On the continuity of positive definite functions on conelike semigroups (Communication in commutative Banach algebras and several fields of mathematics)

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Citation
数理解析研究所講究録 (2006), 1478: 127-132

Issue Date
2006-04

URL
http://hdl.handle.net/2433/58001

Type
Departmental Bulletin Paper

Textversion
publisher

Kyoto University
On the continuity of positive definite functions on conelike semigroups

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Dedicated to the memory of Knud Maack Bisgaard

2000 Mathematics Subject Classification. Primary: 43A35; Secondary: 44A60
Keywards and phrases. continuity, positive definite, moment, conelike semigroup

Abstract

Let $S$ be a conelike semigroup in $\mathbb{Q}^k$. In [5], P. Ressel showed an integral representation of bounded positive definite functions on $S$ which is continuous at 0. In this paper, we will analyze some integral representations of unbounded positive definite functions on $S$ which is continuous at 0.

1 Introduction

Let $S$ be an abelian semigroup with the identity 0. A function $\varphi : S \to \mathbb{R}$ is called positive definite if

$$\sum_{j,k=1}^{n} c_{j} \overline{c_{k}} \varphi(s_{j} + s_{k}) \geq 0$$

for all $n \in \mathbb{N}, s_{1}, \ldots, s_{n} \in S, c_{1}, \ldots, c_{n} \in \mathbb{R}$. A function $\sigma : S \to \mathbb{R}$ is called a character if it is multiplicative and not identically zero. In particular, if $0 \notin \sigma(S)$, $\sigma$ is called zero-free. The set of characters on $S$ is denoted by $S^*$. Denote by $A(S^*)$ the least $\sigma$-ring of subsets of $S^*$ rendering the mapping $S^* \ni \sigma \mapsto \sigma(s) \in \mathbb{R}$ measurable for each $s \in S$. A function $\varphi : S \to \mathbb{R}$ is called a moment function if there is a measure $\mu$ defined on $A(S^*)$ such that

$$\varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma)$$

for all $s \in S$. Note that every moment function is positive definite and every bounded positive definite function on $S$ is a moment function whose representing
measure is unique (see [1], Theorem 4.2.8). But a positive definite function is not necessarily a moment function (see [1], Theorem 6.3.5), and a representing measure is not necessarily unique if any (see [1], Example 6.4.3).

An abelian *-semigroup $S$ is called determinate if whenever $\mu$ and $\nu$ are measures on $\mathcal{A}(S^*)$ such that

$$\int_{S^*} \sigma(s)d\mu(\sigma) = \int_{S^*} \sigma(s)d\nu(\sigma), \ s \in S$$

then $\mu = \nu$. The semigroup $S$ is called semiperfect if every positive definite function $\varphi : S \to \mathbb{R}$ is a moment function, and perfect if $S$ is semiperfect and determinate.

A subset $M$ of a vector space over the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$) is called conelike if for each $s \in M$ there is some $a \in \mathbb{K}$ such that $\alpha s \in M$ for all $\alpha \in \mathbb{K}$ satisfying $\alpha \geq a$.

P. Ressel has proved the following theorem (see [5], Theorem 2):

**Ressel's Theorem** Let $S$ be a conelike semigroup in the real vector space $\mathbb{R}^k$, $k \geq 1$, with $S \neq \emptyset$ and $0 \in \overline{S}$, where $\overline{S} := \{s \in S \mid (\mathbb{R}_+s) \cap s \neq \emptyset\}$. For a bounded positive definite function $\varphi : S \to \mathbb{R}$ the following properties are equivalent:

(i) $\varphi$ is uniformly continuous.

(ii) $\varphi$ is continuous at 0.

(iii) $\exists \{s_n\} \subset \overline{S}$ with $s_n \to 0$ and $\varphi(s_n) \to \varphi(0)$.

(iv) There is a bounded nonnegative measure $\mu$ on $S^\square$ such that $\varphi(s) = \int_{S^\square} e^{-\langle v,s \rangle}d\mu(v), \ s \in S$, where $S^\square := \{v \in \mathbb{R}^k \mid \langle v,s \rangle \geq 0 \text{ for all } s \in S\}$.

It is natural to consider this theorem for unbounded positive definite functions. In general, every unbounded positive definite function is not a moment function. But every conelike semigroup in the rational vector space $\mathbb{Q}^k$, $k \geq 1$, is perfect (see [4], Theorem 3.3, [2], Theorem 6). In section 3, we will prove a Ressel-type theorem for unbounded positive definite functions on conelike semigroups in $\mathbb{Q}$. In section 4, we will show that such a Ressel-type theorem in $(\mathbb{Q}_+ \setminus \{0\})^2 \cup \{(0,0)\}$ does not hold. In section 5, for some conelike semigroups in $\mathbb{Q}^k$, we will prove that the implication (ii) $\Rightarrow$ (iv) holds.

Throughout this paper, an abelian semigroup $S$ in $\mathbb{Q}^k$ (or $\mathbb{R}^k$) is conelike, and the composition on $S$ is the ordinary addition. See [1] for other details on positive definite and moment functions, and see [3] on positive definite functions on conelike semigroups.

## 2 Pleliminaries

In this section, we will determine explicitly the zero-free characters on $S$ with $\overline{S}_Q \neq \emptyset$, where $\overline{S}_Q$ is the interior of $S$ in the rational vector space $\mathbb{Q}^k$ with the relative topology. This argument is similar to P. Ressel's (cf. [5]).
Proposition 1 Let $S$ be a conelike subsemigroup of $\mathbb{Q}^k$ with $\hat{S}_Q \neq \emptyset$. Then every zerofree character $\sigma \in S^*$ is of the form
\[ \sigma(s) = \exp(v, s) \]
for some $v \in \mathbb{R}^k$.

Put $\overline{S}_Q := \{ s \in S \mid (\mathbb{Q}_+ s) \cap \hat{S}_Q \neq \emptyset \}$. The set $\overline{S}_Q$ contains $\hat{S}_Q$. By the similar proof of [5], Lemma 3, we have the following.

Lemma 2 Let $S$ be a conelike subsemigroup of $\mathbb{Q}^k$ with $\hat{S}_Q \neq \emptyset$, and $\sigma \in S^*$ is not zerofree. Then $\sigma \equiv 0$ on $\overline{S}_Q$, in particular on $\hat{S}_Q$.

Define the sets
\[ W := \{ \sigma \in S^* \mid \sigma \text{ is zerofree} \}, \]
\[ N := \{ \sigma \in S^* \mid \sigma \text{ is not zerofree} \}. \]

If $\hat{S}_Q \neq \emptyset$, by Proposition 1, $W$ is topological semigroup isomorphic to $\mathbb{R}^k$ by the correspondence
\[ \varphi : (s \mapsto \exp(v, s)) \mapsto v. \]

Since $S$ is perfect, every positive definite function $\varphi$ on $S$ has the following integral representation with the unique measure $\mu$ on $S^*$:
\[ \varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma), \quad s \in S. \]

Since every character $\sigma \in N$ is identically zero on $\hat{S}_Q$ by Lemma 2, then
\[ \varphi(s) = \int_{\mathbb{R}^k} \exp(v, s) d\nu(v), \quad s \in \hat{S}_Q, \]

where $\nu$ is the image measure defined by $\nu := \mu'$.

3 In the Case of $S$ in $\mathbb{Q}$

In the case of $S \subseteq \mathbb{Q}$ with $\hat{S}_Q \neq \emptyset$, it is easily obtained that $S^* = W \cup N = W \cup \{1_{\{0\}}\}$, where $1_{\{0\}}$ is the indicator function of $\{0\}$. We have the following:

Theorem 3 Let $S$ be a conelike semigroup in the rational vector space $\mathbb{Q}$ with $\hat{S}_Q \neq \emptyset$ and $0 \in \overline{S}_Q$. For a positive definite function $\varphi : S \to \mathbb{R}$ the following properties are equivalent:

(i) $\varphi$ is continuous.

(ii) $\varphi$ is continuous at 0.

(iii) $\exists \{s_n\} \subset \overline{S}_Q$ with $s_n \to 0$ and $\varphi(s_n) \to \varphi(0)$. 

(iv) There is a nonnegative measure \( \nu \) on \( \mathbb{R} \) such that \( \varphi(s) = \int_{\mathbb{R}} e^{uv} d\nu(v), \) \( s \in S. \)

**Corollary 4** Let \( S \) be a conelike semigroup in the real vector space \( \mathbb{R} \) and define \( S_{\mathbb{Q}} := S \cap \mathbb{Q} \). Suppose that \( \bar{S} \neq \emptyset \), \( 0 \in \overline{S_{\mathbb{Q}}} \) and \( S = \overline{S_{\mathbb{Q}}} \). Then a function \( \varphi : S \to \mathbb{R} \) is continuous and positive definite if and only if there exists a nonnegative measure \( \nu \) on \( \mathbb{R} \) such that

\[
\varphi(s) = \int_{\mathbb{R}} e^{uv} d\nu(v), \quad s \in S.
\]

**4 In the Case of \( S \) in \( \mathbb{Q}^{2} \)**

In the case of \( S \) in \( \mathbb{Q} \), we proved a Ressel-type theorem for unbounded positive definite functions. But, in the case of \( S \) in \( \mathbb{Q}^{2} \), a Ressel-type theorem such as Theorem 3 does not hold. In this section, we will show some counterexamples. Throughout this section, let \( S \) be the abelian semigroup \( (\mathbb{Q}_{+} \setminus \{0\})^{2} \cup \{(0,0)\} \).

**Example 1** (Counterexample of (iv) \( \Rightarrow \) (ii)) For each \( k \in \mathbb{N} \), define \( v_{k} \in \mathbb{R}^{2} \) by \( v_{k} = (k, -k^2) \). Let \( m \) be the measure \( \sum_{k=1}^{\infty} \frac{1}{k^2} \delta_{v_{k}} \) on \( \mathbb{R}^{2} \), where \( \delta_{v_{k}} \) is the Dirac measure supported by \( \{v_{k}\} \). Define

\[
\varphi(x,y) := \int_{\mathbb{R}^{2}} e^{(v_{k},(x,y))} dm(v) = \sum_{k=1}^{\infty} \frac{1}{k^{2}} e^{kx-k^{2}y} < \infty, \quad (x,y) \in S.
\]

Now \( \varphi \) is not continuous at \( (0,0) \). In fact, let \( \{x_{n}\} \) be any sequence of positive numbers tending to 0. For each \( n \), since \( \varphi(x_{n},y) \to \infty \) as \( y \to 0 \), we can choose \( y_{n} \) such that \( 0 < y_{n} < \frac{1}{n} \) and \( \varphi(x_{n},y_{n}) > n \). Then \( (x_{n},y_{n}) \to (0,0) \) but \( \varphi(x_{n},y_{n}) \to \infty \).

**Example 2** (Counterexample of (iii) + (iv) \( \Rightarrow \) (ii)) Let \( \varphi \) be the function as above. We only have to show that there is a sequence \( \{s_{n}\} \) in \( S_{\mathbb{Q}} \) such that \( s_{n} \to 0 \) and \( \varphi(s_{n}) \to \varphi(0) \) as \( n \to \infty \). For each \( n \in \mathbb{N} \), define a continuous mapping \( \gamma_{n} \) on \((-1,1)\) by \( \gamma_{n}(-t) = \left( \frac{1-t}{n}, \frac{1}{n} \right) \) and \( \gamma_{n}(t) = \left( \frac{1}{n}, \frac{1-t}{n} \right) \) for \( 0 \leq t < 1 \). We can easily prove that \( \varphi(\gamma_{n}(t)) \downarrow \sum_{k=1}^{\infty} \frac{1}{k^{2}} e^{k(t) \frac{k^2}{n}} < \varphi(0) \) and \( \varphi(\gamma_{n}(t)) \to \infty \) as \( 0 \leq t \uparrow 1 \). By continuity, we can choose \( t_{n} \in (-1,1) \cap \mathbb{Q} \) such that \( \varphi(\gamma_{n}(t_{n})) = \varphi(0) \). Putting \( s_{n} = \gamma_{n}(t_{n}) \in S_{\mathbb{Q}} \), we have that \( s_{n} = \gamma_{n}(t_{n}) \to 0 \) and \( \varphi(s_{n}) = \varphi(\gamma_{n}(t_{n})) = \varphi(0) \). Then we can obtain the result.

**Example 3** (Counterexample of (iii) \( \Rightarrow \) (iv)) Let \( \varphi \) and \( \{s_{n}\} \) be as above, and let \( \mu \) be the representing measure of \( \varphi \) on \( S^{*} \). Choose a number \( \alpha \) such that \( \sum_{k=1}^{\infty} \frac{1}{k^{2}} e^{k(t) \frac{k^2}{n}} < \alpha < \varphi(0) \). Define the function \( \psi \) as follows:

\[
\psi(x,y) = \begin{cases} 
\varphi(x,y) \quad ((x,y) \in S \setminus \{(0,0)\}) \\
\alpha \quad ((x,y) = (0,0))
\end{cases}
\]
Then \( \psi \) is positive definite on \( S \). By the similar argument to take \( \{t_n\} \), we can choose \( t_n \in (-1,1) \cap \mathbb{Q} \) such that \( \psi(\gamma_n(t_n)) = \alpha \). Putting \( s_n = \gamma_n(t_n) \in S^*_\mathbb{Q} \), we have that \( s_n = \gamma_n(t_n) \to 0 \) and \( \psi(s_n) = \psi(\gamma_n(t_n)) = \psi(0) \). But the support of the representing measure of \( \psi \) contains \( \{1_{\{0\}}\} \). In fact, Since \( \psi \) is a moment function on \( S \), there exists the measure \( \mu_0 \) on \( S^* \) such that

\[
\psi(s) = \int_{S^*} \sigma(s) d\mu_0(\rho), \quad s \in S.
\]

Put \( H := S \setminus \{(0,0)\} \). By [6], Lemma 2.2, the mapping \( f : \sigma \mapsto \sigma|_H \) is a one-to-one correspondence between \( S^* \setminus \{1_{\{0\}}\} \) and \( H^* \). Let \( \tilde{\mu} \) and \( \overline{\mu}_0 \) be the images of \( \mu \) and \( \mu_0 \), respectively, i.e., \( \tilde{\mu} = \mu^f \) and \( \overline{\mu}_0 = \mu_0^f \). For \( s \in H \),

\[
\int_{H^*} \sigma(s) d\overline{\mu}_0(\sigma) = \int_{S^*} \sigma(s) d\mu_0(\sigma) = \psi(s) = \varphi(s) = \int_{S^*} \sigma(s) d\mu(\sigma) = \int_{H^*} \sigma(s) d\tilde{\mu}(\sigma).
\]

By [6], Theorem 3.2, \( H \) is perfect (see [6] for the definition of perfectness of \( H \)). By [6], Proposition 3.1, \( \tilde{\mu} = \overline{\mu}_0 \) on \( H^* \). Suppose \( \mu_0(\{1_{\{0\}}\}) = 0 \), then \( \mu = \mu_0 \) on \( S^* \), hence \( \varphi = \psi \) on \( S \). This contradicts to \( \varphi \neq \psi \). Therefore \( \mu_0(\{1_{\{0\}}\}) \neq 0 \).

5 In the case of \( S \) in \( \mathbb{Q}^k \)

In the case of \( S \) in \( \mathbb{Q}^2 \), a Ressel-type theorem such as Theorem 3 does not hold. But, under an assumption of \( S \), we will show the implication (ii) \( \Rightarrow \) (iv).

**Proposition 5** Let \( S \) be a conelike semigroup in the rational vector space \( \mathbb{Q}^k \), \( k \geq 2 \), such that \( S^*_\mathbb{Q} \neq \emptyset \) and there exists a sequence \( \{s_n\} \) of \( S^*_\mathbb{Q} \) satisfying \( \lim_{n \to \infty} s_n = 0 \) and \( \dim(\text{linspan}\{s_n\}) = 1 \). For a continuous and positive definite function \( \varphi \) on \( S \) there exists the nonnegative measure \( \nu \) on \( \mathbb{R}^k \) such that

\[
\varphi(s) = \int_{\mathbb{R}^k} e^{(v,s)} d\nu(v), \quad s \in S.
\]

Acknowledgements. The second-mentioned author was partly supported by Grant-in-Aid for Scientific Research of Japan Society for the Promotion of Science.

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