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BOUNDS FOR THE RATIO AND DIFFERENCE BETWEEN PARALLEL SUM AND SERIES AND NONCOMMUTATIVE KANTOROVICH INEQUALITIES

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ABSTRACT. In this report, upper bounds for the ratio and the difference between parallel sum and series of operator connections in the sense of Anderson-Duffin-Trapp are obtained, in which the Mond-Pečarić method for convex functions is applied: Let A_i be positive operators on a Hilbert space such that $0 < mI \leq A_i \leq MI$ for some scalars $m < M$ and $i = 1, 2, \cdots, n$. Then we show an upper bound of the difference of parallel sum and series :

$$(A_1 + A_2 + \cdots + A_n) - (A_1^{-1} + A_2^{-1} + \cdots + A_n^{-1})^{-1} \leq \left(n(M+m) - 2\sqrt{Mm}\right) I.$$

As an application, we show a noncommutative Kantorovich inequality: For positive operators A_i such that $0 < mI \leq A_i \leq MI$ for some scalars $m < M$ and $i = 1, 2, \cdots, n$,

$$\frac{1}{n}(A_1 + A_2 + \cdots + A_n) \leq \frac{(M+m)^2}{4Mm} \left(\frac{A_1^{-1} + \cdots + A_n^{-1}}{n}\right)^{-1}$$

and

$$\frac{1}{n}(A_1 + A_2 + \cdots + A_n) - \left(\frac{A_1^{-1} + \cdots + A_n^{-1}}{n}\right)^{-1} \leq (\sqrt{M} - \sqrt{m})^2 I.$$

1. INTRODUCTION

This report is based on [4].

Motivated by a study of electrical network connection, Anderson and Duffin [1] introduced the concept of parallel sum of two positive semidefinite matrices and sequently Anderson and Trapp [2] have extended this notion to positive operators on a Hilbert space H. If A and B are impedance matrices of two resistive n-port networks, then their parallel sum $A : B$ defined by

$$A : B = (A^{-1} + B^{-1})^{-1}$$

is the impedance matrix of parallel connection and their series

$$A + B$$

is the impedance matrix of series connection. Some properties of parallel sum of two positive semidefinite matrices are discussed. For example, Anderson and Duffin [1] showed the following estimate of two impedance above: If $A_1, \cdots, A_n$ are positive semidefinite, then

$$\sum_{i=1}^{n} A_i \geq n^2 \prod_{i=1}^{n} : A_i, \tag{1}$$

where

$$\prod_{i=1}^{n} : A_i = A_1 : A_2 : \cdots : A_n.$$

In fact, the inequality (1) is a generalization of the classical inequality between the arithmetic mean and the harmonic mean.

Thus we consider upper bounds for the ratio and the difference between two impedance matrices above. We attempt to determine an upper estimate α such that

$$\sum_{i=1}^{n} A_i \ \leq \ \alpha \prod\nolimits_{i=1}^{n} : A_i$$

and an upper estimate β such that

$$\sum_{i=1}^{n} A_i \ - \ \prod\nolimits_{i=1}^{n} : A_i \ \leq \ \beta I.$$

The following estimation gives us a unified view to the above two inequalities: For a given real number α, there exists the most suitable estimate β such that

$$\sum_{i=1}^{n} A_i \ \leq \ \alpha \prod\nolimits_{i=1}^{n} : A_i + \beta I.$$

We regard these constants as two types of energy loss of two impedance matrices.

Throughout this report, we discuss parallel sum and series in the framework of operator theory on a Hilbert space.

Our purpose in this report is to give upper bounds for two types of energy loss of two impedances in terms of the spectra for given positive operators on a Hilbert space, in which the Mond-Pečarić method for convex functions [5] is applied. As an application, we show a noncommuatative Kantorovich inequality.

2. Mond-Pečarić method

A capital letter means a bounded linear operator on a Hilbert space H. An operator A is said to be positive ($A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. We denote by $B(H)$ the algebra of all bounded linear operators on H.

In this section, we prove a few lemmas on positive linear maps to obtain upper bounds for the ratio and the difference between parallel sum and series of operator connections in the sense of Anderson-Duffin-Trapp [1, 2].

Let Φ be a normalized positive linear map on $B(H)$. Then it follows from [3, Corollary 4.2] that Jensen's operator inequality implies Kadison's Schwarz inequality as follows:

$$\Phi(A^{-1})^{-1} \leq \Phi(A) \tag{2}$$

for every positive invertible operator A.

By using the Mond-Pečarić method [5], we have the following reverse inequality of (2) without the assumption of the normalization of Φ.

Lemma 1. *Let Φ be a positive linear map on $B(H)$ such that $\Phi(I) = kI$ for some positive scalar k. If A is a positive operator on a Hilbert space H such that $0 < mI \leq A \leq MI$ for some scalars $m < M$, then for each $\alpha > 0$*

$$\Phi(A) \leq \alpha \Phi(A^{-1})^{-1} + \beta(m, M, \alpha, k) I, \tag{3}$$

where

$$(4)\qquad \beta(m,M,\alpha,k)=\begin{cases} k(m+M)-2\sqrt{\alpha mM} & \text{if}\quad m\le \frac{\sqrt{\alpha Mm}}{k}\le M,\\ (k-\frac{\alpha}{k})M & \text{if}\quad \frac{\sqrt{\alpha Mm}}{k}\le m,\\ (k-\frac{\alpha}{k})m & \text{if}\quad M\le \frac{\sqrt{\alpha Mm}}{k}. \end{cases}$$

By Lemma 1, we have the following upper bounds for the ratio and the difference in the inequality (2):

Lemma 2. *Let Φ be a positive linear map on $B(H)$ such that $\Phi(I)=kI$ for some positive scalar k. If A is a positive operator on a Hilbert space H such that $0<mI\le A\le MI$ for some scalars $m<M$, then*

$$(5)\qquad \Phi(A)\le \frac{k^2(M+m)^2}{4Mm}\Phi(A^{-1})^{-1}$$

and

$$(6)\qquad \Phi(A)-\Phi(A^{-1})^{-1}\le (k(m+M)-2\sqrt{Mm})I.$$

Remark 3. If Φ is normalized, that is, $\Phi(I)=I$, then by Lemma 2 we have the following results due to Mond-Pečarić [9], cf. [5, Theorem 1.32]:

$$(7)\qquad \Phi(A)\le \frac{(M+m)^2}{4Mm}\Phi(A^{-1})^{-1}$$

and

$$(8)\qquad \Phi(A)-\Phi(A^{-1})^{-1}\le (\sqrt{M}-\sqrt{m})^2 I.$$

3. Main result

We state our main theorem, in which upper bounds for the ratio and the difference between parallel sum and series of operator connections are given.

Theorem 4. *If A and B are positive operators on H such that $0<mI\le A,B\le MI$ for some scalars $m<M$, then for each $\alpha>0$*

$$(9)\qquad A+B\ \le \alpha(A:B)+\beta(m,M,\alpha,k=2)I,$$

where

$$(10)\qquad \beta(m,M,\alpha,k=2)=\begin{cases} 2(m+M)-2\sqrt{\alpha mM} & \text{if}\quad m\le \frac{\sqrt{\alpha Mm}}{2}\le M,\\ (2-\frac{\alpha}{2})M & \text{if}\quad \frac{\sqrt{\alpha Mm}}{2}\le m,\\ (2-\frac{\alpha}{2})m & \text{if}\quad M\le \frac{\sqrt{\alpha Mm}}{2}. \end{cases}$$

In particular,

$$(11)\qquad A+B\ \le \frac{(M+m)^2}{Mm}(A:B)$$

and

$$(12)\qquad A+B-(A:B)\le 2\left(M+m-\sqrt{Mm}\right)I.$$

Proof. Let a map $\Psi : B(H) \oplus B(H) \mapsto B(H) \oplus B(H)$ be defined by

$$\Psi \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix}.$$

Then Ψ is a positive linear map such that $\Psi(I) = 2I$. Since

$$m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \leq M \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

it follows from Lemma 1 that for each $\alpha > 0$

$$\Psi \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} - \alpha \Psi \begin{pmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{pmatrix}^{-1} \leq \beta(m, M, \alpha, k = 2) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

We have the desired inequality (9) by rearranging the expression above.

If we choose α such that $2((M+m) - \sqrt{\alpha Mm}) = 0$ in (9), then it follows that $\alpha = \frac{(M+m)^2}{Mm}$ and α satisfies the condition $m \leq \frac{\sqrt{\alpha Mm}}{2} \leq M$. Thus we have (11). Also, if we put $\alpha = 1$ in (9), then it follows that

$$\beta(m, M, \alpha = 1, k = 2) \leq 2\left(M + m - \sqrt{Mm}\right)$$

and hence we have (12). □

Similarly, we have the following n-variable version of Theorem 4.

Theorem 5. *If A_i are positive operators on H such that $0 < mI \leq A_i \leq MI$ for some scalars $m < M$ for $i = 1, 2, \cdots, n$, then for each $\alpha > 0$*

$$\sum_{i=1}^{n} A_i \leq \alpha \prod_{i=1}^{n} : A_i + \beta(m, M, \alpha, k = n) I, \tag{13}$$

where $\beta(m, M, \alpha, k = n)$ is defined as (4) in Lemma 1.

In particular,

$$\sum_{i=1}^{n} A_i \leq n^2 \frac{(M+m)^2}{4Mm} \prod_{i=1}^{n} : A_i \tag{14}$$

and

$$\sum_{i=1}^{n} A_i - \prod_{i=1}^{n} : A_i \leq \left(n(M+m) - 2\sqrt{Mm}\right) I. \tag{15}$$

Proof. Let a map $\Psi : B(H) \oplus \cdots \oplus B(H) \mapsto B(H) \oplus \cdots \oplus B(H)$ be defined by

$$\Psi \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix} = \begin{pmatrix} A_1 + \cdots + A_n & & 0 \\ & \ddots & \\ 0 & & A_1 + \cdots + A_n \end{pmatrix}.$$

Then we can prove (13) by the same way as Theorem 4. □

4. Noncommutative Kantorovich inequality

Motivated by a study of parallel sum due to Anderson and Duffin [1], and Anderson and Trapp [2], Kubo and Ando [8] introduced the notion of operator mean. A map $(A, B) \to A \sigma B$ in the cone of positive invertible operators is called an operator mean if the following conditions are satisfied:

monotonicity: $A \leq C$ and $B \leq D$ imply $A \sigma B \leq C \sigma D$,
upper continuity: $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n \sigma B_n \downarrow A \sigma B$,
transformer inequality: $T^*(A \sigma B)T \leq (T^*AT) \sigma (T^*BT)$ for every operator T,
normalized condition: $A \sigma A = A$.

A key for the theory is that there is a one-to-one correspondence between an operator mean σ and a nonnegative operator monotone function $f(x)$ on $[0, \infty)$ through the formula

$$f(x) = 1 \sigma x \qquad \text{for all } x > 0,$$

or

$$A \sigma B = A^{\frac{1}{2}}(1 \sigma A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \quad \text{for all } A, B > 0.$$

We say that f is the representing function for σ. In this case, notice that $f(t)$ is operator monotone if and only if it is operator concave. The operator mean with representing function $tf(t^{-1})$ is called the transpose of σ and denoted by σ°:

$$A \sigma^\circ B = B \sigma A \qquad \text{for every positive } A \text{ and } B.$$

An operator mean is called symmetric if $\sigma = \sigma^\circ$. The operator mean with representing function $f(t^{-1})^{-1}$ is called the adjoint of σ and denoted by σ^*:

$$A \sigma^* B = (A^{-1} \sigma B^{-1})^{-1} \qquad \text{for every positive invertible } A \text{ and } B.$$

Simple examples of operator means are the arithmetic mean, in symbol ∇,

$$A \nabla B = \frac{A+B}{2}.$$

The normalized parallel sum is called the harmonic mean, in symbol !,

$$A \,!\, B = 2(A : B).$$

For invertible A, B, the geometric mean $A \sharp B$ is

$$A \sharp B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}.$$

Also, the representing function of the logarithmic mean λ is $(t-1)/\log t$. Then the following harmonic-geometric-logarithmic-arithmetic mean inequality holds

$$A \,!\, B \leq A \sharp B \leq A \lambda B \leq A \nabla B.$$

Furthermore, the arithmetic mean is the maximum of all symmetric operator means while the harmonic mean is the minimum.

Next, for positive numbers $a_i > 0$ $(i = 1, 2, \cdots, n)$, the following harmonic-geometric-arithmetic mean inequality holds

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{-1}\right)^{-1} \leq \sqrt[n]{\prod_{i=1}^{n} a_i} \leq \frac{1}{n}\sum_{i=1}^{n} a_i.$$

On the other hand, Kantorovich [7] proved the following inequality which is considered as a ratio type reverse inequality of harmonic - arithmetic mean inequality. If the sequence $\{a_i\}$ ($i = 1, 2, \cdots, n$) of positive numbers has the property

$$0 < m \leq a_i \leq M,$$

then the inequality

$$1 \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} a_i^{-1}\right) \leq \frac{(M+m)^2}{4Mm} \tag{16}$$

holds. Also, Shisha and Mond [11] proved the following difference type reverse inequality:

$$0 \leq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right) - \left(\frac{1}{n}\sum_{i=1}^{n} a_i^{-1}\right)^{-1} \leq \left(\sqrt{M} - \sqrt{m}\right)^2.$$

The following theorem is the harmonic-arithmetic operator mean inequality.

Theorem (A-H inequality) If A_i are positive operators on H for $i = 1, 2, \cdots, n$, then

$$\left(\frac{1}{n}\sum_{i=1}^{n} A_i^{-1}\right)^{-1} \leq \frac{1}{n}\sum_{i=1}^{n} A_i.$$

Proof. Let a map $\Psi : B(H) \oplus \cdots \oplus B(H) \mapsto B(H) \oplus \cdots \oplus B(H)$ be defined by

$$\Psi \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix} = \begin{pmatrix} \frac{A_1+\cdots+A_n}{n} & & 0 \\ & \ddots & \\ 0 & & \frac{A_1+\cdots+A_n}{n} \end{pmatrix}.$$

Then it follows that Ψ is a normalized positive linear map. By Kadison's Schwarz inequality we have $\Psi(\mathbf{A}^{-1})^{-1} \leq \Psi(\mathbf{A})$ for $\mathbf{A} = A_1 \oplus \cdots \oplus A_n$ and hence we have A-H inequality. □

Prof. S. Izumino suggested that Theorem 4 implies the following noncommutative Kantorovich inequality:

Theorem 6. *If A and B are positive operators on H such that $0 < mI \leq A, B \leq MI$ for some scalars $m < M$, then for all $\alpha > 0$*

$$A \nabla B \leq \frac{\alpha}{4} A \,!\, B + \frac{1}{2}\beta(m, M, \alpha, 2) I \tag{17}$$

where $\beta(m, M, \alpha, 2)$ is defined as (4) in Lemma 1.

In particular,

$$A \nabla B \leq \frac{(M+m)^2}{4Mm} A \,!\, B \tag{18}$$

and

$$A \nabla B - A \,!\, B \leq (\sqrt{M} - \sqrt{m})^2 I. \tag{19}$$

As an application of Theorem 5, we have the following n-variable version of a noncommutative Kantorovich inequality. We use the notation

$$\prod_{i=1}^{n} ! A_i = A_1 ! A_2 ! \cdots ! A_n = \left(\frac{A_1^{-1} + \cdots + A_n^{-1}}{n}\right)^{-1}.$$

Theorem 7. *If A_i are positive operators on H such that $0 < mI \leq A_i \leq MI$ for some scalars $m < M$ for $i = 1, 2, \cdots, n$, then*

$$\frac{1}{n}\sum_{i=1}^{n} A_i \leq \frac{(M+m)^2}{4Mm} \prod_{i=1}^{n} ! A_i \tag{20}$$

and

$$\frac{1}{n}\sum_{i=1}^{n} A_i - \prod_{i=1}^{n} ! A_i \leq (\sqrt{M} - \sqrt{m})^2 I. \tag{21}$$

Proof. The inequality (20) follows from (14) in Theorem 5. If we put $\alpha = n^2$ in (13) of Theorem 5, then the condition $m \leq \frac{\sqrt{\alpha Mm}}{k} \leq M$ satisfies and $\beta(m, M, \alpha = n^2, k = n) = n(m + M - 2\sqrt{mM})$. Therefore we have the desired inequality (21). □

Remark 8. Prof. T. Furuta kindly pointed out that Theorem 7 is the special case where $r = -1$ and $s = 1$ in [10, Theorem 1] due to Pečarić and Mićić, also where $p = -1$ in [6, Theorem E] due to Furuta and Pečarić, which is one of typical examples applying the Mond-Pečarić method.

Furthermore we show a generalization of Theorem 6 by means of symmetric operator means.

Theorem 9. *Let σ be a symmetric operator mean with the representing function f. If A and B are positive operators on H such that $0 < mI \leq A, B \leq MI$ for some scalars $m < M$, then*

$$\frac{m \,\sigma\, M}{m \,\nabla\, M} A \,\nabla\, B \leq A \,\sigma\, B \tag{22}$$

and

$$A \,\sigma^*\, B \leq \frac{m \,\nabla\, M}{m \,\sigma\, M} A \,!\, B. \tag{23}$$

Also,

$$A \,\nabla\, B - A \,\sigma\, B \leq M\left(\frac{m \,\nabla\, M}{m \,\sigma\, M} - 1\right) I \tag{24}$$

and

$$A \,\sigma^*\, B - A \,!\, B \leq M\left(\frac{m \,\nabla\, M}{m \,\sigma\, M} - 1\right) I. \tag{25}$$

To prove it, we need the following lemma.

Lemma 10. *Let m and M be positive scalars. Then*

$$\frac{m \,\sigma^*\, M}{m \,!\, M} = \frac{m \,\nabla\, M}{m \,\sigma\, M}$$

for every symmetric operator mean σ.

Proof. Let f be the representing function for σ. Then it follows that

$$\frac{m\ \sigma^*\ M}{m\ !\ M}=\frac{(m^{-1}\ \sigma\ M^{-1})^{-1}}{(m^{-1}\ \nabla\ M^{-1})^{-1}}=\frac{m^{-1}\ \nabla\ M^{-1}}{m^{-1}\ \sigma\ M^{-1}}=\frac{M\ \nabla\ m}{M\ \sigma\ m}=\frac{m\ \nabla\ M}{m\ \sigma\ M}.$$

The last equality holds since σ is symmetric. □

Proof of Theorem 9. Since the representing function f is concave on $(0,\infty)$, it follows that

$$f(t)\geq\frac{f\left(\frac{M}{m}\right)-f\left(\frac{m}{M}\right)}{\frac{M}{m}-\frac{m}{M}}\left(t-\frac{m}{M}\right)+f\left(\frac{m}{M}\right)\qquad\text{for all } t\in[\tfrac{m}{M},\tfrac{M}{m}].$$

Since $\frac{m}{M}I\leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\leq\frac{M}{m}I$, we have

$$f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\geq\frac{f(\frac{M}{m})-f(\frac{m}{M})}{\frac{M}{m}-\frac{m}{M}}\,(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}-\frac{m}{M}I)+f\left(\frac{m}{M}\right)I$$

and hence

$$\begin{aligned}A\ \sigma\ B=A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}&\geq\frac{f(\frac{M}{m})-f(\frac{m}{M})}{\frac{M}{m}-\frac{m}{M}}\,(B-\frac{m}{M}A)+f\left(\frac{m}{M}\right)A\\&=\frac{f(\frac{M}{m})-f(\frac{m}{M})}{\frac{M}{m}-\frac{m}{M}}\,B+\frac{\frac{M}{m}f(\frac{m}{M})-\frac{m}{M}f(\frac{M}{m})}{\frac{M}{m}-\frac{m}{M}}\,A\\&=\frac{2(f(\frac{M}{m})-f(\frac{m}{M}))}{\frac{M}{m}-\frac{m}{M}}\,A\ \nabla\ B.\end{aligned}$$

The last equality holds since σ is symmetric, that is, $f(t)=tf(t^{-1})$. This relation also implies

$$\begin{aligned}\frac{2(f(\frac{M}{m})-f(\frac{m}{M}))}{\frac{M}{m}-\frac{m}{M}}&=\frac{2Mm}{M^2-m^2}\left(1-\frac{m}{M}\right)f\left(\frac{M}{m}\right)=\frac{2}{M+m}\,mf\left(\frac{M}{m}\right)\\&=\frac{m\ \sigma\ M}{m\ \nabla\ M},\end{aligned}$$

and hence we have the desired inequality (21).

Replacing A by A^{-1} and B by B^{-1} in (22), it follows from $\frac{1}{M}I\leq A^{-1},B^{-1}\leq\frac{1}{m}I$ that

$$\frac{m^{-1}\ \sigma\ M^{-1}}{m^{-1}\ \nabla\ M^{-1}}\,A^{-1}\ \nabla\ B^{-1}\leq A^{-1}\ \sigma\ B^{-1}.$$

Taking inverse of both sides, we have

$$\left(\frac{m^{-1}\ \sigma\ M^{-1}}{m^{-1}\ \nabla\ M^{-1}}\right)^{-1}\ (A^{-1}\ \nabla\ B^{-1})^{-1}\geq(A^{-1}\ \sigma\ B^{-1})^{-1}$$

and it follows from Lemma 10 that

$$A\ \sigma^*\ B\leq\frac{m\ \sigma^*\ M}{m\ !\ M}\ A\ !\ B=\frac{m\ \nabla\ M}{m\ \sigma\ M}\ A\ !\ B$$

as desired.

It follows from the inequality (22) that

$$A \nabla B - A \sigma B \le \left(\frac{m \nabla M}{m \sigma M} - 1\right) A \sigma B$$
$$\le M\left(\frac{m \nabla M}{m \sigma M} - 1\right) I.$$

Similarly we have (25). □

As a special case of Theorem 9, we have the following refinement of Theorem 6.

Theorem 11. *If A and B are positive operators on H such that $0 < mI \le A, B \le MI$ for some scalars $m < M$, then*

$$\frac{2\sqrt{Mm}}{M+m} A \nabla B \le A \sharp B \le \frac{M+m}{2\sqrt{Mm}} A \,!\, B \tag{26}$$

and

(27)

$$A \nabla B - \frac{(\sqrt{M}-\sqrt{m})^2}{2}\sqrt{\frac{M}{m}} I \le A \sharp B \le A \,!\, B + \frac{(\sqrt{M}-\sqrt{m})^2}{2}\sqrt{\frac{M}{m}} I$$

Proof. Since the geometric mean $\sharp$ is symmetric and selfadjoint, that is, $(\sharp)^* = \sharp = (\sharp)^\circ$, it follows from Theorem 9 if we put the representing function $f(t) = \sqrt{t}$. □

Remark 12. The inequality (26) in Theorem 11 is a refinement of Corollary 5.39 in [5] if Φ is the identity map.

The power means $A \, m_r \, B$ is defined by

$$A \, m_r \, B = A^{\frac{1}{2}} \left(\frac{1 + (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r}{2}\right)^{\frac{1}{r}} A^{\frac{1}{2}} \quad \text{for } r \in [-1, 1].$$

Then we have the following theorem:

Theorem 13. *If A and B are positive operators on H such that $0 < mI \le A, B \le MI$ for some scalars $m < M$, then*

$$A \, m_r \, B \ge \frac{m \, m_r \, M}{m \sharp M} A \sharp B \qquad \text{for} \quad 0 \ge r \ge -1 \tag{28}$$

and

$$A \sharp B \ge \frac{m \sharp M}{m \lambda M} A \lambda B. \tag{29}$$

Proof. If we put

$$F(t) = \frac{1}{\sqrt{t}}\left(\frac{1+t^r}{2}\right)^{\frac{1}{r}} \quad \text{and} \quad G(t) = \frac{\sqrt{t}\log t}{t-1},$$

then it follows that $F(t)$ and $G(t)$ are monotone decreasing. □

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