Green functions and heat kernels of second order ordinary differential operators with discontinuous complex coefficients

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Abstract

We consider the operator $B u \equiv (r(x))^{-1} A u$ where

$$(Au)(x) \equiv \frac{d}{dx} \left( a(x) \frac{du}{dx} + b_1(x)u \right) + b_2(x) \frac{du}{dx} + c(x)u, \quad (-\infty < x < \infty)$$

with discontinuous bounded complex-valued coefficients. Under some additional condition, we estimate the kernel function (Green functions) of $(B - \lambda)^{-1}$ and the kernel for $e^{-tB}$.

1 Basic Assumptions and Notations

Consider an ordinary differential operator $A \in L^2(\mathbb{R})$:

$$(Au)(x) \equiv -(a(x)u' + b_1(x)u') + b_2(x)u' + c(x)$$

$$\equiv -\frac{d}{dx} \left( a(x) \frac{du}{dx} + b_1(x)u \right) + b_2(x) \frac{du}{dx} + c(x)u \quad (-\infty < x < \infty)$$

(1)

with

$\text{Dom}(A) = \{u \in H^1(\mathbb{R}); a(x)du/dx + b(x)u \in H^1(\mathbb{R})\}$

Here

$a(\bullet), b_1(\bullet), b_2(\bullet), c(\bullet) \in L^\infty(\mathbb{R})$

are complex-valued and may be discontinuous and we assume there exist two positive constants $\theta_a \in (0, \pi/2)$ and $a_0 > 0$ such that

$|\arg(a(x))| \leq \theta_a, \quad \Re(a(x)) \geq a_0$

We also consider another operator $B$ with the same domain:

$$(Bu)(x) \equiv \frac{(Au)(x)}{r(x)}, \text{Dom}(B) = \text{Dom}(A)$$

where $r(x) \in L^\infty(\mathbb{R})$ is a scale function for which there exist also two positive constants $\theta_r \in (0, \pi/2)$ and $r_0 > 0$ such that

$|\arg(r(x))| \leq \theta_r, \quad \Re(r(x)) \geq r_0$
We will further assume later that $0 < \theta_a + \theta_r < \pi/2$.

Our problem is the solvability of $Bu - \lambda u = f \in L^2(\mathbb{R})$ and the representation of the solution by a Green function. Equivalently, we have only to consider the solvability of

$$Au - \lambda r(x)u = r(x)f(x) \in L^2(\mathbb{R}).$$

We also consider the kernel of the analytic semigroup $e^{-tB}$.

We sometimes omit $(\mathbb{R})$ of $L^1(\mathbb{R}), L^\infty(\mathbb{R}), H^1(\mathbb{R}), \cdots$ for simplicity. And we generally denote constants by $k, k_0, k_1, \cdots$.

## 2 Functions with compact support in Dom$(A)$

Just as the domain $H^2$ of the operator $-d^2/dx^2$ is itself a Hilbert space, the domain Dom$(A)$ of $A$ can be regarded as the Banach space (actually a Hilbert space).

**Definition** For $u \in \operatorname{Dom}(A)$, we define

$$\|u\|_{\operatorname{Dom}(A)} \equiv \left\{ (\|u\|_{H^1})^2 + (\|a(x)u' + b_1(x)u\|_{H^1})^2 \right\}^{1/2}$$

**Theorem 1** The domain Dom$(A)$ of $A$ is itself a Banach space with norm $\| \bullet \|_{\operatorname{Dom}(A)}$.

**Proof.** We have only to consider the completeness. Let $\{u_n\}$ be a Cauchy sequence with $\| \bullet \|_{\operatorname{Dom}(A)}$. Then $u_n$ and $a(x)u_n' + b_1(x)u_n$ are both Cauchy sequences in $H^1$. Hence there exist $u, v \in H^1$ such that

$$u_n \rightarrow u, \quad a(x)u_n' + b_1(x)u_n \rightarrow v \text{ in } H^1.$$  

The first one means $a(x)u_n' + b_1(x)u_n \rightarrow a(x)u' + b_1(x)u$ in $L^2$. Therefore we have $a(x)u' + b_1(x)u = v \in H^1$ and $u \in \operatorname{Dom}(A)$. Q.E.D.

We will prove $C_0(\mathbb{R}) \cap \operatorname{Dom}(A)$ is dense in $\operatorname{Dom}(A)$ with norm $\| \bullet \|_{\operatorname{Dom}(A)}$.

We first define cut-off functions in the next three lemmas.

**Lemma 2** Fix $\rho(x) \in C_0^\infty$ such that

$$\rho(x) = \begin{cases} \geq 0 & (0 < x < 1) \\ = 0 & (x \leq 0, x \geq 1). \end{cases}$$

Then

$$c_n = \int_{-\infty}^{\infty} \frac{\rho(x-n)}{a(x)} dx \neq 0, \quad n = 0, \pm 1, \pm 2, \cdots$$

Moreover there exists a constant $k > 1$ such that

$$k^{-1} \leq |c_n| \leq k \quad (n = 0, \pm 1, \pm 2, \cdots)$$
Proof. The assumption on $a(x)$ implies

$$(k_0)^{-1} \leq \Re \frac{1}{a(x)} \leq k_0 \quad (-\infty < x < \infty)$$

with some constant $k_0 > 1$. Taking account of $\rho(x) \geq 0$, we have

$$(k_0)^{-1} \int_{-\infty}^{\infty} \rho(x-n)dx \leq \Re \int_{-\infty}^{\infty} \frac{\rho(x-n)}{a(x)}dx \leq k_0 \int_{-\infty}^{\infty} \rho(x-n)dx,$$

that is,

$$(k_0)^{-1} \int_{-\infty}^{\infty} \rho(x)dx \leq \Re c_n \leq k_0 \int_{-\infty}^{\infty} \rho(x)dx.$$  

or

$$\begin{align*}
(k_0)^{-1} k_1 \leq \Re c_n \leq k_0 k_1 \quad (3)
\end{align*}$$

if we put $k_1 = \int_{-\infty}^{\infty} \rho(x)dx$. On the other hand, $\rho(x) \geq 0$ and the convexity of the set $\{z \in \mathbb{C}; |\arg Z| \leq \theta_a < \pi/2\}$ implies

$$\left| \arg \int_{-\infty}^{\infty} \frac{\rho(x-n)}{a(x)}dx \right| \leq \sup_x \left| \arg \frac{1}{a(x)} \right| \leq \theta_a < \pi/2$$

that is,

$$|\arg c_n| \leq \theta_a < \pi/2 \quad (4)$$

From these, we have the claim of the present lemma. Q.E.D.

Now the next two lemmas are clear.

**Lemma 3** Let $\rho(x)$ and $c_n \ (n = 0, \pm 1, \cdots)$ be the same as in the previous lemma. Then

$$\phi_n(x) \equiv c_n^{-1} \int_{-\infty}^{x} \frac{\rho(y-n)}{a(y)}dy,$$

$$\psi_n(x) \equiv c_n^{-1} \int_{x}^{\infty} \frac{\rho(y-n)}{a(y)}dy$$

satisfy

$$\begin{align*}
\phi_n(x) &= \begin{cases} 
0 & (x \leq n) \\
1 & (x \geq n + 1),
\end{cases} \\
\psi_n(x) &= \begin{cases} 
1 & (x \leq n) \\
0 & (x \geq n + 1).
\end{cases}
\end{align*}$$

In addition, the functions

$$a(x)\phi_n' = (c_n)^{-1} \rho(x-n), a(x)\psi_n' = -(c_n)^{-1} \rho(x-n) \quad (n = 0, \pm 1, \cdots)$$

belong to $C_0^\infty(\mathbb{R})$ and form a bounded set in $B^1(\mathbb{R})$. 

Lemma 4 Let \( \phi_m(x) \) and \( \psi_n(x) \) be the same as in the previous lemma. The functions

\[ \phi_{mn}(x) \equiv \phi_m(x) \psi_n(x) \]

with the integer parameter \( n \geq m + 1 \) satisfies

\[ \phi_{mn}(x) = \begin{cases} 
1 & \text{if } m + 1 \leq x \leq n \\
0 & \text{if } x \leq m, \quad x \geq n + 1
\end{cases} \]

In addition, two families of support compact functions

\[ \{\phi_{mn}(x)\}, \{a(x)\phi_{mn}'\} \]

are bounded subsets in \( W^{1,\infty}(\mathbb{R}) \) and \( B^1(\mathbb{R}) \), respectively.

Using Lemma 4, we can prove the next theorem.

**Theorem 5** The set \( C_0(\mathbb{R}) \cap \text{Dom}(A) \) is dense in \( \text{Dom}(A) \) with norm \( \| \cdot \|_{\text{Dom}(A)} \).

**Proof.** Fix \( u \in \text{Dom}(A) \) arbitrarily. Set

\[ u_{mn} \equiv \phi_{mn}(x)u(x) \in C_0(\mathbb{R}) \cap L^2(\mathbb{R}) \]

where \( \phi_{mn}(x) \) is the function in the previous lemma. Recalling \( \phi_{mn}(x) \in W^{1,\infty} \) and \( \{a(x)\phi_{mn}'\}(x) \in B^1 \), we know

\[
\begin{align*}
  u_{mn}' &= \phi_{mn}'(x)u + \phi_{mn}(x)u' \\
  a(x)u_{mn}' + b_1(x)u_{mn} &= a(x)\phi_{mn}'(x)u + \phi_{mn}(x)\{a(x)u' + b_1(x)u\} \\
  \{a(x)u_{mn}'+b_1(x)u_{mn}\}' &= (a(x)\phi_{mn}'(x))'u + a(x)\phi_{mn}'(x)u'(x) \\
  &\quad + \phi_{mn}(x)\{a(x)u' + b_1(x)u\} \\
  &\quad + \phi_{mn}(x)\{a(x)u' + b_1(x)u\}'
\end{align*}
\]

are all in \( L^2(\mathbb{R}) \), i.e., \( u_{mn} \in \text{Dom}(A) \). The previous lemma states \( \{\phi_{mn}\} \) and \( \{a(x)\phi_{mn}'\} \) are bounded subsets in \( W^{1,\infty}(\mathbb{R}) \) and \( B^1(\mathbb{R}) \), respectively. Note also

\[ \text{supp}\phi_{mn} \subset [m, m+1] \cup [n, n+1] \]

Therefore \( \phi_{mn}(x) \to 1 \) \( (m \to -\infty, \ n \to \infty) \) implies

\[
\begin{align*}
  u_{mn} \to u(x) \\
  u_{mn}' \to u'(x) \\
  a(x)u_{mn}' + b_1(x)u_{mn} \to \{a(x)u' + b_1(x)u\} \\
  \{a(x)u_{mn}'+b_1(x)u_{mn}\}' \to \{a(x)u' + b_1(x)u\}'
\end{align*}
\]
all in \( L^2(\mathbb{R}) \). This means \( u_{mn} \in C_0(\mathbb{R}) \cup \text{Dom}(A) \) converges to \( u \) in the sense of the norm \( \| \cdot \|_{\text{Dom}(A)} \). Q.E.D.

In the later sections, we consider the perturbation \( A^\mu \) of the operator \( A \) which is formally defined

\[
(A^\mu)(x) \equiv e^{-\mu \Phi(x)} A(e^{\mu \Phi(x)} u(x))
\]

where

\[
\Phi(x) \equiv \int_0^x \frac{dy}{a(y)}.
\]

The next theorem partially guarantees the appropriateness of the definition of \( A^\mu \).

**Theorem 6** Let \( \mu \in \mathbb{C} \) be an arbitrarily fixed constant and

\[
\Phi(x) \equiv \int_0^x \frac{dy}{a(y)}.
\]

Suppose \( u \in \text{Dom}(A) \cap C_0(\mathbb{R}) \). Then

\[
v(x) \equiv e^{-\mu \Phi(x)} u(x) \in \text{Dom}(A).
\]

**Proof.** Since \( \Phi(x) \) is absolutely continuous and locally bounded,

\[
\begin{align*}
v(x) &= e^{\mu \Phi(x)} u(x) \in L^2 \\
v'(x) &= \frac{\mu}{a(x)} e^{\mu \Phi(x)} u(x) + e^{\mu \Phi(x)} u'(x) \in L^2
\end{align*}
\]

as \( u \in \text{Dom}(A) \cap C_0(\mathbb{R}) \subset C_0(\mathbb{R}) \cap H^1(\mathbb{R}) \). Moreover,

\[
a(x)v' + b_1(x)v = \mu e^{\mu \Phi(x)} u(x) + e^{\mu \Phi(x)} u'(x) + b_2(x)u(x)(a(x)u' + b_1(x)u) \in H^1(\mathbb{R})
\]

since \( u \in \text{Dom}(A) \cap C_0(\mathbb{R}) \) implies

\[
a(x)u' + b_1(x)u \in H^1(\mathbb{R}) \cap C_0(\mathbb{R})
\]

by definition.

### 3 Sesquilinear form associated with \( A \)

**Theorem 7** The sesquilinear form \( \alpha[u, v] \) defined as

\[
\alpha[u, v] = \int_{-\infty}^{\infty} \{(a(x)u' + b_1(x)u)v' + b_2(x)u'v + c(x)uv\} \, dx,
\]

\[
\text{Dom}(\alpha) = H^1(\mathbb{R})
\]

is a closed sectorial form in \( L^2(\mathbb{R}) \). Moreover, \( A \) is the sectorial operator representing the sectorial form \( \alpha \), i.e.,

\[
\alpha[u, v] = (Au, v)
\]

for any \( u \in \text{Dom}(A) \) and any \( v \in H^1 \).
Proof. First, we prove the sectoriality. We begin with the first part of $\alpha[u, u]$:

$$\int_{-\infty}^{\infty} a(x)|u'|^{2} dx = \gamma(u)||u'||_{L^{2}}^{2}$$

Here $\gamma(u)$ is in the closed convex hull of

$$\{a(x); x \in \mathbb{R} \} \subset \{|\arg(z)| \leq \theta_{a}\} \cap \{|\Re z| \geq a_{0}\} \cap \{|z| \leq |a(\bullet)|_{L^{\infty}}\}.$$

On the other hand,

$$|\int_{-\infty}^{\infty} \{b_{1}(x)u\overline{u}'+b_{2}(x)u'u+\mathrm{c}(x)u\overline{u}\} dx| \leq \epsilon$$

Il $$||u'||^{2}+(k/\epsilon)||u||^{2}$$

with two constant $k > 0$ and $0 < \epsilon < 1$ where $0 < \epsilon < 1$ can be arbitrarily chosen. So, with appropriately chosen constant $K > 0$,

$$\alpha[u, u] + K(u, u)$$

takes values in the sector $\{|\arg z| \leq \theta_{a} < \pi/2\}$. In other words, $\alpha[u, v]$ is a sectorial form. It is also shown that

$$|\alpha[u, u] + K(u, u)| \geq k_{0}(||u||^{2}+||u'||^{2})$$

for some constant $k_{0} > 0$. Therefore Cauchy sequences in the sense of $\alpha[u, v]$ are the one in $H^{1}$ and it is a closed form.

Theorem 8 The dual $A^{*}$ of the operator $A$ is

$$(A^{*}v)(x) \equiv -(\overline{a(x)}v'+\overline{b_{2}(x)}v)'+\overline{b_{1}(x)}v'+\overline{\mathrm{c}(x)}v$$

with

$$\text{Dom}(A^{*}) = \{v \in H^{1}(\mathbb{R}); \overline{a(x)}dv/dx + \overline{b_{2}(x)}u \in H^{1}(\mathbb{R})\}$$

We omit the proof.

In order to obtain later the exponential decay the Green functions, we will need the next perturbation of the operator $A$.

Definition. $A^{\mu}$ is defined to be a perturbation of $A$:

$$(A^{\mu}u)(x) \equiv (Au)(x) - 2\mu u' + \mu (c_{1}(x)u + \mu^{2}c_{2}(x)u$$

with perturbation parameter $\mu \in \mathbb{C}$. where

$$c_{1}(x) = \left. \frac{-b_{1}(x) + b_{2}(x)}{a(x)} \right|, c_{2}(x) = -\frac{1}{a(x)} \in L^{\infty}.$$

The corresponding sesquilinear form is denoted by

$$\alpha^{\mu}[u, v] \equiv \alpha[u, v] - 2\mu(u', v) + \mu (c_{1}(x)u, v) + \mu^{2}(c_{2}(x)u, v).$$
Remark. $A^\mu$ is formally obtained as
\[(A^\mu u)(x) = e^{-\mu\Phi(x)} A(e^{\mu\Phi(x)} u)\]

Next is one of the Sobolev inequalities.

**Lemma 9** For arbitrary $u \in W^{1,2}(\mathbb{R})$,
\[
\|u\|_{L^\infty} \leq \sqrt{2}\|u\|_{L^2}^{1/2}\|u'\|_{L^2}^{1/2}.
\]

**Proof.** For any $x \in \mathbb{R}$,
\[
\{u(x)\}^2 = \int_{-\infty}^{x} 2u(t)u'(t)dt
\]
Hence
\[
|u(x)|^2 \leq 2 \left(\int_{-\infty}^{\infty} |u(t)|^2 \right)^{1/2} \left(\int_{-\infty}^{\infty} |u'(t)|^2 \right)^{1/2}
\]
Q.E.D.

**Lemma 10** Arbitrary $z, w \in \mathbb{C} \setminus \{0\}$ satisfy
\[
|z - w| \geq (\sin(|\theta|/2))(|z| + |w|).
\]
Here
\[
\theta = \arg(z) - \arg(w) = \arg(z/w) \in [-\pi, \pi)
\]

**Proof.** Applying the cosine theorem to the triangle with vertices $0, z, w$, we have
\[
|z - w|^2 = |z|^2 + |w|^2 - 2|z||w|\cos \theta
= \frac{1 - \cos \theta}{2}(|z| + |w|)^2 + \frac{1 + \cos \theta}{2}(|z| - |w|)^2
\geq \sin^2\left(\frac{\theta}{2}\right)(|z| + |w|)^2.
\]
Q.E.D.

**Theorem 11** The sesquilinear form
\[
\alpha_\lambda[u, v] \equiv \alpha[u, v] - \lambda(r(x)u, v)
\]
is a closed form with $\text{Dom}(\alpha_\lambda) = \text{Dom}(\alpha) = W^{1,2}$. Let also $\theta_a + \theta_r < \omega < \pi/2$ for some $\omega \in (0, \pi/2)$. Then
\[
|\alpha_\lambda[u, u]| \geq k_0\|u'\|_{L^2}^2 + k_1|\lambda|\|u\|_{L^2}^2, \quad u \in \text{Dom}(\alpha_\lambda) = W^{1,2}
\]
for $\lambda$ which satisfies
\[
|\arg(\lambda)| \geq \omega, |\lambda| \geq k_2.
\]
Here $k_0, k_1$ and $k_2$ are positive constants which depend only on $\omega, \theta_a, \theta_r, a_0, r_0, \|b_1(\bullet)\|_{L^\infty}, \|b_2(\bullet)\|_{L^\infty}, \|c(\bullet)\|_{L^\infty}$.  

Proof. First notice that

\[ \left| \arg \left( \int a(x)|u'|^2 \, dx \right) \right| \leq \theta_a \]

and

\[ \left| \arg \left( \lambda \int r(x)|u'|^2 \, dx \right) \right| \geq \omega - \theta_r \]

Therefore the previous lemma is applicable and

\[ \left| \int a(x)|u'|^2 \, dx - \lambda \int r(x)|u'|^2 \, dx \right| \geq \sin \frac{\omega - \theta_a - \theta_r}{2} \left( \left| \int a(x)|u'|^2 \, dx \right| + \left| \lambda \right| \left| \int r(x)|u'|^2 \, dx \right| \right) \]

\[ \geq k_0 (||u'||_{L^2}^2 + ||\lambda|| \left| ||u||_{L^2}^2 \right|) \]

for some constant \( k_0 > 0 \) dependent only on \( \theta_a, \theta_r, \omega, a_0, r_0 \). On the other hand,

\[ \int (b_1(x)u\overline{u} + b_2(x)u'u + c(x)|u|^2) \, dx \leq k_1 (||u||_{L^2} \left| ||u'||_{L^2} \right| + (||u||_{L^2}^2)^2) \]

\[ \leq (k_0/2)||u'||_{L^2}^2 + k_2 ||u||_{L^2}^2 \]

for some other constants \( k_2 \) dependent only on \( k_0 \) and \( ||b_1(\bullet)||_{L^\infty}, ||b_2(\bullet)||_{L^\infty}, ||c(\bullet)||_{L^\infty} \).

Combining these two inequalities, we have

\[ |\alpha_{\lambda}[u, u]| = |\alpha[u, u] - \lambda (r(x)u, u)| \geq (k_0/2)||u'||_{L^2}^2 + (k_0||\lambda|| - k_2)||u||_{L^2}^2 \]

We have only to redefine the positive constants \( k_0, k_1, k_2 \).

Corollary The sesquilinear form

\[ \alpha_{\lambda}^\mu[u, v] \equiv \alpha^\mu[u, v] - \lambda (r(x)u, v) \]

is a closed form with \( \text{Dom}(\alpha_{\lambda}^\mu) = \text{Dom}(\alpha) = H^1 \). Let also \( \theta_a + \theta_r < \omega < \pi/2 \) for some \( \omega \in (0, \pi/2) \). Then

\[ |\alpha_{\lambda}^\mu[u, u]| \geq k_0 \left| ||u'||_{L^2}^2 + (k_1||\lambda|| - k_2||\mu||^2)||u||_{L^2}^2 \right|, \quad u \in \text{Dom}(\alpha_{\lambda}) = H^1 \]

for \( \lambda \) and \( \mu \) which satisfy

\[ |\arg(\lambda)| \geq \omega, ||\lambda|| \geq k_3, ||\mu|| \leq k_4||\lambda||^{1/2}. \]

Here \( k_0, k_1, k_2, k_3 \) and \( k_4 \) are positive constants which depend only on \( \omega, \theta_a, \theta_r, a_0, r_0, ||b_1(\bullet)||_{L^\infty}, ||b_2(\bullet)||_{L^\infty}, ||c(\bullet)||_{L^\infty} \).

Proposition 12 Let \( \theta_a + \theta_r < \omega < \pi/2 \). Suppose that \( |\arg(\lambda)| \geq \omega \) and that \( ||\lambda|| \) is sufficiently large. Then, for any \( f(\bullet) \in L^2 \),

\[ (A - \lambda r(\bullet))u(x) \equiv Au(x) - \lambda r(x)u = f(x), \]

has a unique solution \( u \in \text{Dom}(A_{\lambda}) = \text{Dom}(A) \) and it satisfies

\[ ||u||_{L^2} \leq k_1||\lambda||^{-1/2}||f||_{L^2}, ||u'||_{L^2} \leq k_1||\lambda||^{-1/2}||f||_{L^2}, ||u||_{L^\infty} \leq k_1||\lambda||^{-3/4}||f||_{L^2} \]
Proof. By the preceding theorem 11,
\[ k_0 \|u''\|_{L^2}^2 + k_1 |\lambda| \|u\|_{L^2}^2 \leq |\alpha_\lambda [u, u]| = |(f, u)| \leq \|f\| \|u\|. \]
Therefore there exists a unique solution \( u \in \text{Dom}(A) \). We also have
\[ k_1 |\lambda| \|u\|_{L^2}^2 \leq \|f\| \|u\|, \]
hence
\[ \|u\| \leq (k_1)^{-1} |\lambda|^{-1} \|f\|. \]
Back to the original inequality, we obtain
\[ k_0 \|u''\|_{L^2}^2 \leq \|f\| \|u\| \leq (k_1)^{-1} |\lambda|^{-1} \|f\|^2, \]
hence
\[ \|u''\|_{L^2} \leq (k_0 k_1)^{-1/2} |\lambda|^{-1/2} \|f\|. \]
Finally, we have
\[ \|u\|_L^\infty \leq \sqrt{2} \|u\|_{L^2} \|u''\|_{L^2} \leq \sqrt{2} k_0^{-1/4} k_1^{-3/4} |\lambda|^{-3/4} \|f\|_{L^2} \]
Q.E.D.

Corollary Let \( \theta_a + \theta_r < \omega_0 < \omega < \pi/2 \). Suppose that \( |\arg \lambda| \geq \omega \) and \( |\lambda| \) is sufficiently large. Suppose also that \( |\mu| \leq k_0 |\lambda|^{1/2} \) with some constant \( k_0 > 0 \). Then, for any \( f(\bullet) \in L^2 \),
\[ (A^\mu - \lambda r(\bullet))u(x) \equiv A^\mu u(x) - \lambda r(x)u = f(x), \]
has a unique solution \( u \in \text{Dom}(A^\mu) = \text{Dom}(A) \) and it satisfies
\[ \|u\|_{L^2} \leq k_1 |\lambda|^{-1} \|f\|_{L^2}, \|u''\|_{L^2} \leq k_1 |\lambda|^{-1/2} \|f\|_{L^2}, \|u\|_L^\infty \leq k_1 |\lambda|^{-3/4} \|f\|_{L^2} \]

Proposition 13 Let \( \theta_a + \theta_r < \omega < \pi/2 \) for some \( \omega \in (0, \pi/2) \). Suppose that \( |\arg \lambda| > \omega \) and \( |\lambda| \) is sufficiently large. Then, for any \( f(\bullet) \in L^2 \),
\[ (A - \lambda r(\bullet))u(x) \equiv Au(x) - \lambda r(x)u = (f(x))', \]
has a unique solution \( u \in \text{Dom}(\alpha_\lambda) = \text{Dom}(\alpha) = H^1 \) and it satisfies
\[ \|u\|_{L^2} \leq k_1 |\lambda|^{-1/2} \|f\|_{L^2}, \|u''\|_{L^2} \leq k_2 \|f\|_{L^2}, \|u\|_L^\infty \leq k_4 |\lambda|^{-1/4} \|f\|_{L^2} \]
Proof. Note that
\[ k_0 \|u''\|_{L^2}^2 + k_1 |\lambda| \|u''\|_{L^2}^2 \leq |\alpha_\lambda [u, u]| = |(f', u)| = |(f, u')| \leq \|f\| \|u'\| \]
in the present case. Similarly to the preceding theorem, we have first
\[ k_0 \|u''\|_{L^2}^2 \leq \|f\| \|u'\| \]
hence, \[ \|u'\|_{L^2} \leq k_0^{-1}\|f\|. \]

Back to the original inequality, we obtain
\[ k_1|\lambda|\|u\|_{L^2}^2 \leq \|f\|\|u'\| \leq (k_0)^{-1}|\lambda|^{-1}\|f\|^2, \]
hence
\[ \|u\|_{L^2} \leq k_0^{-1}|\lambda|^{-1/2}\|f\|. \]

Finally, we have
\[ \|u\|_{L^\infty} \leq \sqrt{2}\|u\|_{L^2}\|u'\|_{L^2} \leq \sqrt{2}k_0^{-1/4}k_1^{-3/4}|\lambda|^{-1/4}\|f\|_{L^2} \]
\[ \|u'\|_{L^2} \leq k_0^{-1}\|f\|_{L^2}. \]

**Corollary** Let \( \theta_a + \theta_r < \omega_0 < \omega < \pi/2 \). Suppose that \(|\arg \lambda| > \pi - \omega \) and \(|\lambda| \) is sufficiently large. Suppose also that \(|\mu| \leq k_0|\lambda|^{1/2} \) with some constant \( k_0 > 0 \). Then, for any \( f(\bullet) \in L^2 \),
\[(A^\mu - \lambda r(\bullet))u(x) \equiv A^\mu(x) - \lambda r(x)u = (f(x))', \]
has a unique solution \( u \in \mathrm{Dom}(\alpha_\lambda) = \mathrm{Dom}(\alpha) = H^1 \) satisfying
\[ \|u\|_{L^2} \leq k_1|\lambda|^{-1/2}\|f\|_{L^2}, \|u'\|_{L^2} \leq k_2\|f\|_{L^2}, \|u\|_{L^\infty} \leq k_1|\lambda|^{-1/4}\|f\|_{L^2} \]

**Proposition 14** Let the assumption be the same as in the previous two Propositions. Then there exists a kernel function \( R_\lambda(x, \xi) \) which represents the solution \( u = (A - \lambda r)^{-1}f \):
\[ u(x) = \int_{-\infty}^{\infty} R_\lambda(x, \xi)f(\xi)d\xi \]
with the estimate
\[ |R_\lambda(x, \xi)| \leq k_0|\lambda|^{-1/2} \]
for some constant \( k_0 > 0 \).

**Proof.** Since \( u \in H^1 \subset B^0 \) is a continuous function and
\[ |u(x)| \leq \|u\|_{B^0} \leq \|u\|_{H^1} \leq k_1|\lambda|^{-3/4}\|f\|_{L^2} \]
for an arbitrarily fixed \( x \). Thus \( L^2 \rightarrow C : f(\bullet) \rightarrow u(x) \) is turned out to be a bounded functional. Hence the Riesz theorem asserts that there exists \( R_\lambda(x, \bullet) \in L^2 \) dependent on \( x \in \mathbb{R} \) such that
\[ u(x) = \int_{-\infty}^{\infty} R_\lambda(x, \xi)f(\xi)d\xi \]
and $\|R_\lambda(x, \bullet)\|_{L^2} \leq k_1|\lambda|^{-3/4}$

Now we consider the solution $v \in H^1 \subset B^0$ of $(A - \lambda)v = g'$, $g \in L^2$. By the previous theorem, $L^2 \rightarrow C : f(\bullet) \rightarrow v(x)$ with an arbitrarily fixed $x$ is also a bounded functional and

$$|v(x)| \leq \|v\|_{B^0} \leq \|v\|_{H^1} \leq k_2|\lambda|^{-1/4}\|f\|_{L^2}$$

with another constant $k_2 > 0$. So there exists again another kernel $S_\lambda(x, \xi)$ such that

$$v(x) = \int_{-\infty}^{\infty} S_\lambda(x, \xi)g(\xi)d\xi$$

and $\|S_\lambda(x, \bullet)\|_{L^2} \leq k_2|\lambda|^{-1/4}$ We look into the relation of $R_\lambda(x, \xi)$ and $S_\lambda(x, \xi)$. For an arbitrary $g \in C_0^\infty$, the solution $v \in H^1$ of $(a - \lambda r)v = g'$ can be written in two ways.

$$v(x) = \int_{-\infty}^{\infty} R_\lambda(x, \xi)g'(\xi)d\xi,$$

$$v(x) = \int_{-\infty}^{\infty} S_\lambda(x, \xi)g(\xi)d\xi.$$

Since $g \in C_0^\infty$ is arbitrary, $S_\lambda(x, \xi) \in L^2$ is a distribution derivative of $R_\lambda(x, \xi)$ with respect to $\xi$. Thus $R_\lambda(x, \bullet) \in H^1 \subset B^0$. By Lemma?,

$$\|R_\lambda(x, \bullet)\|_{L^\infty} \leq \|R_\lambda(x, \bullet)\|_{L^2}^{1/2}\|S_\lambda(x, \bullet)\|_{L^2}^{1/2} \leq k_2|\lambda|^{-1/2}$$

**Corollary** Let the assumption be the same as in the corollaries of the two previous two propositions. Then there exists a kernel function $R_\lambda^\mu(x, \xi)$ which represents the solution $u = (A^\mu - \lambda r)^{-1}f$:

$$u(x) = \int_{-\infty}^{\infty} R_\lambda^\mu(x, \xi)f(\xi)d\xi$$

with the estimate

$$|R_\lambda^\mu(x, \xi)| \leq k_0|\lambda|^{-1/2}$$

for some constant $k_0 > 0$.

**Theorem 15** Let the assumption be the same as in the two theorems. The kernel function $R_\lambda(x, \xi)$ which represents the solution $u = (A - \lambda r)^{-1}f$:

$$u(x) = \int_{-\infty}^{\infty} R_\lambda(x, \xi)f(\xi)d\xi$$

has the estimate

$$|R_\lambda(x, \xi)| \leq k_1|\lambda|^{-1/2}e^{-k_2|\lambda|^{1/2}|x-\xi|}$$

for some constant $k_1, k_2 > 0$. 


Proof. Let $\mu$ be as in the corollaries of the Theorems. Let $u \in \text{Dom}(A) \cap C_0$ be arbitrarily fixed. Then

$$e^{-\mu \Phi(x)}u(x) \in \text{Dom}(A)$$

where

$$\Phi(x) = \int_0^x \frac{dy}{a(y)}$$

as in Theorem 6. Now putting

$$f = (A - \lambda R)u,$$

we have

$$(A - \lambda R)e^{\mu \Phi(x)}(e^{-\mu \Phi(x)}u(x)) = f(x)$$

$e^{\mu \Phi(x)}(A^\mu - \lambda r)(e^{-\mu \Phi(x)}u(x)) = f(x)$$

$$(A^\mu - \lambda r)(e^{-\mu \Phi(x)}u(x)) = e^{-\mu \Phi(x)}f(x).$$

Hence

$$e^{-\mu \Phi(x)}u(x) = \int_{-\infty}^{\infty} R_\lambda^\mu(x, \xi)e^{-\mu \Phi(\xi)}f(\xi)d\xi$$

$$u(x) = e^{\mu \Phi(x)} \int_{-\infty}^{\infty} R_\lambda^\mu(x, \xi)e^{-\mu \Phi(\xi)}f(\xi)d\xi$$

$$u(x) = \int_{-\infty}^{\infty} e^{\mu(\Phi(x)-\Phi(\xi))}R_\lambda^\mu(x, \xi)f(\xi)d\xi.$$

On the other hand, $u = (A - \lambda r)^{-1}f$ can be written as

$$u(x) = \int_{-\infty}^{\infty} R_\lambda(x, \xi)f(\xi)d\xi.$$ 

Hence

$$\int_{-\infty}^{\infty} e^{\mu(\Phi(x)-\Phi(\xi))}R_\lambda^\mu(x, \xi)f(\xi)d\xi = \int_{-\infty}^{\infty} R_\lambda(x, \xi)f(\xi)d\xi$$

for all $f = (A - \lambda r)u$ with $u \in \text{Dom}(A) \cap C_0$. Such $f$ form a dense subset in $L^2$. Therefore

$$R_\lambda(x, \xi) \equiv e^{\mu(\Phi(x)-\Phi(\xi))}R_\lambda^\mu(x, \xi).$$

Recalling that $\mu$ with $|\mu| \leq k_0|\lambda|^{1/2}$ is arbitrary and using the Corollary of the previous Proposition 14,

$$|R_\lambda(x, \xi)| \leq k_1|\lambda|^{-1/2}e^{-k_2|\lambda|^{1/2}|\Phi(x)-\Phi(\xi)|}.$$ 

Noticing

$$\Re(1/a(y)) \geq k_0$$
with a certain constant \( k_0 > 0 \),

\[
|\Phi(x) - \Phi(\xi)| \geq |\Re \Phi(x) - \Phi(\xi)| = |\Re \int_x^\xi \frac{dy}{a(y)}| \geq k_0 |x - \xi|.
\]

Combining these, we finally obtain

\[
|R_\lambda(x, \xi)| \leq k_1 |\lambda|^{-1/2} e^{-k_2 |\lambda|^{1/2} |x - \xi|}.
\]

Q.E.D.

**Corollary** There exists a kernel function \( \tilde{R}_\lambda(x, \xi) \) of \((B - \lambda)^{-1}\) where \( Bu(x) = (r(x))^{-1}Au(x) \):

\[
(B - \lambda)^{-1}f(x) = \int_{-\infty}^{\infty} \tilde{R}_\lambda(x, \xi)f(\xi)d\xi
\]

Moreover

\[
|\tilde{R}_\lambda(x, \xi)| \leq k_1 |\lambda|^{-1/2} e^{-k_2 |\lambda|^{1/2} |x - \xi|}
\]

with constants \( k_1, k_2 \).

**Proof.** Since \( Bu - \lambda u = f \in L^2 \) is equivalent to

\[
Au - \lambda r(x)u = r(x)f \in L^2,
\]

we have

\[
u(x) = (B - \lambda)^{-1}f(x) = (A - \lambda r)^{-1}(rf) = \int_{-\infty}^{\infty} R_\lambda(x, \xi)r(\xi)f(\xi)d\xi.
\]

Therefore, we have only to put \( \tilde{R}_\lambda(x, \xi) = R_\lambda(x, \xi)r(\xi) \). Q.E.D.

**Theorem 16** Let the assumption be the same as the preceding theorem and its corollary. Then

\[
\left| \frac{\partial R_\lambda}{\partial x}(x, \xi) \right| \leq k_1 e^{-k_2 |\lambda|^{1/2} |x - \xi|}
\]

\[
\left| \frac{\partial \tilde{R}_\lambda}{\partial x}(x, \xi) \right| \leq \tilde{k}_1 e^{-\tilde{k}_2 |\lambda|^{1/2} |x - \xi|}.
\]

for some constants \( k_1, k_2, \tilde{k}_1, \tilde{k}_2 > 0 \).

We omit the proof.
Theorem 17 The analytic semigroup $e^{-tB}$ generated by

$$Bu(x) = (r(x))^{-1}(Au)(x)$$

has a kernel function $G(x, y; t)$ with estimate

$$|G(x, y; t)| \leq k_0 e^{k_1 t} e^{-k_2 |x-y|^2}, (x, y) \in \mathbb{R}^2, |\arg t| \leq \pi/2 - \omega$$

with constants $k_0, k_1, k_2 > 0$.

Proof. The kernel function $\tilde{R}_\lambda(x, y)$ of $(B - \lambda)^{-1}$ is expressed by the kernel $\overline{R}_\lambda(x, y)$ with estimate

$$|\tilde{R}_\lambda(x, y)| \leq k_1 |\lambda|^{-1/2} e^{-k_2 |x-y|^2}$$

for

$$\{\lambda; |\arg \lambda| \geq \omega', |\lambda| \geq k_3\}$$

with constants $\omega' \in (\theta_a + \theta_r \omega), k_1, k_2, k_3 > 0$.

By a standard argument, $B + k_0$ with some $k_0 > 0$ has a kernel which has a similar estimate in

$$\{\lambda; |\arg \lambda| \geq \omega'\}$$

We have only to discuss this $B + k_0$ and $e^{-t(B+k_0)}$, rewriting $B + k_0$ as $B$ from now on. We recall the formula:

$$e^{-tB} = \frac{-1}{2\pi} \int_{\Gamma} e^{-\lambda t} (B - \lambda)^{-1} d\lambda,$$

with the integral path

$$\Gamma = \{\lambda = \rho e^{i\omega'}; \infty > \rho \geq 0\} \cup \{\lambda = \rho e^{i\omega'}; 0 \leq \rho < \infty\}$$

The corresponding kernel function is

$$G(x, \xi; t) = \frac{-1}{2\pi i} \int_{\Gamma} e^{-\lambda t} (\tilde{R}_\lambda(x, \xi)) d\lambda.$$

We modify the integral path to $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$:

$$\Gamma_1 = \{\lambda = \rho e^{i\omega'}; \infty > \rho \geq k|t|^{-2} |x-\xi|^2\}$$

$$\Gamma_2 = \{\lambda = k|t|^{-2} |x-\xi|^2 e^{i\theta}; \omega' \leq \theta \leq 2\pi - \theta\}$$

$$\Gamma_3 = \{\lambda = \rho e^{-i\omega'}; k|t|^{-2} |x-\xi|^2 \leq \rho < \infty\}$$

Here the constant $k > 0$ is chosen so small that

$$|\lambda| |t| = k|t|^{-1} |x-\xi|^2 \leq 2^{-1} |k|^{1/2} k_2 |t|^{-1} |x-\xi|^2 = 2^{-1} k_2 |\lambda|^{1/2} |x-\xi|$$
holds on the path $\Gamma_2$. We estimate the integral on each path.

\[
\left| \frac{-1}{2\pi i} \int_{\Gamma_1} e^{-\lambda t} \tilde{R}_\lambda(x, \xi) d\lambda \right| \leq k_0 \int_{\frac{|x-\xi|^2}{|t|^2}}^{\infty} e^{-k_1 |\rho|^2} |\rho|^{-1/2} e^{-k_1 |\rho|^{1/2} |x-\xi|^2} d\rho
\]

\[
\leq k_0 e^{-k_1 |t|^{-1/2} |x-\xi|^2} \int_{\frac{|x-\xi|^2}{|t|^2}}^{\infty} e^{-k_1 |t| |\rho|} |\rho|^{-1/2} d\rho
\]

\[
\leq k_0 e^{-k_1 |t|^{-1/2} |x-\xi|^2} O(|t|^{-1/2})
\]

Similarly,

\[
\left| \frac{-1}{2\pi i} \int_{\Gamma_3} e^{-\lambda t} \tilde{R}_\lambda(x, \xi) d\lambda \right| \leq k_0 |t|^{-1/2} e^{-k_1 |t|^{-1} |x-\xi|^2}
\]

with some constants $k_0, k_1 > 0$. Finally, holds on the path $\Gamma_2$. We estimate the integral on each path.

\[
\left| \frac{-1}{2\pi i} \int_{\Gamma_2} e^{-\lambda t} \tilde{R}_\lambda(x, \xi) d\lambda \right| \leq k_0 \int_{\Gamma_2} e^{2^{-1} k_2 |\lambda|^2 |x-\xi|^2} |\rho|^{-1/2} e^{-k_2 |\lambda|^2 |x-\xi|^2} d|\lambda|
\]

\[
\leq k_0 \int_{2\pi - \omega'}^{2\pi - \omega'} e^{-k_3 |t|^{-1} |x-\xi|^2} (|t|^{-1} |x-\xi|^2)^{1/2} |T|^{-1/2} d\theta
\]

\[
\leq k_0 |t|^{-1/2} e^{-k_4 |t|^{-1} |x-\xi|^2}
\]