数理解析研究所講究録
1480 巻 2006 年 179-191



cond-mat/0506613

# States with $v_1 = \lambda, v_2 = -\lambda$ and reciprocal equations in the six-vertex model

M.J. Rodríguez-Plaza[1]

*Departamento de Física Teórica I, Facultad de Ciencias Físicas*
*Universidad Complutense de Madrid, 28040 Madrid, Spain*

Abstract

The eigenvalues of the transfer matrix in a six-vertex model (with periodic boundary conditions) can be written in terms of $n$ constants $v_1, \ldots, v_n$, the zeros of the function $Q(v)$. A peculiar class of eigenvalues are those in which two of the constants $v_1, v_2$ are equal to $\lambda, -\lambda$, with $\Delta = -\cosh\lambda$ and $\Delta$ related to the Boltzmann weights of the six-vertex model by the usual combination $\Delta = (a^2 + b^2 - c^2)/2ab$. The eigenvectors associated to these eigenvalues are Bethe states (although they seem not). We count the number of such states (eigenvectors) for $n = 2, 3, 4, 5$ when $N$, the columns in a row of a square lattice, is arbitrary. The number obtained is independent of the value of $\Delta$, but depends on $N$. We give the explicit expression of the eigenvalues in terms of $a, b, c$ (when possible) or in terms of the roots of a certain reciprocal polynomial, being very simple to reproduce numerically these special eigenvalues for arbitrary $N$ in the blocks $n$ considered. For real $a, b, c$ such eigenvalues are real.

*PACS numbers: 05.50.+q 75.10.Hk*
*Keywords: Statistical mechanics; six-vertex model; transfer matrix; Bethe ansatz; $Q(v)$ function; reciprocal polynomial*

## 1 The problem

Some time ago the author of this note read in the paper *Completeness of the Bethe Ansatz for the Six and Eight-Vertex Models* by R.J Baxter [1, Sect. 4] the following sentence concerning certain proper states of the transfer matrix in the six-vertex model at zero-field:

> The other problem that we encountered first occurs for $N = 4$ and $n = 2$, then for even $N$ and $2 \leq n \leq N-2$. It is referred to by Bethe himself and has been considered by others since[2]. For some eigenvalues with momentum $\pm 1$, [3] i.e. $k_1 + \cdots + k_n = 0$ or $\pi$, we found that
>
> $$Q(v) = \prod_{j=1}^{n} \sinh[(v - v_j)/2]$$
>
> had a pair of zeros $v_1, v_2$ such that $v_1 = \lambda, v_2 = -\lambda$.

[1] E-mail: mjrplaza@fis.ucm.es

[2] [2, after eq. (23)],[3], [4], [5]

[3] Baxter means $e^{i(k_1 + \cdots + k_n)} = \pm 1$

The lines continued later as follows:

> For $N = 4$ there was just one such eigenvalue $\Lambda$, in the $n = 2$ central block. For $N = 6$ there was one in the $n = 2$ block, two in the $n = 3$ block, and one in the $n = 4$ block. For $N = 8$ there were $1, 2, 5, 2, 1$ in the $n = 2, 3, 4, 5, 6$ blocks, respectively. This suggests (tentatively) that the Catalan numbers may count such eigenvalues.[4] The momenta were $-1$ except for a single eigenvalue with momentum $+1$ in each block with $3 \leq n \leq N - 3$.

If the author had understood properly the eigenvectors associated to such eigenvalues and how to obtain them from Bethe ansatz, probably would have not detained so long when reading these sentences. But that was not the case: we were calculating at that time the free-energy per site of a vertex model whose ground state was a state of this type, and the value of the free-energy that we were deriving was once and again the incorrect one. We decided in consequence to put aside the free-energy problem for a time and study instead these states in the six-vertex model. We ignore the correct name that we shall use for them. In the literature they have received the name of *singular* Bethe states or *singularities* of the Bethe solutions [3, 4, 6], and also *non-Bethe* eigenvectors [7]. We might even remember some references in which they are alluded as *improper* states. Since they need a name and no other states are considered in this paper we will refer to them as *bound pairs* merely.

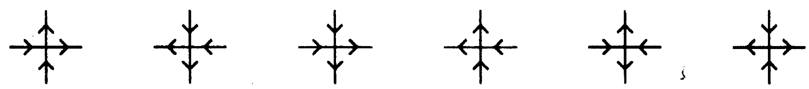
This note communicates some results of the study and answers the interrogation suggested in Baxter's paper: *Are Catalan numbers counting the bound pair states of a square six-vertex model with periodic boundary conditions?*

## 2 The model

The model to be considered is a six-vertex model in a square lattice [8, 9]. In this model to each site of the lattice is associated one of the six arrangements of arrows shown in figure 1, where each of these arrangements has an energy $\varepsilon_1, \ldots, \varepsilon_6$ and a Boltzmann weight given by

$$\omega_j = \exp(-\varepsilon_j / k_B T), \qquad j = 1, \ldots, 6.$$

The configurations of arrows satisfy the 'ice rule', because at each site of the lattice there are two arrows in and two arrows out.



1 2 3 4 5 6

Figure 1. The six configurations allowed at a vertex. At each site of the lattice there are two arrows in and two arrows out. This is known as the 'ice-rule'.

Suppose that the lattice has dimensions $M \times N$, that is $N$ sites horizontally and $M$ vertically, with the imposition of periodic boundary conditions in both directions. The state of an arbitrary row of $N$ vertical edges is then specified by the configuration of up and down arrows on the edge. Let $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N)$ denote the state ($\sigma_j = +1$ for an up arrow at vertex $j$, $\sigma_j = -1$ for a down arrow). If $\boldsymbol{\sigma}$ is the state of a row and $\boldsymbol{\sigma}'$ the state of the row bellow, the two adjacent states are coupled by the transfer matrix $T_{\sigma\sigma'}$, whose entries are given by a trace of $2 \times 2$ matrices

$$T_{\sigma\sigma'} = \operatorname{trace} R_{\sigma_1\sigma_1'} R_{\sigma_2\sigma_2'} \cdots R_{\sigma_N\sigma_N'}, \tag{2.1}$$

where

$$R_{++} = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_4 \end{pmatrix}, \quad R_{+-} = \begin{pmatrix} 0 & \omega_5 \\ 0 & 0 \end{pmatrix}, \quad R_{-+} = \begin{pmatrix} \omega_6 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_{--} = \begin{pmatrix} \omega_3 & 0 \\ 0 & \omega_2 \end{pmatrix}.$$

[4]Catalan numbers are $1, 2, 5, 14, 132, 429, \ldots$

A consequence of the 'ice rule' together with the horizontal periodicity of the lattice is that the number $n$ of down (or up) arrows in a row is a conserved quantity from row to row, and $T$, a $2^N \times 2^N$ matrix, breaks up into $N+1$ diagonal blocks with one block for each value $n = 0, 1, \dots, N$. The dimension of block $n$ is $\binom{N}{n}$. The transfer matrix is used to calculate in statistical mechanics the partition function of the lattice $Z = \mathrm{trace}\, T^M$, and this has implied the diagozalization of matrix $T$. In the case of a zero electrical field (the case treated here) where

$$a = \omega_1 = \omega_2, \quad b = \omega_3 = \omega_4, \quad c = \omega_5 = \omega_6, \tag{2.2}$$

the eigenvalues $\Lambda$ of the transfer matrix are known to be [9]

$$\Lambda(v) = (-1)^n \frac{\phi(\lambda - v)\, Q(v + 2\lambda) + \phi(\lambda + v)\, Q(v - 2\lambda)}{q(v)}, \tag{2.3}$$

where functions $\phi(v)$, $Q(v)$ are

$$\phi(v) = \rho^N \sinh^N(v/2) \tag{2.4}$$

$$Q(v) = \prod_{j=1}^{n} \sinh\left[(v - v_j)/2\right], \tag{2.5}$$

and $\rho$, $\lambda$, $v$ are defined so that

$$a = \rho \sinh \frac{1}{2}(\lambda - v), \quad b = \rho \sinh \frac{1}{2}(\lambda + v), \quad c = \rho \sinh \lambda. \tag{2.6}$$

To write the eigenvalues (2.3) we have to locate $v_1, \dots, v_n$ for these eigenvalues. There are many solutions, corresponding to the different eigenvalues.

## 3 Catalan numbers or not

*Do Catalan numbers count the bound pair states of a square six-vertex model with periodic boundary conditions?* The answer is *no*.

As was communicated in ref. [1], it is true that for $N = 4$ there is one bound pair in the $n = 2$ central block, that for $N = 6$ there are $1, 2, 1$ in the $n = 2, 3, 4$ blocks, respectively, and that for $N = 8$ there are $1, 2, 5, 2, 1$ in the $n = 2, 3, 4, 5, 6$ blocks. However, if the counting started in the previous reference had continued it would have found that there are $1, 2, 6, 10$ in $n = 2, 3, 4, 5$ [5] for $N = 10$, and $1, 2, 7, 12$ in $n = 2, 3, 4, 5$ for $N = 12$. In fact our calculations here show that for general even $N$ the number is exactly $1, 2, N/2 + 1, N$ in the blocks $n = 2, 3, 4, 5$. The numbers of states for $n = 6$ and beyond wont be studied in this paper.

## 4 Bound pairs and Bethe Ansatz

To obtain the eigenfunctions of the transfer matrix one can either diagonalize exactly the matrix (impossible when the size is not reasonable) or use the Bethe ansatz, the trial form that Bethe used for diagonalizing the quantum-mechanical Hamiltonial of the one-dimensional Heisenberg model [2]. The ansatz suggests that the eigenstate of $T(v)$, $T(v)|\psi\rangle = \Lambda(v)|\psi\rangle$, can be written as $|\psi\rangle = \sum_{x_1 < \dots < x_n} f(x_1, \dots, x_n)\, |x_1, \dots, x_n\rangle$, where the coefficients $f(x_1, \dots, x_n)$ are

$$f(x_1, \dots, x_n) = \sum_P A_{p_1, \dots, p_n}\, e^{ik_{p_1} x_1} \dots e^{ik_{p_n} x_n}. \tag{4.1}$$

[5] We omit to mention the blocks $n = N/2 + 1$ to $N - 2$ since the number of these states is the same as in the blocks $2, \dots, N/2 - 1$ but in reverse order

The numbers $x_1, \dots, x_n$ indicate the positions of $n$ down arrows on the lower vertical edges of a row of the lattice, and are ordered so that $1 \le x_1 < x_2 < \dots x_n \le N$. We have experienced that the coefficients $A_{p_1,\dots,p_n}$ suitable to construct bound pairs[6] are given by

$$A_{p_1,\dots,p_n} = \epsilon_P / C \prod_{1 \le i < j \le n} s_{p_j,p_i}, \tag{4.2}$$

where $\epsilon_P = \pm 1$ is the sign of the permutation $\{p_1, \dots, p_n\}$ of $\{1, \dots, n\}$ and $C$ is a non-zero constant to be fixed later in the most convenient manner (usually normalization). The vertex model defined by activities $a, b, c$ so that

$$\Delta = \frac{a^2 + b^2 - c^2}{2\,ab} \tag{4.3}$$

enters in $s_{ij}$, defined as

$$s_{ij} = 1 - 2\,\Delta\, e^{ik_j} + e^{i(k_i + k_j)}. \tag{4.4}$$

To write the eigenstates is only necessary then to know the factors $e^{ik_1}, \dots, e^{ik_n}$ that appear in (4.1) and (4.4). These factors are the solutions of the equations

$$e^{iNk_{p_1}} A_{p_2,\dots,p_n,p_1} = A_{p_1,\dots,p_n}, \tag{4.5}$$

that impose the periodic boundary conditions on the problem making that $f(x_1, x_2, \dots, x_N) = f(x_2, \dots, x_N, x_1 + N)$ what identifies the $N+1$ and $1$ vertices. To be (4.5) consistent equations among themselves, it is necessary that

$$e^{iN(k_1 + \dots + k_n)} = 1. \tag{4.6}$$

Equations (4.1), (4.2), (4.4), (4.5) and (4.6), are *sufficient* equations to write bound pair eigenfunctions, and when needed we will refer to them as "the Bethe ansatz equations for bound pairs". However, and this is not less important, it is also necessary a *correct normalization* of the eigenfunction. Without it, the state cannot be obtained. We have learned the correct normalization in ref. [1, Sect. 4], and show an example later for $N = 6$ and $n = 3$. Equations (8.2), (8.3) and (8.2), (8.3), (8.6) are deduced taking into account such normalization.

It is important to write, before finishing, the relation between $k_1 \dots, k_n$ and $v_1, \dots, v_n$ in (2.5) (or better between $e^{ik_j}$ and $e^{v_j}$)

$$e^{ik_j} = \frac{e^{\lambda} - e^{v_j}}{e^{\lambda + v_j} - 1}, \tag{4.7}$$

as mentioned in many papers. This relation permits to move from the eigenvalue (2.3) to the eigenvector (4.1) of the transfer matrix when we precise it.

## 5 A change of variables

Before describing any eigenvalue we make a useful change of variables concerning $v$ and $\lambda$ in (2.6). The change is convenient for those (the author in this specific problem among them) who prefer to work with polynomials rather than with hyperbolic functions as in (2.5). Define the variables

$$z = e^{-v}, \qquad y = e^{-\lambda} \tag{5.1}$$

[6] Because they give the same result that when the transfer matrix is directly diagonalized. When the size of the matrix allows it it is possible to carry many numerical experiments and they confirm this choice for the amplitudes

instead of $v$ and $\lambda$, then (2.5) is essentially[7] the polynomial in $z$ and $1/z$ given by

$$Q(z) = \frac{1}{z^{n/2}} \prod_{j=1}^{n} (z - z_j), \tag{5.2}$$

with $z_j = e^{-v_j}$, $j = 1 \ldots, n$. To be correct we should have defined another symbol for (5.2), $\tilde{Q}(z)$ for instance, however we will use the same letter with the understanding that $Q(v)$ stands for (2.4) and $Q(z)$ for (5.2). In terms of these variables and together with definitions (2.4) and (2.5), relation (2.3) becomes

$$(2/\rho)^N \Lambda(v)\, Q(z) = \frac{(-1)^n}{(zy)^{N/2}} \left[ (z-y)^N\, Q(zy^2) + (1-zy)^N\, Q(z/y^2) \right], \tag{5.3}$$

where multiplicative constant factors in $Q$ cancel out of the calculations. To operate in a computer we prefer to work with this relation more than with (2.3).

## 6 n=2

This is the simplest case to study because the transfer matrix of $N$ edges (with $N$ even) has only one bound pair state in this block for arbitrary $\Delta$ defined in (4.3). Since bound pairs are characterized by $v_1 = \lambda$, $v_2 = -\lambda$ as mentioned in Sec. 1, function (5.2) factorizes as

$$Q(z) = (z\,y - 1)(z - y)/z, \tag{6.1}$$

the zeros of $Q(z)$ being $z_1 = y$ and $z_2 = 1/y$. Introduced this function in (5.3) and noting that the r.h.s. is *exactly* divided by $Q(z)$ in the l.h.s, the quotient affords the eigenvalue[8]

$$\Lambda(v) = a^2 b^2 \left(a^{N-4} + b^{N-4}\right) - c^2 \left(a^{N-2} + b^{N-2}\right), \qquad N \geq 4, \quad n = 2, \tag{6.2}$$

that is valid for generic $N$ even. It can be checked numerically that (6.2) is always an eigenvalue of the transfer matrix for all values of $a, b, c$ real or complex,[9] and since the block $n = 2$ is among the blocks of smallest dimensions, it can be done even for $N$ not too small. The eigenvector associated to (6.2) was known to Bethe himself [2, also after eq. (23)] and is proportional to

$$|\psi\rangle = \sum_{l=1}^{N} (-1)^l\, |l, l+1\rangle, \tag{6.3}$$

after appropriate normalization. We do not reproduce here this eigenvector with the Bethe ansatz (the example that we reproduce is for $N = 6$, $n = 3$ later), but want to comment about $e^{ik_1}$ and $e^{ik_2}$. The product of these two factors is for the eigenfunction (6.3) equal to $-1$, since from (4.1) derives the relation

$$f(x_1 + 1, x_2 + 1) = e^{i(k_1 + k_2)} f(x_1, x_2), \quad \text{with} \quad N + 1 \equiv 1, \tag{6.4}$$

which is simply a consequence of the translation invariance of the transfer matrix (2.1). But also $v_1 = \lambda$ in (4.7) fixes $e^{ik_1} = 0$, what obliges to set

$$e^{i\,k_1} = -e^{-i\,k_2} = 0, \tag{6.5}$$

[7] Essentially means up to multiplicative constants that do not depend on $z$ (they may depend on $y$ because $y$ is regarded as a constant: after all $y$ is fixed by the value that we choose for $\Delta$, and viceversa). The constants are not relevant because do not change the value of $\Lambda(v)$, as commented after equation (5.3)

[8] $\Lambda(v)$ in (5.3) is obtained in terms of $z$ and $y$, of course. We have reexpressed the result in terms of $a, b, c$ to write (6.2)

[9] $a, b, c$, the Boltzmann weights (2.2) of the vertex model, are real and positive, but when diagonalization of a matrix is considered in general, with no restriction to physical values only, they can also be negative or complex

as was done in [1]. This happens for all bound pairs that we have obtained no matter the values of $N$ and $n$: it is simply a *fact* that for these states in this model

$$e^{i\,(k_1 + k_2)} = -1. \tag{6.6}$$

This condiction, together with the two identities in (6.5) mark how to work appropiately with bound pairs.

## 7 n=3

The trial function (5.2) is now of the form

$$Q(z) = (z\,y - 1)(z - y)(z - A)/z^{3/2}, \tag{7.1}$$

with $A$ a constant (numerical or depending on $y$) to be determined. Substituting (7.1) in (5.3), the r.h.s. of this equation is *exactly* divided by $Q(z)$ in the l.h.s. if and only if $A = 0, -1, 1$ or $A$ is the solution of a certain polynomial whose coefficients depend only on $\Delta$. The root $A = 0$ is not an admissible solution because (7.1) has not the required expansion (5.2); on the contrary, roots $A = -1, 1$ yield admissible functions $Q(z)$ because the associated $\Lambda(v)$ by (5.3) are always in the spectrum of the transfer matrix, as we have verified in numerous experiments. For example, the numbers

$$\Lambda_+ \equiv 2\,a^3b^3 - abc^2\,(a^2 + ab + b^2) + c^4\,(a^2 - ab + b^2), \tag{7.2}$$
$$\Lambda_- \equiv 2\,a^3b^3 - abc^2\,(a^2 - ab + b^2) - c^4\,(a^2 + ab + b^2), \tag{7.3}$$

are eigenvalues of the $N = 6$ transfer matrix for arbitrary values of $a, b, c$. The first is for $A = -1$, the second for $A = 1$. We present some of these numerical tests in Table 1. Regarding the situation in which $A$ is the solution of a certain polynonial, when $N = 6$ such polynomial is

$$A^4 + \left(8\Delta^3 - 4\Delta\right)\,A^3 + \left(20\Delta^2 - 14\right)\,A^2 + \left(8\Delta^3 - 4\Delta\right)\,A + 1 = 0, \tag{7.4}$$

but it has to de discarded because none of the four roots of (7.4) is linked to an eigenvalue of the transfer matrix for arbitrary $\Delta$ (it can be checked also with Table 1). There are only two $Q$'s (that is, two bound pairs in the block) and two eigenvalues.

$N = 6,\, n = 3$

| $z = \exp(i\,\pi/5)$<br>$y = \exp(i\,\pi/3)$<br>$\rho = i$<br>$\Delta = -1/2$ | $z = \exp(i\,2\pi/5)$<br>$y = \exp(i\,3\pi/4)$<br>$\rho = i$<br>$\Delta = \sqrt{2}/2 = 0.707\ldots$ | $z = 64$<br>$y = 25$<br>$\rho = 1$<br>$\Delta = -313/25 = -12.52$ |
|---|---|---|
| | | $\times 10^6$ |
| 0.779 508 | 2.028 398 | 24.899 897 |
| 0.270 722 | 0.421 472 | $10.523\,295^+$ |
| 0.168 519 | 0.305 370 | 8.909 579 |
| $0.168\,519^+$ | 0.074 622 2 | 3.719 286 |
| −0.105 922 | 0.065 240 7 | −3.724 777 |
| −0.229 696 | $0.001\,298\,21^+$ | $-8.840\,959^-$ |
| $-0.465\,600^-$ | $-0.349\,829^-$ | −10.587 344 |
| −0.527 034 | −0.448 262 | −24.913 779 |
| $0.265\,632 \pm 0.336\,454\,i$ | $0.851\,314 \pm 0.535\,651\,i$ | $12.449\,161 \pm 21.567\,683\,i$ |
| $0.086\,642\,0 \pm 0.196\,887\,i$ | $0.395\,856 \pm 0.293\,548\,i$ | $4.891\,466 \pm 7.822\,856\,i$ |
| $0.084\,086\,1 \pm 0.159\,490\,i$ | $0.184\,229 \pm 0.002\,347\,62\,i$ | $4.750\,077 \pm 8.912\,538\,i$ |
| $-0.084\,223\,2 \pm 0.124\,642\,i$ | $0.098\,987\,5 \pm 0.098\,842\,6\,i$ | $-4.756\,978 \pm 8.869\,031\,i$ |
| $-0.131\,171 \pm 0.269\,132\,i$ | $0.039\,950\,5 \pm 0.112\,727\,i$ | $-4.887\,688 \pm 7.863\,040\,i$ |
| $-0.176\,703 \pm 0.299\,057\,i$ | $0.003\,395\,86 \pm 0.451\,325\,i$ | $-12.457\,140 \pm 21.571\,004\,i$ |
| $a = \sin(\pi/15) = 0.207\ldots$<br>$b = \sin(4\pi/15) = 0.743\ldots$<br>$c = \sqrt{3}/2 = 0.866\ldots$ | $a = \sin(7\pi/40) = 0.522\ldots$<br>$b = \sin(23\pi/40) = 0.972\ldots$<br>$c = \sqrt{2}/2$ | $a = 39/80 = 0.4875$<br>$b = -1599/80 = -19.9875$<br>$c = -312/25 = -12.48$ |

Table 1. In vertical are shown the 20 eigenvalues of the transfer matrix block $N=6, n=3$ for different values of $a,b,c$. The eigenvalues are obtained by numerical diagonalization of the matrix in (2.1), and each result approximated to the number arrayed in the table with the rule of 5. In all the examples we have fixed $z, y, \rho$, and $a,b,c$ are derived from them through (2.6). The values marked with $+$ and $-$ coincide, no matter the number of digits of accuracy demanded in the computation, with the theoretical values (7.2), (7.3) obtained in this paper solving (5.3). In the third column it is necessary to multiply by $10^6$ to obtain the correct eigenvalue. Notice that when $\Delta=-1/2$ the bound pair (7.9) is degenerated and the transfer matrix has another linearly independent proper state with the same eigenvalue $a^6+b^6$. This degeneration happens for all values of $a,b,c$ and not only for the particular value listed here.

The situation is the same for arbitrary $N$ even: there are only two bound pairs in the block and the generalization of (7.2) and (7.3) is

$$\Lambda_+ = a^3b^3\left(a^{N-6}+b^{N-6}\right) - abc^2\left(a^{N-4}+b^{N-4}+ab\,\frac{a^{N-5}+b^{N-5}}{a+b}\right) + c^4\left(\frac{a^{N-3}+b^{N-3}}{a+b}\right),$$
$$\Lambda_- = a^3b^3\left(a^{N-6}+b^{N-6}\right) - abc^2\left(a^{N-4}+b^{N-4}-ab\,\frac{a^{N-5}-b^{N-5}}{a-b}\right) - c^4\left(\frac{a^{N-3}-b^{N-3}}{a-b}\right),$$

that correspond to

$$Q^+(z) = (zy-1)(z-y)(z+1)/z^{3/2} \quad \text{and} \quad Q^-(z) = (zy-1)(z-y)(z-1)/z^{3/2}, \tag{7.5}$$

respectively. The quotients written in $\Lambda_\pm$ above are fictitious[10] because the divisions can be performed exactly giving as result polynomials in $a,b,c$ with no denominators.

Each eigenvector of the transfer matrix has associated a given $Q(z)$, we now calculate as an example the eigenvector associated to $Q^+$ in (7.5) for $N=6$ using Bethe ansatz.[11] For such state the product

$$e^{i\,(k_1+k_2+k_3)} = 1, \tag{7.6}$$

that can be justified in several manners: one, if the eigenvalue is known, (7.2) in this case, it is enough to set $b=0$, $a=c$ in the eigenvalue. The coefficient of $c^N$ is precisely $e^{i\,(k_1+\cdots+k_n)}$ [9]; or two, evaluating $(-1)^n Q(zy^2)/Q(z)$ at the point $z=1/y$ [1]. This gives also such product. Since the third zero of the function $Q^+$ is at $z_3=-1$, relation (4.7) indicates that $e^{i\,k_3}=-1$, that substituted in (7.6) gives the product $e^{i\,(k_1+k_2)}=-1$, something that seems to be shared by all bound pairs of the model as we remarked in (6.6). For our pair holds again (6.5) what makes that the factor $s_{21}$ vanishes according to (4.4). To obtain the correct bound pair state the rule is[12]: calculate the $s_{ij}$ that do not vanish (in the present case there are five of them) with (4.4), keeping only the dominant term as $e^{ik_1}$ goes to zero, and calculate $s_{21}$ with (4.5). In this manner, instead of writting '$s_{21}=0$' in the formulae, $s_{21}$ takes the expression that vanishes most rapidly as $e^{ik_1}$ goes to zero. This expression is

$$s_{21} = 2\,\Delta\,(1+2\,\Delta)\,e^{i(N-1)k_1}, \tag{7.7}$$

while

$$\begin{aligned} s_{12} &= 2\,\Delta\, e^{-ik_1}, & s_{13} &= 1+2\,\Delta, & s_{31} &= 1, \\ s_{23} &= e^{-ik_1}, & s_{32} &= (1+2\,\Delta)\,e^{-ik_1}. \end{aligned} \tag{7.8}$$

Note that the amplitudes obtained with (4.2) after the substitution of (7.7) and (7.8) do satisfy exactly equations (4.5), as expected. Take now $N=6$. Inserting the values (4.2) into (4.1) we find that for

[10] I.e., introduced by the author to make the expressions compact

[11] We insist on the words *Bethe ansatz* because some authors refer to bound pair states as *non-Bethe states*, and they are Bethe states

[12] We have taken this rule from [1, Sect. 4]

example, $f(1,2,3) = -2\,\Delta\,(1+2\,\Delta)^2/C$ and $f(1,2,4) = 2\,\Delta\,(1+2\,\Delta)\,e^{-ik_1}/C$. In the case $N=6$ two more components are necessary to write the eigenvector, namely

$$f(1,2,5) = -2\,\Delta\,(1+2\,\Delta)\,e^{-ik_1}/C, \qquad f(1,3,5) = -6\,\Delta\,(1+2\,\Delta)/C,$$

since the remaining components are deduced from these four with the generalization of property (6.4) to the case $n=3$. Clearly $f(1,2,4)$, $f(1,2,5)$ are the elements that grow most rapidly as $e^{ik_1}$ vanishes and the sensible choice here is to take $C$ so that $f(1,2,4)=1$. The result is the right eigenvector associated to $Q^+$ in (7.5)[13]

$$\begin{aligned}|\psi\rangle =& |1,2,4\rangle + |2,3,5\rangle + |3,4,6\rangle + |1,4,5\rangle + |2,5,6\rangle + |1,3,6\rangle \\ &- |1,2,5\rangle - |2,3,6\rangle - |1,3,4\rangle - |2,4,5\rangle - |3,5,6\rangle - |1,4,6\rangle,\end{aligned} \tag{7.9}$$

which coincides with the vector found in [3, eq. (22)] using different methods.

## 8 n=4 and n=5

There is no problem in repeating the same steps as in $n=3$ to deduce the number of bound pairs when $n=4$ or $n=5$. In fact introducing

$$Q(z) = (z\,y-1)(z-y)(z^2 + A\,z + B)/z^2 \tag{8.1}$$

into (5.3), it is possible to find constants $A$ and $B$ so that the function $\Lambda(v)$ is an eigenvalue of the transfer matrix block $n=4$ for arbitrary $a,b,c$ activities. However, we follow a different method in this section with the intention of obtaining a better trial function $Q$ not as general as in (8.1): we solve directly Bethe ansatz equations (4.5) instead[14]. The equations are already solved for $e_1$ and $e_2$ (for brevity we will use from now the notation $e_1$ to denote the number $e^{ik_1}$, $e_2$ to denote $e^{ik_2}$, and so on), since we know that $e_1=0$, $e_2=-1/e_1$, with the product $e_1e_2$ equal to $-1$ as a characteristic of bound pairs. It remains to solve for $e_3, e_4$ in the case $n=4$, and for $e_3,e_4,e_5$ in the case of $n=5$. And when resolving the same care about $s_{ij}$ has to be taken that when the eigenfunction (7.9) was constructed in the previous section: $s_{21}$ that vanishes has to be evaluated with (4.5), taking then the expression that vanishes most rapidly as $e_1$ goes to zero, and the remaining $s_{ij}$ with (4.4). With these remarks taken into consideration the equations to solve are

$$e_3^{N-1} = -\left(\frac{1-2\Delta e_3}{e_3-2\Delta}\right)\left(\frac{1-2\Delta e_3+e_3e_4}{1-2\Delta e_4+e_3e_4}\right), \tag{8.2}$$

$$e_4^{N-1} = -\left(\frac{1-2\Delta e_4}{e_4-2\Delta}\right)\left(\frac{1-2\Delta e_4+e_3e_4}{1-2\Delta e_3+e_3e_4}\right), \quad N\geq 8, \tag{8.3}$$

in the block $n=4$, and

$$e_3^{N-1} = \left(\frac{1-2\Delta e_3}{e_3-2\Delta}\right)\left(\frac{1-2\Delta e_3+e_3e_4}{1-2\Delta e_4+e_3e_4}\right)\left(\frac{1-2\Delta e_3+e_3e_5}{1-2\Delta e_5+e_3e_5}\right), \tag{8.4}$$

$$e_4^{N-1} = \left(\frac{1-2\Delta e_4}{e_4-2\Delta}\right)\left(\frac{1-2\Delta e_4+e_3e_4}{1-2\Delta e_3+e_3e_4}\right)\left(\frac{1-2\Delta e_4+e_4e_5}{1-2\Delta e_5+e_4e_5}\right), \quad N\geq 10 \tag{8.5}$$

$$e_5^{N-1} = \left(\frac{1-2\Delta e_5}{e_5-2\Delta}\right)\left(\frac{1-2\Delta e_5+e_3e_5}{1-2\Delta e_3+e_3e_5}\right)\left(\frac{1-2\Delta e_5+e_4e_5}{1-2\Delta e_4+e_4e_5}\right), \tag{8.6}$$

when $n=5$. Remember that $\Delta$ is given by (4.3) and $N$ is an even number.

[13] For general $N$ the eigenvector is $|\psi\rangle = \sum_{l=1}^{N}(|l,l+1,l+3\rangle - |l,l+2,l+3\rangle)$. The state that accompanies to $Q^-$ is $|\psi\rangle = \sum_{l=1}^{N}(-1)^l(|l,l+1,l+3\rangle + |l,l+2,l+3\rangle)$. It has some similary with (6.3) but in the block $n=3$

[14] Once $e^{ik_3},\ldots,e^{ik_n}$ are found solving Bethe equations, we use (4.7) to write $Q$ given by (5.2)

Consider the equations relative to $n = 5$ for a moment. Notice that if $(e_3, e_4, e_5)$ is a solution of equations (8.4)-(8.6) for given $N$ and $\Delta$ [15], also $(e_4, e_3, e_5)$, the interchange of $e_3$ with $e_4$, is a solution; and also it is $(e_3, e_5, e_4)$. Equations (8.4)-(8.6) do not distinghish a solution from any of its permutations. It is for this reason that two solutions are considered the same if coincide up to permutations.

There is another relevant property of the equations: if $(e_3, e_4, e_5)$ is a solution, $\left(\frac{1}{e_3}, \frac{1}{e_4}, \frac{1}{e_5}\right)$ is also a solution for the same $N$ and $\Delta$. This feature brings considerable insight into the resolution of (8.4)-(8.6). For example, if $e_3$ is in the solution so does $1/e_3$, as this property establishes, therefore $1/e_3$ is one of the numbers in $(e_3, e_4, e_5)$. If it is equal to its inverse, $e_3$ is 1 or $-1$, but if not, the inverse of $e_3$ has to be say, $e_4$, and thus $e_3e_4 = 1$. The argument is repeated with $e_4$ to conclude that $e_4$ is 1 or $-1$ or the inverse of $e_3$. Finally, it is the turn of $e_5$, that can be only $\pm 1$ and not the inverse of any other number because there are no more left numbers to be paired with. In conclusion: $(e_3, e_4, e_5)$ are $(1, 1, 1)$, $(-1, -1, -1)$ or $(e_3, e_4, \pm 1)$, with $e_3e_4 = 1$. There are no more possibilities for arbitrary $\Delta$. Something similar happens when $n = 4$: the only solutions $(e_3, e_4)$ of (8.2), (8.3) with $\Delta$ arbitrary are $(1, -1)$ or the combinations $(e_3, e_4)$ that satisfy $e_3e_4 = 1$. Obviously this is so because the two properties explained above, permutation and inversion, hold for equations (8.2), (8.3) as well[16].

**Lemma 8.1** ($n = 4$) *The numbers $e_3, e_4$ given by equations (8.2), (8.3) subject to the condition $e_3e_4 = 1$, are the roots of the quadratic polynomial*

$$x^2 - (r + 1/r)\,x + 1 = 0, \tag{8.7}$$

*where $r$ is, in turn, the solution of the polynomial of degree $N$ with coefficients fixed by $\Delta$ given by*

$$r^N - 3\,\Delta r^{N-1} + 2\,\Delta^2(r^{N-2} + r^2) - 3\,\Delta r + 1 = 0. \tag{8.8}$$

**Proof** Very simple. Just substitute directly $e_3 = r$, $e_4 = 1/r$ in (8.2) and write the relation that results. Zero solutions $r = 0$ are not wanted[17]. □

Surprisingly, the polynomial in (8.8) has the same coefficients when $N = 8$, say, that when $N = 100$, only that in this case the coefficients are distributed according to a degree 100. Equality (8.8) belongs to the class of *reciprocal equations* [10] because the coefficient of $r^N$ is the same as the independent term, the coefficient of $r^{N-1}$ the same as the coefficient of $r$, and so on. If $R$ is a root of a reciprocal equation, so it is its reciprocal $1/R$. This cannot be a surprise, merely it is an expected consequence of the second property of the Bethe equations remarked a few paragraphs above.

**Lemma 8.2** ($n = 5$) *The numbers $e_3, e_4, e_5$ given by equations (8.4)-(8.6) with the additional requirement $e_3e_4 = 1$, $e_5 = -1$, are the roots of the cubic polynomial*

$$(x + 1)\left(x^2 + (r + 1/r)\,x + 1\right) = 0, \tag{8.9}$$

*where $r$ is the solution of (for simplicity we write the polynomial when $N = 10$)*

$$\begin{aligned} r^{10} + (5\,\Delta + 2)r^9 &+ 2\,(2\,\Delta + 1)^2 r^8 + 2\,(2\,\Delta + 1)(\Delta + 1)^2(r^7 + r^6 + r^5 + r^4 + r^3) \\ &+ 2\,(2\,\Delta + 1)^2 r^2 + (5\,\Delta + 2)r + 1 = 0. \end{aligned} \tag{8.10}$$

[15] $\Delta$ fixed though arbitrary

[16] Observe that for all bound pairs obtained so far the product $e_1 \cdots e_n = \pm 1$, something already mentioned in [1] and [3]. The momentum of these states, the sum of the $k$ 's, is therefore 0 or $\pi$ (mod $2\pi$)

[17] We want $e_3e_4 = 1$ with $e_3$ and $e_4$ *finite* numbers. Therefore none of them vanishes. We do not want more *special* objects like the pair $e_1e_2 = -1$ with $e_1 = 0$

This is a reciprocal equation too. When $N$ is arbitrary, the polynomial that generalizes (8.10) is a polynomial of degree $N$: $r^{10}$, $r^9$, $r^8$ above change into $r^N$, $r^{N-1}$, $r^{N-2}$, respectively, and $r^7+\cdots+r^3$ into $r^{N-3}+\cdots+r^3$. Nothing else changes. With these directions we avoid to write the generalization explicitly.

When the requirement is $e_3e_4 = 1, e_5 = 1$, the solution $(e_3, e_4, e_5)$ of equations (8.4)-(8.6) is given by $(x-1)\ x^2+(r+1/r)\,x+1 = 0$, i.e., $e_3 = -r, e_4 = -1/r, e_5 = 1$, with $r$ the roots of the polynomial obtained changing $r$ by $-r$ and $\Delta$ by $-\Delta$ in (8.10). The polynomial thus obtained is generalized to other $N$'s with the directions explained in the previous lines.

**Proof** The substitution of $e_3e_4 = 1$ and $e_5 = -1$ in (8.6) gives no information because the l.h.s. of (8.6) reduces to a number and the r.h.s. to the same number. However, substituted in (8.4) (or in (8.5)) is obtained a relation between the sum $e_3+\dfrac{1}{e_3} = e_4+\dfrac{1}{e_4} = u$ and $\Delta$. This relation depends on $N$ and, for example, when $N = 10$ is given by

$$\begin{aligned} u^5-(5\,\Delta+2)\,u^4&+\left(8\,\Delta^2+8\,\Delta-3\right)u^3-\left(4\,\Delta^3+10\,\Delta^2-12\,\Delta-6\right)u^2\\ &+\left(4\,\Delta^3-14\,\Delta^2-16\,\Delta+1\right)u+2\,(2\,\Delta-1)\left(\Delta^2+3\,\Delta+1\right)=0. \end{aligned} \tag{8.11}$$

It is hard to see any recurrence in this equation but if $u$ is decomposed into a number and its inverse, i.e., as $u=-(r+1/r)$, $r$ is a root of (8.10), which is a much simpler equation than the previous one. The numbers $e_3=-r$, $e_4=-1/r$, $e_5=-1$, are therefore roots of (8.9) with $r$ given by (8.10) if $N = 10$. $\square$

Now we count states. Starting with $n = 4$, we have the state characterized by $(e_1, e_2, e_3, e_4) = (e_1, -1/e_1, 1, -1)$ obtained before Lemma 8.1. For this state $e_1e_2e_3e_4 = 1$, and $Q$ and $\Lambda$ are given by

$$Q(z) = (z\,y-1)(z-y)(z^2-1)/z^2, \tag{8.12}$$

$$\begin{aligned} \Lambda = a^4b^4\left(a^{N-8}+b^{N-8}\right)-a^2b^2c^2\left(a^{N-6}+b^{N-6}-2\,a^2b^2\,\frac{a^{N-8}-b^{N-8}}{a^2-b^2}\right)\\ -3\,a^2b^2c^4\left(\frac{a^{N-6}-b^{N-6}}{a^2-b^2}\right)+c^6\left(\frac{a^{N-4}-b^{N-4}}{a^2-b^2}\right),\quad N\geq 8 \end{aligned} \tag{8.13}$$

as deduced from (4.7), (5.2) and the relation (5.3). As in $\Lambda_\pm$ obtained in Sect. 7, the quotients in (8.13) are artificial, and the divisions can be performed exactly giving for $\Lambda$ an homogeneous expression of order $N$ in $a, b, c$ with constant coefficients. Regarding the solution $(e_1, -1/e_1, r, 1/r)$ of Lemma 8.1, notice that since the roots of (8.8) are single or at most double[18], there are $N/2$ different solutions because of the reciprocity of (8.7) and (8.8). For these $N/2$ solutions (i.e., states) $e_1e_2e_3e_4 = -1$, and $Q$ is given by

$$Q(z) = (z\,y-1)(z-y)\left(z^2-(t+1/t)\,z+1\right)/z^2, \tag{8.14}$$

with

$$t+\frac{1}{t} = -\frac{2\,\Delta\,(r+1/r)-4}{r+1/r-2\,\Delta},\qquad \Delta\neq\pm1. \tag{8.15}$$

The number $\Lambda(v)$ is obtained inserting (8.14) and (8.15) into (5.3). This result shows also that (8.12) and (8.14) are more accurate trial functions to solve (5.3) than the general (8.1). Contrary to what

[18]The discriminant of (8.8) in $r$ vanishes only for $\Delta = \pm1/2, \pm1$, thus indicating multiplicity of the roots $r$ more than 1 only for these values. Why for these values? Notice that the bilinear transformation $e_3 \to \frac{1-2\Delta e_3}{e_3-2\Delta}$ in the r.h.s. of (8.2) (and in the r.h.s. of (8.3) for $e_4$) collapses to a constant when $\Delta = \pm1/2$ instead of being a one-to-one mapping. This justifies the multiplicities at $\Delta = \pm1/2$. A similar reason happens when $e_3e_4 = 1$ and $\Delta = \pm1$ to the second factor in the r.h.s. of equations (8.2), (8.3)

we have done along this paper, we do not write the function $\Lambda(v)$ associated to (8.14) and (8.15) for general $N$, but we write it when $N=8$, which is

$$\Lambda(v) = 2\,a^4b^4 + c^2\left(2\,\lambda_2\,a^3b^3 - \lambda_3\,a^2b^2\,(a^2+b^2) - 2\,\lambda_1\,ab\,(a^4+b^4-a^2b^2) - a^6 - b^6\right), \tag{8.16}$$

with $\lambda_1, \lambda_2, \lambda_3$ certain numbers depending on $\Delta$ that we do not specify. The object to present (8.16) is to comment about the excluded cases $\Delta = \pm 1$ pointed in (8.15). We have excluded these two points for mathematical reasons only. Let us fix $\Delta = 1$ (we center the discussion in this value because the polynomial (8.8) indicates that the situation when $\Delta = -1$ is the same just negating $r$). Subtituting $\Delta = 1$ in (8.15), the r.h.s. reduces either to the constant $-2$ or to the indetermination $0/0$ [19]: which is then the function (8.14) and how many of them can one write when $\Delta = 1$? We wont be more explicit in this point now, however we want to convince the reader that for $N = 8, \Delta = 1$ there are four (eventually $N/2$ for general $N$, if things go as they shall) bound pair states with $e_1e_2e_3e_4 = -1$: we have just constructed the states (4.1) with (4.2), (4.4) and (4.5) imposing the conditions (6.5) and (6.6); we have obtained exactly four states, and have checked (diagonalizing numerically the matrix block) that they are eigenvectos of the transfer matrix (2.1) when $N = 8, n = 4$. The associated eigenvalues are precisely (8.16) with $\lambda_1 = 0, -3.69963, -1.76088, 0.460505$ [20], and $\lambda_2, \lambda_3$ given in terms of $\lambda_1$ by

$$\lambda_2 = \frac{2 - 3\,\lambda_1^2 - 4\,\lambda_1}{2+\lambda_1}, \qquad \lambda_3 = 2\,\lambda_1^2 + 2\lambda_1 - 1, \qquad \Delta = 1. \tag{8.17}$$

In conclusion, for each real value of $\Delta$ in the vertex model, there are $N/2+1$ bound pair states in the $n = 4$ block of the $N$-site transfer matrix. The number of such states is correct[21] because exact diagonalization of the block corroborates it: our numerical experiments carried up to $N = 12$ with different but arbitrary values of the activities $a, b, c$ confirm that the numbers $\Lambda(v)$ obtained substituting $Q$ by (8.14) with (8.15) and (8.8) into (5.3) are true eigenvalues of the transfer matrix. The number (8.13) is also an eigenvalue. We have no reason then to doubt that they are eigenvalues for general $N$ as well. The author thus admits the number $N/2+1$ as absolutely right.

For $n = 5$, we count a total of $N$ bound pairs. This is so because the solutions $(e_3, e_4, e_5) = (1,1,1), (-1,-1,-1)$ of equations (8.4)-(8.6)[22] do not afford eigenvalues of the transfer matrix for $\Delta$ generic. We noticed this fact from our numerical tests carried with different values of $a, b, c$ and $N = 10, 12$: the numbers $\Lambda$ obtained with (5.3) and $Q$ as in (5.2) with zeros at $z_1 = y, z_2 = 1/y, z_3 = z_4 = z_5 = \pm 1$ and $y$ arbitrary, do not correspond to eigenvalues of the transfer matrix[23]. Unlike this, the solutions in Lemma 8.2 that satisfy $e_3e_4e_5 = -1$ afford $N/2$ bound pairs for each $\Delta$, and the solutions that satisfy $e_3e_4e_5 = 1$ afford another $N/2$ bound pairs (even for $\Delta = \pm 1$ in both cases). The corresponding numbers $\Lambda$ were checked numerically. These eigenvalues are obtained with

$$Q(z) = (z\,y - 1)(z - y)\left(z^2 - (t + 1/t)\;z + 1\right)(z \pm 1)/z^{5/2}, \tag{8.18}$$

the plus sign in $\pm$ is for $e_1e_2e_3e_4e_5 = 1$ (i.e., $e_3e_4e_5 = -1$), the minus sign for $e_1e_2e_3e_4e_5 = -1$. In both functions written in (8.18)

$$t + \frac{1}{t} = -\frac{2\,\Delta\,(r+1/r)+4}{r+1/r+2\,\Delta}, \qquad \Delta \neq \pm 1, \tag{8.19}$$

but $r$ is the root of different polynomials, as stated in Lemma 8.2.

[19] $r = 1$ is solution of (8.8) when $\Delta = 1$

[20] Approximated to the nearest six digit number the last three data

[21] In reference [1] were found 5 states when $N = 8, n = 4$, as we mentioned in Sect. 1. Our result agrees with that number

[22] We mentioned these solutions in the paragraph before Lemma 8.1

[23] The reason is that the states derived from these solutions proceeding as in Sections 4 and 8 are the zero vector

We write an example for $N = 10$ and $\Delta = -1/2$, with the choice of $z, y, \rho$ as in the left column of Table 1. After diagonalizing numerically the blocks $n = 4, 5$ of the transfer matrix, the eigenvalues corresponding to bound pairs (we have recognized them because they match exactly our predicted values) approximated to the nearest six digit number are[24]:

| $n=4$ | [deg] | | $n=5$ | [deg] | | | |
|---|---|---|---|---|---|---|---|
| $0.111\,240^+$ | [2] | | $0.223\,385^+$ | [2] | $-0.065\,746\,4^-$ | [2] | $(r=2)$ |
| $-0.040\,142\,3^-$ | [2] | $(r=-1)$ | $0.138\,383^+$ | [2] | $-0.086\,922\,0^-$ | [2] | $(r=1)$ |
| $-0.047\,993\,4^-$ | [1] | | $0.074\,747\,8^+$ | [2] | $-0.116\,844^-$ | [1] | |
| $-0.080\,403\,8^-$ | [1] | | $0.043\,830\,8^+$ | [2] | $-0.226\,567^-$ | [1] | |
| $-0.158\,869^-$ | [1] | | $0.032\,535\,7^+$ | [2] | $-0.463\,488^-$ | [1] | |
| $-0.280\,662^-$ | [1] | | | | | | |

Table 2. Each eigenvalue listed is followed by a sign $+$ or $-$ : the sign $+$ indicates that $e_1e_2e_3e_4 = 1$ (or that $e_1e_2e_3e_4e_5 = 1$ if $n = 5$), the sign $-$ that the product of the Bethe roots is $-1$. The degeneration of the eigenvalue is [deg]. In the column corresponding to $n = 4$, the number $0.111\,240$ coincides with (8.13), and the remaining five values agree with the theoretical $\Lambda$ obtained inserting (8.14) and (8.15) into (5.3). The eigenvalue that corresponds to $r = -1$, remember that in this column $r$ is a solution of (8.8), is degenerated. This degeneration is not a surprise, because it is a case in which two Bethe roots coincide ($e_3 = e_4 = -1$), and when it is true that the eigenvector associated to such cases is usually the zero vector, when $\Delta = -1/2$ it is not. Regarding the list when $n = 5$, the values with a $+$ correspond to solutions $r$ of (8.10), and the values with a $-$ to solutions $r$ of the polynomial that is obtained changing in (8.10) the variables $r, \Delta$ by $-r, -\Delta$. Totally expected is the degeneration of the eigenvalue $-0.086\,922\,0$ since $e_3 = e_4 = -1, e_5 = 1$. But the degeneration of $-0.065\,746\,4$ which happens for $e_3 = -2, e_4 = -1/2, e_5 = -1$ is less expected.

The last comment of the paper: the numerators of (6.1), (7.5), (8.12), (8.14) and (8.18) are polynomials in $z$ with a reciprocal property: if $R$ is a solution, so it is $1/R$. When looking for other $Q's$ in $n = 7$ (say) one has to restrict to numerators with this property.

## Acknowledgments

I am pleased to thank Prof. J. Shiraishi and the organizers of the RIMS 2004 Symposium, *Recent progress in Solvable Lattice Models*, held in Kyoto for allowing me to expose these ideas. In my work I am grateful to G. Álvarez Galindo for resolving some of my doubts. But to whom I feel inevitably grateful every day is to Pepe Aranda: seventy times seven I have knocked on his door asking about polynomials, roots and other matters of Calculus, and seventy times seven he has received me without ever showing the slightest unwelcome gesture in his face or manners that prevented me from knocking on his door again.

This work is financially supported by the Ministerio de Educación y Ciencia of Spain through grant No. BFM2002-00950.

## References

[1] R.J. Baxter, *Completeness of he Bethe ansatz for the six and eight-vertex models*, J. Statist. Phys. **108** (2002) 1. cond-mat/0111188.

[2] H. Bethe, *Zur Theorie der Metalle*, Z. Physik **71** (1931) 205. English translation: *On the theory of metals, I: Eigenvalues and eigenfunctions of a linear chain of atoms* in "The many-body problem", ed. D.C. Mattis, World Scientific, Singapore, 1993. pages 689-716.

[3] R. Siddharthan, *Singularities in the Bethe solution of the XXX and XXZ Heisenberg spin chains*, cond-mat/9804210.

[4] J.D. Noh, D-S. Lee and D. Kim, *Origin of the singular Bethe ansatz solutions for the Heisenberg XXZ spin chain*, Physica **A287** (2000) 167. cond-mat/0001175.

[24] An interesting question is if they can be recognized in another manner

[5] M.T. Batchelor, *Finite lattice methods in statistical mechanics*, Ph.D. thesis, Australian National University, Canberra, 1987.

[6] N. Beisert, J.A. Minahan, M. Staudacher and K. Zarembo, *Stringing Spins and Spinning Strings*, JHEP**09** (2003) 010. hep-th/0306139.

[7] T. Fujita, T. Kobayashi and H. Takahashi, *Large $N$ behavior of string solutions in the Heisenberg model*, J. Phys. A: Math. Gen. **36** (2003) 1553. cond-mat/0207117.

[8] E.H. Lieb, *Exact solution of the problem of the entropy of two-dimensional ice*, Phys. Rev. Lett. **18** (1967) 692; *Residual entropy of square ice*, Phys. Rev. **162** (1967) 162. B. Sutherland, *Exact solution of a two-dimensional model for hydrogen-bonded crystals*, Phys. Rev. Lett. **19** (1967) 103. C.P. Yang, *Exact solution of a model of two-dimensional ferroelectrics in an arbitrary external electric field*, Phys. Rev. Lett. **19** (1967) 586. B. Sutherland, C.N. Yang and C.P. Yang, *Exact solution of a model of two-dimensional ferroelectrics in an arbitrary external field*, Phys. Rev. Lett. **19** (1967) 588.

[9] R.J. Baxter, *Exactly solved models in statistical mechanics*, Academic Press, London, 1982.

[10] J.V. Uspensky, *Theory of equations*, McGraw-Hill, New York, 1948.