ON THE FUNCTIONAL DERIVATIVES OF THE
GENERATING FUNCTIONAL FOR CORRELATION
FUNCTIONS

WATARU ICHINOSE*

Department of Mathematical Science, Shinshu University, Matsumoto 390-8621, Japan.
E-mail: ichinose@math.shinshu-u.ac.jp

ABSTRACT. Let a Lagrangian function be given in $[0,T] \times \mathbb{R}^n$. We
add a so-called source term $x \cdot J(t)$ to the Lagrangian function, where
$J(t) = (J_1(t), \ldots, J_n(t))$ is an $\mathbb{R}^n$-valued continuous function in the in-
terval $[0,T]$. Then the Feynman path integral with this new Lagrangian
function can be defined rigorously by the time-slicing method through bro-
ken line paths. This path integral is a functional of $J(t)$ into weighted
Sobolev spaces and called the generating functional. In the present paper
it is rigorously proved that the generating functional is some times con-
tinuously differentiable in the Fréchet sense correspondingly to a weighted
Sobolev space and its derivatives give correlation functions. This result has
been well known in physics roughly.

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1. INTRODUCTION.

This paper is a continuation of [4, 5, 6]. We consider some charged non-relativistic particles in an electromagnetic field. For the sake of simplicity we suppose the charge and the mass of every particle to be one and $m > 0$, respectively. Let $0 < T < \infty$ be arbitrary. We consider $x \in \mathbb{R}^n$ and $t \in [0, T]$. Let $E(t, x) = (E_1, \ldots, E_n) \in \mathbb{R}^n$ and $(B_{jk}(t, x))_{1 \leq j < k \leq n} \in \mathbb{R}^{n(n-1)/2}$ denote electric strength and magnetic strength tensor, respectively and $(V(t, x), A(t, x)) = (V, A_1, \ldots, A_n) \in \mathbb{R}^{n+1}$ an electromagnetic potential, i.e.

$$E = -\frac{\partial A}{\partial t} - \frac{\partial V}{\partial x},$$

$$d(\sum_{j=1}^{n} A_j dx_j) = \sum_{1 \leq j < k \leq n} B_{jk} dx_j \wedge dx_k \text{ on } \mathbb{R}^n,$$

where $\partial V/\partial x = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n)$. Then the Lagrangian function $\mathcal{L}(t, x, \dot{x})$ ($\dot{x} \in \mathbb{R}^n$) is given by

$$\mathcal{L}(t, x, \dot{x}) = \frac{m}{2} |\dot{x}|^2 + \dot{x} \cdot A - V$$

and the Hamiltonian operator $H(t)$ is defined by

$$H(t) = \frac{1}{2m} \sum_{j=1}^{n} \left( \frac{1}{i} \frac{\partial}{\partial x_j} - A_j \right)^2 + V,$$

where we set the Planck constant $\hbar = 1$ in the present paper. Let $U(t, s)f$ be the solution to the Schrödinger equation

$$i \frac{\partial}{\partial t} u(t) = H(t)u(t), \quad u(s) = f.$$  

We define the multiplication operator $\hat{x}_j$ ($j = 1, \ldots, n$) by $(\hat{x}_j f)(x) = x_j f(x)$ and $\hat{x}_j(t)$ by $U(t, 0)^{-1} \hat{x}_j U(t, 0)$. Let $(\mathbb{R}^n)^{[s,t]}$ be the space of all paths $q = (q_1, \ldots, q_n) : [s, t] \ni \theta \rightarrow q(\theta) \in \mathbb{R}^n$. Then the classical action $S(t, s; q)$ for some $q \in (\mathbb{R}^n)^{[s,t]}$ is given by

$$S(t, s; q) = \int_{s}^{t} \mathcal{L}(\theta, q(\theta), \dot{q}(\theta))d\theta, \quad \dot{q}(\theta) = \frac{dq}{d\theta}(\theta).$$
We denote the Banach space of all continuous paths \( q : [s, t] \ni \theta \mapsto q(\theta) \in \mathbb{R}^n \) with the norm \( \|q\|_{\infty} := \sum_{j=1}^{n} \max_{s \leq \theta \leq t} |q_j(\theta)| \) by \( C([s, t]; \mathbb{R}^n) \). Let \( J \in C([s, t]; \mathbb{R}^n) \). We add a so-called source term \( x \cdot J(t) \) to \( \mathcal{L}(t, x, \dot{x}) \). We write the classical action for this new Lagrangian function

\[
S^J(t, s; q) = \int_{s}^{t} \left( \mathcal{L}(\theta, q(\theta), \dot{q}(\theta)) + x \cdot J(\theta) \right) d\theta.
\]  

(1.6)

For the sake of simplicity let \( n = 1 \). In physics \( |x> \) is an eigenfunction of \( \hat{x} \) with an eigenvalue \( x \). We set \( |x, t> = U(t, 0)^{-1}|x> \) and denote its complex conjugate by \( <x, t| \). Let \( t < t' \) and \( t \leq \theta_j \leq t' \) \((j = 1, \ldots, k)\). Then correlation functions \( <x', t'|T[\hat{x}(\theta_1) \cdots \hat{x}(\theta_k)]|x, t> \) are formally defined by the integration of the product of \( <x', t'| \) and \( T[\hat{x}(\theta_1) \cdots \hat{x}(\theta_k)]|x, t> \), where \( T[\hat{x}(\theta_1) \cdots \hat{x}(\theta_k)] \) is the product of the operators in decreasing time order, regardless of their order as written (cf. page 223 in [9] or page 182 in [10]). Then we have the path integral formula formally

\[
<x', t'|T[\hat{x}(\theta_1) \cdots \hat{x}(\theta_k)]|x, t> = \int e^{iS^{x', t}|q(\theta_1) \cdots q(\theta_k)Dq},
\]  

(1.7)

where the path integral is taken over the space \( \{q \in (\mathbb{R}^n)^{[t, t']}; q(t) = x, q(t') = x'\} \) (cf. (6.13) in [9]). The generating functional \( \tilde{F}(J) \) of \( J \in C([t, t']; \mathbb{R}^n) \) is formally defined by the path integral

\[
\tilde{F}(J) := \int e^{iS^{x', t}|q(\theta_1) \cdots q(\theta_k)Dq}
\]  

(1.8)

over the space \( \{q \in (\mathbb{R}^n)^{[t, t']}; q(t) = x, q(t') = x'\} \) (cf. page 223 in [9]). Let’s take formally its functional derivative. Then we have

\[
<x', t'|T[\hat{x}(\theta_1) \cdots \hat{x}(\theta_k)]|x, t>
= (-i)^k \frac{\delta^k \tilde{F}(J)}{\delta J(\theta_1) \cdots \delta J(\theta_k)}|_{J=0} \quad (k = 0, 1, \ldots)
\]  

(1.9)

(cf. (6.14) in [9]).
Our aim in the present paper is to show (1.9) rigorously. We note that we write (1.8) a slightly different form to obtain the rigorous result. The author does not know that such a result has been published (cf. [3, 8, 11]). We also note that (1.7) has been already proved rigorously in [6] in a slightly different form.

2. THE MAIN RESULT

Let $X$ and $Y$ be Banach spaces with norms $\| \cdot \|_X$ and $\| \cdot \|_Y$ respectively. We denote the space of all linear bounded operators from $X$ into $Y$ by $B(X; Y)$. Let $\Omega$ be a subset of $X$ and $G : \Omega \to Y$ Fréchet differentiable on $\Omega$. That is, there exists a Fréchet derivative $D_xG(x_0) \in B(X; Y)$ of $G$ at any $x_0 \in \Omega$ such that we have

$$\|G(x_0 + h) - G(x_0) - D_xG(x_0)h\|_Y = o(1)\|h\|_X$$

as $\|h\|_X \to 0$. In the same way a $k$ times ($k = 0, 1, \ldots$) Fréchet derivative $D_x^kG(x_0) \in B(X; B(X; \cdots; B(X; Y)) \cdots)$ at $x_0 \in \Omega$ can be defined (cf. Chapter I in [12]). Let $G : \Omega \to Y$ be a $k$ times Fréchet differentiable on $\Omega$. We write $D_x^kG(x_0)[h^{(1)}, \ldots, h^{(k)}] = (\cdots((D_x^{k-1}G(x_0)h^{(k)})h^{(k-1)})\cdots)h^{(1)}$ for $h^{(j)} \in X$ ($j = 0, \ldots, k$). Then we can identify $D_x^kG(x_0)$ with a $k$-linear bounded operator from $X \times \cdots \times X$ into $Y$ (cf. Chapter I in [12]). Let $x_0 \in \Omega$, $h \in X$ and $h' \in X$. Then we can easily have

$$\|D_xG(x_0 + h') - D_xG(x_0) - D_x^2G(x_0)h'\|_{B(X; Y)} = o(1)\|h'\|_X$$

and so

$$\|D_xG(x_0 + h')h - D_xG(x_0)h - (D_x^2G(x_0)h')h\|_Y = o(1)\|h'\|_X.$$  

Consequently we have

$$(D_x(D_xG(x_0))h')|_{x=x_0} = (D_x^2G(x_0)h')h = D_x^2G(x_0)[h, h'].$$
In the same way we obtain

\[ D^k_x G(x_0)[h^{(1)}, \ldots, h^{(k)}] = (D_x(D_x(\cdots(D_x(D_xG(x)h^{(1)})h^{(2)})\cdots)h^{(k)}|_{x=x_0}. \quad (2.1) \]

For an \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) and a multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \) we write \( |\alpha| = \sum_{j=1}^{n} \alpha_j \), \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) and \( \partial_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n} \). Hereafter we write \( X = C([0, T]; \mathbb{R}^n) \). Let \( \Delta : 0 = \tau_0 < \tau_1 < \ldots < \tau_\nu = T \) be a subdivision of the interval \([0, T] \). We set \( |\Delta| = \max_{1 \leq j \leq \nu}(\tau_j - \tau_{j-1}) \). Let \( x^{(j)} \in \mathbb{R}^n \) \( (j = 0, \ldots, \nu - 1) \) and \( x \in \mathbb{R}^n \). We denote by \( q_\Delta(\theta; x^{(0)}, \ldots, x^{(\nu-1)}, x) \) in \((\mathbb{R}^n)^[0,T]\) the broken line path joining points \( x^{(j)} \) \( (j = 0, \ldots, \nu, x^{(\nu)} = x) \) at \( \tau_j \) in order. We set

\[ q_{x,y}^{t,s}(\theta) = y + \frac{\theta - s}{t - s}(x - y) \in (\mathbb{R}^n)^{[s,t]}. \quad (2.2) \]

For \( J \in X \) we write

\[ (C^J(t, s)f)(x) = \begin{cases} \sqrt{m/(2\pi i(t-s))}^n \int (\exp iS^J(t, s; q_{x,y}^{t,s})) f(y)dy, & s < t, \\ f(x), & s = t \end{cases} \quad (2.3) \]

for \( f \in C_0^\infty(\mathbb{R}^n) \), where \( C_0^\infty(\mathbb{R}^n) \) denotes the space of all infinitely differentiable functions in \( \mathbb{R}^n \) with compact support. Let \( \chi \in C_0^\infty(\mathbb{R}^n) \) such that \( \chi(0) = 1 \) and \( \epsilon > 0 \). Then we have

\[ \left( \prod_{j=1}^{\nu} \sqrt{\frac{m}{2\pi i(\tau_j - \tau_{j-1})}} \right)^n \int \cdots \int (\exp iS^J(t, s; q_\Delta)) \chi(\epsilon x^{(\nu-1)}) \times \chi(\epsilon x^{(\nu-2)}) \cdots \chi(\epsilon x^{(0)}) f(q_\Delta(0))dx^{(0)} \cdots dx^{(\nu-1)} = C^J(T, \tau_\nu-1)x(\epsilon \cdot)C^J(\tau_{\nu-1}, \tau_{\nu-2}) \cdots x(\epsilon \cdot)C^J(\tau_1, \tau_0)x(\epsilon \cdot)f \quad (2.4) \]

for \( f \in C_0^\infty(\mathbb{R}^n) \).

We have from Lemma 6.1 in [5]
Lemma 2.1. Let $\partial^\alpha_x E_j(t, x)$ $(j = 1, \ldots, n)$, $\partial^\alpha_x B_{jk}(t, x)$ and $\partial_t B_{jk}(t, x)$ $(1 \leq j < k \leq n)$ be continuous in $[0, T] \times \mathbb{R}^n$ for all $\alpha$. We assume

$$|\partial^\alpha_x E_j(t, x)| \leq C_\alpha, \ |\alpha| \geq 1, \ j = 1, \ldots, n$$

(2.5)

and

$$|\partial^\alpha_x B_{jk}(t, x)| \leq C_\alpha < x >^{-(1+\delta)}, \ |\alpha| \geq 1, \ 1 \leq j < k \leq n$$

(2.6)

in $[0, T] \times \mathbb{R}^n$, where $< x > = \sqrt{1 + |x|^2}$ and $\delta > 0$ are constants that may depend on $\alpha$. Then there exists a continuous potential $(V, A)$ in $[0, T] \times \mathbb{R}^n$ such that

$$|\partial^\alpha_x A_j(t, x)| \leq C_\alpha, \ |\alpha| \geq 1, \ j = 1, \ldots, n$$

(2.7)

and

$$|\partial^\alpha_x V(t, x)| \leq C_\alpha < x >, \ |\alpha| \geq 1$$

(2.8)

in $[0, T] \times \mathbb{R}^n$.

We note that two of the Maxwell equations

$$d(\sum_{j=1}^{n} E_j dx_j) = - \sum_{1 \leq j < k \leq n} \partial_t B_{jk} dx_j \wedge dx_k \quad \text{on } \mathbb{R}^n,$$

$$d \left( \sum_{1 \leq j < k \leq n} B_{jk} dx_j \wedge dx_k \right) = 0 \quad \text{on } \mathbb{R}^n$$

are used in the proof of Lemma 2.1. The lemma below follows from Lemma 2.1 in [5].

Lemma 2.2. Suppose the assumptions of Lemma 2.1 and take a potential $(V, A)$ satisfying (2.7) and (2.8). Let $f \in C_0^{\infty}(\mathbb{R}^n)$. Then $\partial^\alpha_x (C^J(t, s)f)(x)$ exist for all $\alpha$ and are continuous in $0 \leq s \leq t \leq T$ and $x \in \mathbb{R}^n$. 


Let $L^2 = L^2(R^n)$ be the space of all square integrable functions in $R^n$ with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$. We introduce weighted Sobolev spaces $B^a := \{ f \in L^2; \| f \|_{B^a} := \| f \| + \sum_{|\alpha|=a}(\| x^\alpha f \| + \| \partial_\alpha f \|) < \infty \} (a = 1, 2, \ldots)$. We set $B^0 = L^2$.

**Lemma 2.3.** Suppose the assumptions of Lemma 2.1 and take a potential $(V, A)$ satisfying (2.7) and (2.8). Then there exist a $\rho^* > 0$, which is determined from the constants $C_\alpha$ of (2.5)-(2.7), and constants $K_a \geq 0 (a = 0, 1, \ldots)$ independent of $J \in X$ such that we have

$$
\| C^J(t, s)f \|_{B^a} \leq e^{K_a(t-s)}\| f \|_{B^a}, \quad 0 \leq t - s \leq \rho^*
$$

(2.9) for all $J \in X$.

**Proof.** Lemma 2.3 follows from the proofs of Theorem 3.3 in [5] and Proposition 3.4 in [6]. Q.E.D.

Let $|\Delta| \leq \rho^*$. Suppose the assumptions of Lemma 2.1 and take a potential $(V, A)$ satisfying (2.7) and (2.8). The multiplication operator $\chi(\epsilon \cdot)$ is a bounded operator from $B^a (a = 0, 1, \ldots)$ into $B^a$. So it follows from Lemma 2.3 that the operator on $C_0^\infty(R^n)$ defined by (2.4) can be extended to a bounded operator from $B^a$ into $B^a$. Moreover we have for $f \in B^a$

$$
C^J(T, \tau_{\nu-1})\chi(\epsilon \cdot)C^J(\tau_{\nu-1}, \tau_{\nu-2})\chi(\epsilon \cdot)\cdots\chi(\epsilon \cdot)C^J(\tau_1, 0)\chi(\epsilon \cdot)f
= C^J(T, \tau_{\nu-1})C^J(\tau_{\nu-1}, \tau_{\nu-2})\cdots C^J(\tau_1, 0)f
= \sum_{j=0}^{\nu-1} C^J(T, \tau_{\nu-1})\chi(\epsilon \cdot)\cdots\chi(\epsilon \cdot)C^J(\tau_{j+1}, \tau_{j})\chi(\epsilon \cdot) - 1)C^J(\tau_{j}, \tau_{j-1}) \cdot C^J(\tau_{j-1}, \tau_{j-2}) \cdots C^J(\tau_1, 0)f.
$$
Consequently we get from the Lebesgue dominated convergence theorem

\[
\lim_{\epsilon \to 0} C^J(T, \tau_{\nu-1}) \chi(\epsilon \cdot) C^J(\tau_{\nu-1}, \tau_{\nu-2}) \chi(\epsilon \cdot) \cdots C^J(\tau_1, 0) \chi(\epsilon \cdot)f
\]

\[
= C^J(T, \tau_{\nu-1}) C^J(\tau_{\nu-1}, \tau_{\nu-2}) \cdots C^J(\tau_1, 0) f
\]  
\[(2.10)\]

in \(B^a\). We write the operator defined by (2.10) as \(G_\Delta(J)f\) or \(\int (\exp i S^J(T, 0; q_\Delta)) f(q_\Delta(0)) Dq_\Delta\).

**Proposition 2.4.** Let \(J \in X\) and \(f \in B^a\) \((a = 0, 1, \ldots)\). Suppose the assumptions of Lemma 2.1 and take a potential \((V, A)\) satisfying (2.7) and (2.8). Then there exists a limit \(G(J)f\) of \(G_{\Delta}(J)f\) in \(B^a\) as \(|\Delta| \to 0\).

**Proof.** This proposition follows from Theorem 1 in [6]. We note that \(G(J)f\) gives the solution to the Schrödinger equation (1.4) where \(V\) is replaced by \(V - J(t) \cdot x\). Q.E.D.

**Remark 2.1.** Suppose the assumptions of Lemma 2.1 and take a potential \((V', A')\) satisfying (2.7) and (2.8). Let \((V, A)\) be an arbitrary potential such that \(V, \partial V/\partial x_j, \partial A_j/\partial t\) and \(\partial A_j/\partial x_k (j, k = 1, 2, \ldots, n)\) are continuous in \([0, T] \times R^n\). Then it follows from the proof of Theorem in [5] that there exists a continuously differentiable function \(\psi(t, x)\) in \([0, T] \times R^n\) satisfying

\[
(-V dt + A \cdot dx) - (-V' dt + A' \cdot dx) = d\psi.
\]

This gives the gauge invariance of (2.4) and so \(G_{\Delta}(J)f\), i.e.

\[
G_{\Delta}(J)f = e^{i\psi(T, \cdot)} G'_\Delta(J) e^{-i\psi(0, \cdot)}.
\]  
\[(2.11)\]

Hence we get the same assertion as in Proposition 2.4 for \(a = 0\).

We sometimes write

\[
\int (\exp i S^J(T, 0; q)) f(q(0)) Dq = G(J)f.
\]  
\[(2.12)\]
Let \(0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq T\) \((k = 0, 1, \ldots)\), \(\epsilon > 0\) a constant and \(|\Delta| \leq \rho^*\). We consider for \(f \in C_0^\infty(\mathbb{R}^n)\)

\[
\left(\prod_{j=1}^\nu \frac{m}{2\pi i(\tau_j - \tau_{j-1})}\right)^n \int \cdots \int (\exp iS^J(t, s; q_{\Delta}))\chi(\epsilon x^{(\nu-1)})\chi(\epsilon x^{(\nu-2)}) \ldots \chi(\epsilon x^{(0)})\left(\prod_{j=1}^k (q_{\Delta})_{l_j}(t_j)\right)f(q_{\Delta}(0))dx^{(0)} \cdots dx^{(\nu-1)}.
\]  

(2.13)

**Proposition 2.5.** Suppose the assumptions of Lemma 2.1 and take a potential \((V, A)\) satisfying (2.7) and (2.8). Then the operator (2.13) for \(f \in C_0^\infty(\mathbb{R}^n)\) can be extended to a bounded operator from \(B^{a+k}\) \((a = 0, 1, \ldots)\) into \(B^a\). For \(f \in B^{a+k}\) there exists its limit in \(B^a\) as \(\epsilon \to 0\), which we write \(\int (\exp iS^J(T, 0; q_{\Delta}))\left(\prod_{j=1}^k (q_{\Delta})_{l_j}(t_j)\right)f(q_{\Delta}(0))Dq_{\Delta}\). Moreover, for \(f \in B^{a+k}\) there exists its limit \(\int (\exp iS^J(T, 0; q))\left(\prod_{j=1}^k (q)_{l_j}(t_j)\right)f(q(0))Dq\) in \(B^a\) as \(|\Delta| \to 0\), which is equal to \(U^J(T, t_k)\hat{x}_{l_k}U^J(t_k, t_{k-1}) \cdots \hat{x}_{l_1}U^J(t_1, 0)f\).

**Proof.** This proposition follows from Theorem 2 in [6]. Q.E.D.

Let \(l\) be an non-negative integer and \(f \in B^{a+l}\) \((a = 0, 1, \ldots)\). We can consider

\[G(\cdot)f : X \to B^a\ (\subseteq B^{a+l})\]

from Proposition 2.4. Let \(k = 0, 1, \ldots, l, h^{(j)} \in X\) \((j = 1, 2, \ldots, k)\) and \(0 \leq \theta_1, \ldots, \theta_k \leq T\). We know from Proposition 2.5 in the present paper and Theorem 1 in [6] that the path integral \(\int (\exp iS^J(T, 0; q))\left(\prod_{j=1}^k h^{(j)}(\theta_j) \cdot q(\theta_j)\right)\times f(q(0))Dq\) of variables \(\theta_j \in [0, T]\) \((j = 1, 2, \ldots, k)\) is continuous in \(B^a\). Then we have the following as the main theorem in the present paper.

**Theorem 2.6.** Suppose the assumptions of Lemma 2.1 and take a potential \((V, A)\) satisfying (2.7) and (2.8). Then for \(k = 0, 1, \ldots, l\) the functional \(G(\cdot)f\)
from $X$ into $B^a$ has a $k$-times Fréchet derivative $D^k_J G(J)f[h^{(1)}, \ldots, h^{(k)}]$, which is equal to

$$i^k \int_0^T d\theta_1 \int_0^T d\theta_2 \cdots \int_0^T d\theta_k \int e^{iS_j(T,0;\theta)} \left( \prod_{j=1}^k h^{(j)}(\theta_j) \cdot q(\theta_j) \right) f(q(0)) Dq.$$  

(2.14)

**Remark 2.2.** Suppose the assumptions of Remark 2.1. Then we have the gauge invariance (2.11) of $G_\Delta(J) f$. Hence, taking the appropriate spaces in place of $B^a (a = 0, 1, \ldots)$, the same assertions as in Proposition 2.5 and Theorem 2.6 hold.

We write the integrand of (2.14) with respect to the integration variables $\theta_j (j = 1, 2, \ldots, k)$ as

$$\frac{\delta^k G(J)}{\delta J(\theta_1) \cdots \delta J(\theta_k)} [h^{(1)}, \ldots, h^{(k)}],$$  

(2.15)

which is called the functional derivative of $G$ with respect to variation of the function $J(t)$ at $\theta_1, \ldots, \theta_{k-1}$ and $\theta_k$ (cf. §7-2 in [2]). Then we get the result corresponding to (1.9) from Theorem 2.6 in the present paper and Corollary in [6].

**Corollary 2.7.** Suppose the assumptions of Lemma 2.1 and take a potential $(V, A)$ satisfying (2.7) and (2.8). Let $l$ be a non-negative integer and $f \in B^{a+l} (a = 0, 1, \ldots)$. Then there exist (2.15) in $B^a$ for $k = 0, 1, \ldots, l$. Setting $J = 0$, these (2.15) are equal to $i^k \int (\exp iS(T,0;\theta)) \left( \prod_{j=1}^k h^{(j)}(\theta_j) \cdot q(\theta_j) \right) \times f(q(0)) Dq = U(T,0) \cdot T[h^{(1)}(\theta_1) \cdot \hat{q}(\theta_1), \ldots, h^{(k)}(\theta_k) \cdot \hat{q}(\theta_k)] f$.  

Let $C^J(t,s)$ be the operator defined by (2.3) and $C(t,s) = C^0(t,s)$. The following is the key lemma for the proof of Theorem 2.6.
Lemma 2.8. Suppose the assumptions of Lemma 2.1 and take a potential $(V, A)$ satisfying (2.7) and (2.8). Then we have for $f \in B^{a+2}$ ($a = 0, 1, \ldots$)

$$C^J(t, s)f = C(t, s)f + i \int_s^t d\theta \left( \frac{m}{2\pi i(t-s)} \right)^n \int (\exp iS(t, s; q_{x,y}^{t,s})) \times$$

$$\left( J(\theta) \cdot q_{x,y}^{t,s}(\theta) \right) f(y) dy + R^J(t, s)f$$

(2.16)

and

$$\|R^J(t, s)f\|_{B^a} \leq C^a(t-s)^2 \|J\|_\infty^2 \|f\|_{B^{a+2}},$$

(2.17)

where $C^a \geq 0$ are constants.

Proof. We can easily have

$$e^{i\tau} - 1 - i\tau = -\int_0^1 (1 - \theta)e^{i\theta\tau} d\theta \tau^2,$$

where

$$\tau = \int_s^t J(\theta) \cdot q_{x,y}^{t,s}(\theta) d\theta = \int_s^t J(\theta) \cdot \left( x - \frac{t - \theta}{\sqrt{t-s}} w \right) d\theta, \quad w = \frac{x-y}{\sqrt{t-s}}.$$

We can prove Lemma 2.8 from the above by means of Theorem 4.4 in [5]. Q.E.D.

The detailed proof of Theorem 2.6 will be given in [7].

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