dot{\psi}_i^T \psi_i). \end{aligned} \tag{5.55}$$

Now we consider the case $\hat{b}_i > b_0$ in (5.51). From (5.50), (5.53) and (5.51), (5.55) becomes

$$\dot{V}(e, e_c, \phi_i, \psi_i) \leq -e^T Q_1 e - e_c^T Q_c e_c + 2e^T PBw_0 + 2e_1 w_f + 2\sum_{i=1}^{r}(-k_1 \hat{a}_i^T \phi_i - k_2 \hat{b}_i \psi_i) + 2e_1 \sum_{i=1}^{r} \epsilon_i. \quad (5.56)$$

Since $|\hat{x}(t)| < c_1$, $|u(t)| < c_2$, $|x(t)| < c_3$, w_0, w_f and ϵ_i can be bounded as

$$\begin{aligned} |w_0| \leq & (c_1 + c_3) \sum |a_i| \triangleq m, \\ |w_f| \leq & m, \\ \sum_{i=1}^{r} |\epsilon_i| \leq & 2c_1 \sum_{i=1}^{r} |\phi_i| + 2c_2 \sum_{i=1}^{r} |\psi_i|. \end{aligned}$$

Using this bound, (5.56) can be bounded as

$$\begin{aligned} \dot{V}(e, e_c, \phi_i, \psi_i) \leq & -e^T Q_1 e - e_c^T Q_c e_c + 2\Bigg[m\|P\|\|B\||e| + m|e_1| - k_1 \sum_{i=1}^{r} (\phi_i + a_i)^T \phi_i \\ & - k_2 \sum_{i=1}^{r} (\psi_i + b_i)^T \psi_i + 2c_1 \sum_{i=1}^{r} |\phi_i||e_1| + 2c_2 \sum_{i=1}^{r} |\psi_i||e_1| \Bigg] \end{aligned} \quad (5.57)$$

From (5.49), $|e_1| \leq c|e_c|$, and hence

$$\begin{aligned} \dot{V}(e, e_c, \phi_i, \psi_i) \leq & -\lambda_{Q_1}|e|^2 - \lambda_{Q_c}|e_c|^2 + 2m\|P\|\|B\||e| + 2mc|e_c| \\ & - 2k_1 \sum_{i=1}^{r} (|\phi_i|^2 - |\phi_i||a_i|) \\ & - 2k_2 \sum_{i=1}^{r} (|\psi_i|^2 - |\psi_i||b_i|) \\ & + 4cc_1 \sum_{i=1}^{r} |\phi_i||e_c| + 4cc_2 \sum_{i=1}^{r} |\psi_i||e_c| \end{aligned}$$

$$
\begin{aligned}
\leq & -|e|\Big[\lambda_{Q_1}|e| - 2m\|P\|\|B\|\Big] - |e_c|\Big[(\lambda_{Q_c} - 2c^2)|e_c| - 2mc\Big] \\
& -\Big(c|e_c| - 2c_1\sum_{i=1}^{r}|\phi_i|\Big)^2 - \Big(c|e_c| - 2c_2\sum_{i=1}^{r}|\psi_i|\Big)^2 \\
& -\sum_{i=1}^{r}|\phi_i|\Big[2(k_1 - 2c_1^2)|\phi_i| - 2k_1|a_i|\Big] \\
& -\sum_{i=1}^{r}|\psi_i|\Big[2(k_2 - 2c_2^2)|\psi_i| - 2k_2|b_i|\Big] \\
\leq & -\Big(|e|\sqrt{\lambda_{Q_1}} - \frac{m\|P\|\|B\|}{\sqrt{\lambda_{Q_1}}}\Big)^2 + \frac{m^2\|P\|^2\|B\|^2}{\lambda_{Q_1}} \\
& -\Big(|e_c|\sqrt{\lambda_{Q_c} - 2c^2} - \frac{m}{\sqrt{\lambda_{Q_c} - 2c^2}}\Big)^2 + \frac{m^2}{\lambda_{Q_c} - 2c^2} \\
& -\Big(c|e_c| - c_1\sum_{i=1}^{r}|\phi_i|\Big)^2 - \Big(c|e_c| - c_2\sum_{i=1}^{r}|\psi_i|\Big)^2 \\
& -\sum_{i=1}^{r}\Big(|\phi_i|\sqrt{2(k_1 - 2c_1^2)} - \frac{k_1|a_i|}{\sqrt{2(k_1 - 2c_1^2)}}\Big)^2 + \sum_{i=1}^{r}\frac{k_1^2|a_i|^2}{2(k_1 - 2c_1^2)} \\
& -\sum_{i=1}^{r}\Big(|\psi_i|\sqrt{2(k_2 - 2c_2^2)} - \frac{k_2|b_i|}{\sqrt{2(k_2 - 2c_2^2)}}\Big)^2 + \sum_{i=1}^{r}\frac{k_2^2|b_i|^2}{2(k_2 - 2c_2^2)}
\end{aligned}
$$

where λ_Q means the minimum eigenvalues of Q. Under the condition (5.52), the requirement $\dot{V} \leq 0$ holds outside an ellipsoid in the space of the error variables e, e_c, ϕ_i and ψ_i. In the case $\hat{b}_i = b_0$ and $e_1 u_f \leq 0$, additional term $\sum_{i=1}^{r}\psi_i e_1 u_f$ appears in (5.57). Since $\sum_{i=1}^{r}\psi_i e_1 u_f \leq 0$, $\dot{V} \leq 0$ also holds outside an ellipsoid in the space of the error variables e, e_c, ϕ_i and ψ_i. It follows that e, e_c, e_1, ϕ_i and ψ_i are bounded for all $t \geq 0$. □

Now, we consider the output feedback stabilization of the nonlinear system (5.40). Let Q be positive-definite. Since $\Big(A + B\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{a}_i,\ \sum_{i=1}^{r}\lambda_i(\hat{x})\hat{b}_i\Big)$ is stabilizable and (C, $A + B\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{a}_i$) is observable for each t there exists a positive stabilizing solution $X(\hat{x}, \theta)$ of the algebraic Riccati equation

$$
(A + B\sum_{i=1}^{r}\lambda_i\hat{a}_i)^T X + X(A + B\sum_{i=1}^{r}\lambda_i\hat{a}_i) + Q - X(\sum_{i=1}^{r}\lambda_i\hat{b}_i)(\sum_{i=1}^{r}\lambda_i\hat{b}_i)^T X = 0. \quad (5.58)
$$

Now we set

$$A + B\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{a}_i \triangleq \hat{A}, \quad B\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{b}_i \triangleq \hat{B}.$$

and introduce the control law

$$u = -\hat{B}^T(t)X(t)\hat{x} \tag{5.59}$$

and consider the stability of the adaptive control system.

Theorem 5.4.1. *For sufficiently small $\hat{x}(0)$, $e(0)$ and $\theta(0)$, $x(t)$, $\hat{x}(t)$, $e(t)$, $e_1(t)$, $\theta(t)$ and $y(t)$ are bounded for all $t \geq 0$, provided (5.52) hold.*

Proof. Substituting (5.59) into (5.40) and (5.42), we have

$$\begin{aligned}
\dot{x} =&(\hat{A} - \hat{B}_2\hat{B}_2^T X)x - B\Big[\sum_{i=1}^{r}\lambda_i(\hat{x})\phi_i + \sum_{i=1}^{r}(\lambda_i(\hat{x}) - \lambda_i(x))a_i\Big]x \\
&+\Big[\sum_{i=1}^{r}(\lambda_i(x)b_i - \lambda_i(\hat{x})\hat{b}_i)\Big]\Big[\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{b}_i\Big]^T X\hat{x}+\Big[\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{b}_i\Big]\Big[\sum_{i=1}^{r}\lambda_i(\hat{x})\hat{b}_i\Big]^T Xe. \\
\triangleq&\hat{A}_f x + E_1 x + E_2\hat{x} + E_3 e \qquad (5.60)\\
\dot{\hat{x}} =&(\hat{A} - \hat{B}\hat{B}^T X)\hat{x} - JCe, \\
\triangleq&\hat{A}_f\hat{x} - J_1 e, \qquad (5.61)
\end{aligned}$$

where we have suppressed $\hat{x}$ and t in $\hat{A}$ and $\hat{B}$ and X and so on. Consider $V_1 = x^T X(\hat{x},\theta)x$. The time derivative of V_1 along the solution of (5.60) is given by

$$\begin{aligned}
\dot{V}_1 \leq& - x^T\big[Q - (\nabla_{\hat{x}}\otimes X)(I_n\otimes\hat{x})(\hat{A}_f\hat{x} - J_1 e) - (\nabla_\theta\otimes X)(I_{2r}\otimes\theta)\dot{\theta}\big]x \\
&+ 2x^T XE_1 x + 2x^T XE_2\hat{x} + 2x^T XE_3 e \\
\leq& - x^T\big[Q - (\nabla_{\hat{x}}\otimes X)(I_n\otimes\hat{x})(\hat{A}_f\hat{x} - J_1 e) - (\nabla_\theta\otimes X)(I_{2r}\otimes\theta)\dot{\theta} \\
&- XE_1 - \beta_1\|XE_2E_2^T X\| - \beta_2\|XE_3E_3^T X\|\big]x + \frac{1}{\beta_1}|\hat{x}|^2 + \frac{1}{\beta_2}|e|.
\end{aligned}$$

There exist ϵ, α, β_1, $\beta_2 > 0$ such that $|e|, |\theta| \leq \epsilon$ and $V_1(x) \leq \alpha$ imply

$$\dot{V}_1 \quad\leq\quad -\delta V_1 + \frac{1}{\beta_1}\hat{x}^T\hat{x} + \frac{1}{\beta_2}|e|^2 \leq -\delta V_1 + \frac{\gamma}{\beta_1}\hat{x}^T X\hat{x} + \frac{1}{\beta_2}|e|^2 \tag{5.62}$$

and $\dfrac{\epsilon^2}{\beta_2\delta} < \alpha$, where $\gamma > 0$ is a positive constant. Then there exists a $\rho > 0$ such that $\dfrac{\gamma\rho}{\beta_1\delta} + \dfrac{\epsilon^2}{\beta_2\delta} < \alpha$ and (5.62) implies

$$V_1 \leq -e^{-\delta t}V_1(x_0) + \frac{\gamma}{\beta_1\delta}\hat{x}^T X\hat{x} + \frac{1}{\beta_2\delta}|e|^2 \leq V_1(x_0) + \frac{\gamma\rho}{\beta_1\delta} + \frac{\epsilon^2}{\beta_2\delta}. \tag{5.63}$$

Now choose α_0 such that $\alpha_0 + \dfrac{\gamma\rho}{\beta_1\delta} + \dfrac{\epsilon^2}{\beta_2\delta} < \alpha$ and x_0 such that $V_1(x_0) \leq \alpha_0$.

To guarantee $\hat{x}^T X \hat{x} < \rho$, consider also $V_2 = \hat{x}^T X(\hat{x}, \theta)\hat{x}$. The time derivative of V_2 along the solution of (5.61) is given by

$$\begin{aligned}\dot{V}_2 \leq &- \hat{x}^T \big[Q - (\nabla_x \otimes X)(I_n \otimes x)(\hat{A}_f \hat{x} - J_1 e) - (\nabla_\theta \otimes X)(I_{2r} \otimes \theta)\dot{\theta}\big]\hat{x} + 2\hat{x}^T X J_1 e \\ \leq &- \delta V_2 + \frac{1}{\beta_3}|e|^2, \end{aligned} \tag{5.64}$$

where $\beta_3 > 0$ and conditions for (5.62) are assumed. Choose ρ_0 such that $\rho_0 + \dfrac{\epsilon^2}{\beta_3\delta} < \rho$ and $\hat{x}_0$ such that $V_2(\hat{x}_0) \leq \rho_0$. Then by (5.64),

$$V_2 \leq V_2(\hat{x}_0) + \frac{|e|^2}{\beta_3\delta} \leq \rho_0 + \frac{\epsilon^2}{\beta_3\delta} < \rho \tag{5.65}$$

Combining this and (5.63), we obtain $V_1(x) \leq \alpha$. Thus the solution starting from x_0, $\hat{x}_0$ stays in $\Omega = \{x, \hat{x} \in \mathbb{R}^n |\ V_1(x) \leq \alpha, V_2(\hat{x}) \leq \rho\}$. The boudedness of x guarantees that $y = Cx$ is also bounded. $\square$

5.4.1 Example

Example 5.4.1

Consider the nonlinear mass-spring system:

$$\begin{aligned} \ddot{\xi} &= -0.01\xi - 0.67\xi^3 + u, \\ y &= \xi. \end{aligned}$$

It can be represented by the following two linear models:

$$\dot{x} = \sum_{i=0}^{2} \lambda_i(x)(A_i x + B_i u), \tag{5.66}$$

where

$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ a_1 & a_2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 0 & 0 \\ a_3 & a_4 \end{bmatrix}, B_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix},$$

$$\lambda_1(x) = 1 - x_1^2, \ \lambda_2(x) = x_1^2,$$

$$a_1 = -0.01, \ a_2 = 0, \ a_3 = -0.68, \ a_4 = 0.$$

Here a_1 and a_3 are regarded unknown and $\lambda_i(x)$ are given functions. When the state is available, we can design adaptive stabilization controller by (5.24), (5.25) and (5.29) with $Q = I_2$. The simulation result with $x(0) = [0.8 \quad 0]^T$, $[\hat{a}_1(0), \hat{a}_3(0)] = [0, 0]$ is shown in Figure 5.10.

When available information is the measurement $y = \xi$ only, we have to design observer (5.42). We choose observer gain in (5.42) as $J = [2 \quad 1]$. In this case $H(s)$ becomes $\dfrac{1}{s^2 + 2s + 1}$ and we choose $L(s) = s + 1$. Then calculate the filtered state and input by (5.46) and use these in adaptive laws (5.51). Adaptive stabilization by (5.42), (5.51) and (5.59) with $Q = I_2$ is considered. The simulation result with $x(0) = [0.8 \ \ 0]^T$, $[\hat{a}_1(0), \hat{a}_3(0)] = [0, 0]$ is shown in Figure 5.11.

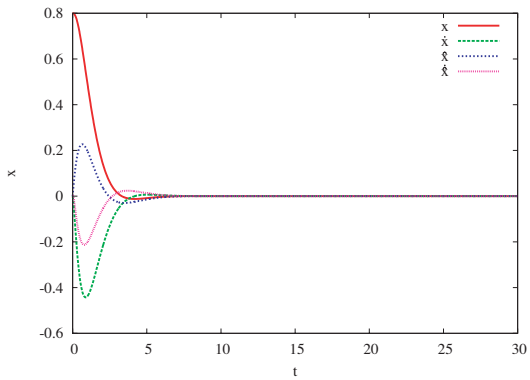


Figure 5.10: The trajectories of the state and its estimate by state feedback.

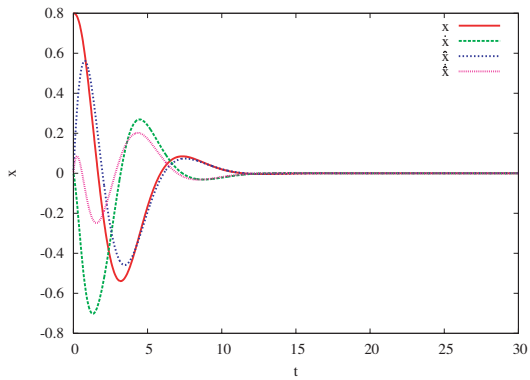


Figure 5.11: The trajectories of the state and its estimate by output feedback.

Example 5.4.2

We can extend Theorem 5.3.1 and 5.4.1to the system (5.22) where $\lambda_i(x)$ are continuously differentiable functions of x but does not satisfy $\sum_{i=1}^{r} \lambda_i(x) = 1,\ \lambda_i(x) \geq 0$. For example, van der Pol equation:

$$\ddot{\xi} \;=\; 0.2(1-\xi^2)\dot{\xi} - \xi + u.$$

can be written as the three linear models:

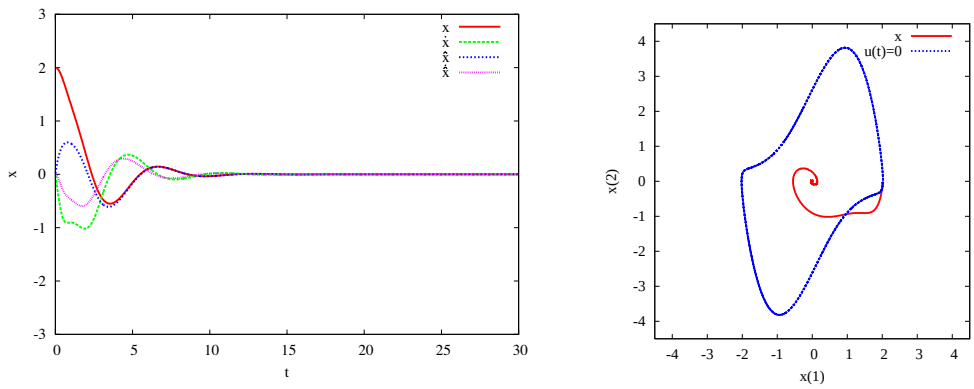
$$\dot{x} = \Big(A_0 + \sum_{i=1}^{r} \lambda_i(x) A_i\Big) x + Bu, \tag{5.67}$$

where

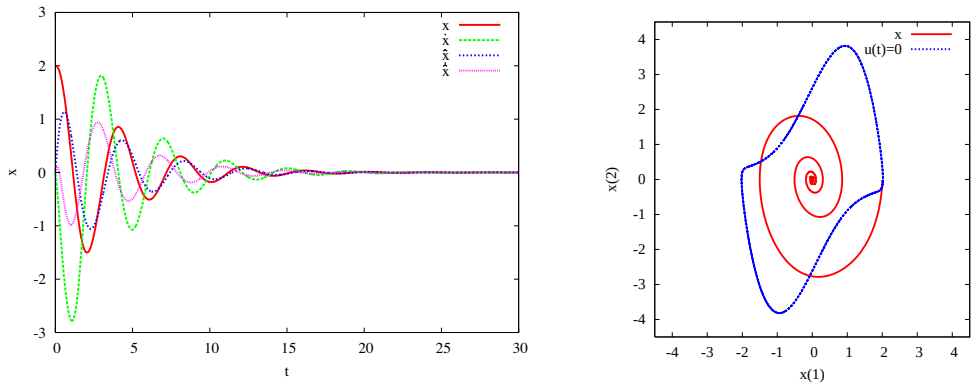
$$A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ a_1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & a_2 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \xi \\ \dot{\xi} \end{bmatrix},\ \lambda_1(x) = 1,\ \lambda_2(x) = 1 - x_1^2, a_1 = -1,\ a_2 = 0.2.$$

Here a_1 and a_2 are regarded unknown and $\lambda_i(x)$ are given functions. When the state is available, the simulation results with $x(0) = [2\ \ 0]^T$, $[\hat{a}_1(0), \hat{a}_3(0)] = [0, 0]$ by (5.24), (5.25) and (5.29) with $Q = I_2$ is shown in Figure 5.12(a).

When available information is the measurement y only, We choose observer gain in (5.42) as $J = [2\ \ 1]$ and $L(s) = s + 1$. The simulation results with $x(0) = [0.8\ \ 0]^T$, $[\hat{a}_1(0), \hat{a}_2(0)] = [0, 0]$ by (5.42), (5.51) and (5.59) with $Q = I_2$ is shown in Figure 5.12(b).



(a) Adaptive regulation by state feedback



(b) Adaptive regulation by output feedback

Figure 5.12: The trajectories of the state (left) and the trajectories of the state in the phase plane (right).

5.5 Conclusion

In this chapter, adaptive output regulation for nonlinear systems described by multiple linear models with unknown parameters is considered. First we designed a local stabilizing controller for affine nonlinear system using the solution of the state dependent Riccati equation. Then local output regulation was established using a state dependent regulator equation. As an example, we designed an controller for the nonlinear system described by van der Pol equation.

Second, locally stabilizing adaptive state-feedback controllers for nonlinear systems described by multiple linear models with unknown parameters are designed based on the Lyapunov stability theory. Then local adaptive output regulation is established using the property of the solution of the state dependent regulator equation.

Finally, we extend our method to output feedback control. Adaptive regulation by output feedback is considered for a class of single-input/single-output nonlinear systems described by multiple linear models. The adaptive laws are derived from Lyapunov stability analysis which guarantees that observer error and parameter estimation error are bounded provided that the state and the control are bounded.

We designed an adaptive controller for the nonlinear mass-dumper system and van der Pol equation.

Chapter 6

Concluding Remarks

This thesis has considered the adaptive control theory and its applications, where adaptive control is useful. The main contributions are summarized as follows.

In Chapter 2, we propose new learning algorithm based on Lyapunov design methods applicable in practical problems. We tested the performance of the proposed method both in simulations and experiments. It is shown that multiple modules are successfully trained and specialized for different domains in the state space in a cooperative way. Furthermore, the control system which consists of several online modules is applied to the autonomous flight control system of aero-robot, and we evaluated our method by a flight experiment. These results show that the proposed method can be applied to control various autonomous robots.

In Chapter 3, we review Riccati equation and regulator equation from linear theory and show the regularity properties of solution of the algebraic Riccati equation and the regulator equation with respect to system matrices.

In Chapter 4, the output regulation problem for linear time-invariant systems with unknown parameters was considered. Based on the Lyapunov stability theory, a stabilizing adaptive controller was derived. It was shown that an adaptive controller can be designed using the solution of the Riccati equation if the derivative of the solution is sufficiently small. Then sufficient conditions for the output regulation problem with full information to be solvable are established. Furthermore, the condition on the solution of the Riccati equation imposed above was relaxed introducing normalized adaptive laws. Simulation results were given to illustrate the theory.

In Chapter 5, adaptive output regulation for nonlinear systems described by multiple linear models with unknown parameters is considered. First we designed a local stabilizing controller for affine nonlinear system using the solution of the state dependent Riccati equation. Then local output regulation was established using a state dependent regulator equation. As an example, we designed an controller for the non-

linear system described by van der Pol equation.

Second, locally stabilizing adaptive state-feedback controllers for nonlinear systems described by multiple linear models with unknown parameters are designed based on the Lyapunov stability theory. Then local adaptive output regulation is established using a state dependent regulator equation.

Finally, we extend our method to output feedback control. Adaptive regulation by output feedback is considered for a class of single-input/single-output nonlinear systems described by multiple linear models. The adaptive laws are derived from Lyapunov stability analysis which guarantees that observer error and parameter estimation error are bounded provided that the state and the control are bounded. Simulation results are given to illustrate the theory.

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Published papers

A. Refereed Journal Papers

- M. Bando and A. Ichikawa: Adaptive Regulation of Nonlinear Systems by Output Feedback, (submitted)
- M. Bando and A. Ichikawa: Adaptive Output Regulation of Nonlinear Systems described by Multiple Linear Models, (submitted)
- M. Bando and A. Ichikawa: Adaptive Output Regulation for Linear Systems, (submitted)
- M. Bando and H. Nakanishi: A Stable Approach for Modular Learning and Its Application to Autonomous Aero-Robot, Journal of Robotics and Mechatronics, Vol. 18, No. 1, pages 44–50, 2006

B. Refereed Conference and Workshop Papers

- M. Bando and A. Ichikawa: Adaptive Output Regulation of Nonlinear Systems described by Multiple Linear Models Proceedings of 9th IFAC Workshop on Adaptation and Learning in Control and Signal Processing 2007 (ALCOSP'2007), 2007
- M. Bando and A. Ichikawa: Adaptive Output Regulation for Linear Systems Proceedings of SICE-ICASE International Joint Conference 2006 (SICE-ICCAS 2006), pages 3217–3222, 2006
- M. Bando, H. Nakanishi: Modular Learning Based on Lyapunov Design Method for Control System of Autonomous Robot, Proceedings of the IASTED International Conference on Artificial Intelligence and Applications (AIA 2005), pages 609–614, 2005
- M. Bando, H. Nakanishi, and K. Inoue: A Study on Designing Control System by Modular Learning, Proceedings of SICE Annual Conference 2004, pages 1071–1074, 2004

C. Conference and Workshop Papers

- M. Bando and H. Nakanishi: Adaptive Formulation of Module's Relations in Maneuvring Control of UAVs, Symposium on FRONTIER SCIENCES Kyoto University and Technische Universitat Munchen WS6, Oct. 7, 2005
- M. Bando and H. Nakanishi: Design of Adaptive and Learning Controllers for Autonomous Flight of an Aero-robot, Robotics and Mechatronics Conference 2005 (ROBOMEC2005), CD-ROM, 2005 (in Japanese)
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