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A RECURSION FORMULA FOR THE CORRELATION FUNCTIONS OF AN INHOMOGENEOUS XXX MODEL

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This article is based on the joint work [1] with H. Boos, M. Jimbo, T. Miwa and F. Smirnov.

1. Introduction

One of recent interesting topics in the study of integrable quantum systems is explicit calculation of correlators of the spin chains. In this article we consider the XXX model with the Hamiltonian

$$H_{\mathrm{XXX}} = \frac{1}{2}\sum_{n}\left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \sigma_n^z\sigma_{n+1}^z\right).$$

Let $\{v_+, v_-\}$ be the basis of the two-dimensional space $\mathbb{C}^2$, and $E_{\epsilon,\bar{\epsilon}}\,(\epsilon,\bar{\epsilon}=\pm)$ the matrix unit defined by $E_{\epsilon,\bar{\epsilon}}v_\mu = \delta_{\bar{\epsilon},\mu}v_\epsilon$. We consider general matrix elements

$$\langle \mathrm{vac}|(E_{\epsilon_1,\bar{\epsilon}_1})_1\cdots(E_{\epsilon_n,\bar{\epsilon}_n})_n|\mathrm{vac}\rangle, \tag{1.1}$$

where $(E_{\epsilon,\bar{\epsilon}})_j$ is the matrix unit acting on the j-th cite.

About the problem of explicit calculation of correlators, some results were obtained in the case of emptiness formation probabilities:

$$P(n) = \langle \mathrm{vac}|(E_{++})_1\cdots(E_{++})_n|\mathrm{vac}\rangle.$$

We have $P(1)=1/2$ because of the symmetry of spin reversal. The value $P(2)$ can be calculated from the ground state energy [8], and the result is

$$P(2) = \frac{1}{3}(1-\log 2).$$

The third one $P(3)$ is interesting from a mathematical point of view. Takahashi obtained the following formula in the study of the Hubbard model [12]:

$$P(3) = \frac{1}{4} - \log 2 + \frac{3}{8}\zeta(3),$$

where $\zeta(s)$ is the Riemann zeta function $\zeta(s)=\sum_{n=1}^{\infty}n^{-s}$. After more than twenty years from this result the explicit value of $P(4)$ was obtained by Boos and Korepin [2]:

$$P(4) = \frac{1}{5} - 2\log 2 + \frac{173}{60}\zeta(3) - \frac{11}{6}\zeta(3)\log 2 - \frac{51}{80}\zeta(3)^2 - \frac{55}{24}\zeta(5) + \frac{85}{24}\zeta(5)\log 2.$$

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They obtained the above formula from the integral representation of correlation functions, which was conjecturally obtained by Jimbo, Miki, Miwa and Nakayashiki [10, 9], and rigorously derived by Kitanine, Maillet and Terras [11]. In the integral representation $P(n)$ is given in terms of n-fold integrals. The value of $P(4)$ is calculated by rewriting the integrand in a suitable way. In the same manner the value of $P(5)$ was also obtained [3]:

$$\begin{aligned}P(5) &= \frac{1}{6} - \frac{10}{3}\log 2 + \frac{281}{24}\zeta(3) - \frac{45}{2}\zeta(3)\log 2 - \frac{489}{16}\zeta(3)^2 - \frac{6775}{192}\zeta(5)\\ &+ \frac{1225}{6}\zeta(5)\log 2 - \frac{425}{64}\zeta(3)\zeta(5) - \frac{12125}{256}\zeta(5)^2 + \frac{6223}{256}\zeta(7)\\ &- \frac{11515}{64}\zeta(7)\log 2 + \frac{42777}{512}\zeta(3)\zeta(7).\end{aligned}$$

In the above results we observe that the value of $P(n)$ $(n = 1, \ldots, 5)$ is given in terms of a polynomial of $\log 2$ and the Riemann zeta functions with odd arguments $\zeta(2a+1)$ with rational coefficients. An explanation for this phenomenon is given by Boos, Korepin and Smirnov [4, 5, 6] as follows. Consider the inhomogeneous XXX model. Denote the spectral parameter associated with the j-th cite by λ_j $(j = 1, \ldots, n)$. Then the correlators are functions in $\lambda_1, \ldots, \lambda_n$:

$$\langle \mathrm{vac}|(E_{\epsilon_1,\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n,\bar{\epsilon}_n})_n|\mathrm{vac}\rangle(\lambda_1, \ldots, \lambda_n). \tag{1.2}$$

The correlators (1.1) are obtained from (1.2) by taking the homogeneous limit $\forall\lambda_j \to \lambda$. It is known that the functions (1.2) can be obtained from a certain solution of the quantum Knizhnik-Zamolodchikov (qKZ) equation [7]. In [5] it is claimed that there exists a new integral formula for the qKZ equation. The claim leads to the following conjecture:

Conjecture. *The functions* (1.2) *are given in the form*

$$\sum \prod \omega(\lambda_i - \lambda_j) f(\lambda_1, \ldots, \lambda_n). \tag{1.3}$$

Here the function $\omega(\lambda)$ is defined by

$$\begin{aligned}\omega(\lambda) &= (\lambda^2 - 1)\frac{d}{d\lambda}\log\left(-\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(-\frac{\lambda}{2}+\frac{1}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}\right) + \frac{1}{2}\\ &= \sum_{k=1}^{\infty}(-1)^k\frac{2k(\lambda^2-1)}{\lambda^2 - k^2} + \frac{1}{2},\end{aligned} \tag{1.4}$$

and $f(\lambda_1, \ldots, \lambda_n)$ is a rational function such that $\prod_{i<j}(\lambda_i - \lambda_j) f(\lambda_1, \ldots, \lambda_n)$ is a polynomial in $\lambda_1, \ldots, \lambda_n$ with rational coefficients.

Note that all the coefficients of the Taylor expansion in $\omega(\lambda)$ at $\lambda = 0$ are given in terms of $\log 2$ and the Riemann zeta functions with odd arguments. From this fact and (1.3), we find that all the correlators are given in terms of a polynomial of $\log 2$ and $\zeta(2a+1)$ $(a = 1, 2, \ldots)$ with rational coefficients.

If we admit the conjecture the next problem is to determine the rational function f. This problem was solved [6] in the cases of $n \le 6$. However the procedure of the calculation in [6] is very complicated.

Our new result stated in this article is as follows. Consider the inhomogeneous XXX model. Let λ_j $(j = 1, \dots, n)$ be the spectral parameters. Set

$$h_n(\lambda_1, \dots, \lambda_n)^{\epsilon_1,\dots,\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} = \prod_{j=1}^{n}(-\bar\epsilon_j)\langle \mathrm{vac}|(E_{-\epsilon_1,\bar\epsilon_1})_1 \cdots (E_{-\epsilon_n,\bar\epsilon_n})_n|\mathrm{vac}\rangle(\lambda_1, \dots, \lambda_n)$$

and define a $\mathbb{C}^{\otimes 2n}$-valued function h_n by

$$(1.5)\quad h_n(\lambda_1, \dots, \lambda_n) = \sum_{\epsilon_1,\dots,\epsilon_n,\bar\epsilon_1,\dots,\bar\epsilon_n=\pm} h_n(\lambda_1, \dots, \lambda_n)^{\epsilon_1,\dots,\epsilon_n,\bar\epsilon_n,\dots,\bar\epsilon_1} v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n} \otimes v_{\bar\epsilon_n} \otimes \cdots \otimes v_{\bar\epsilon_1}.$$

Set $h_0 = 1$ by definition. Then the functions h_n $(n = 0, 1, \dots)$ satisfy a recursion equation of the form "$h_n = h_{n-1} + Z_n h_{n-2}$" (see (2.2) for the explicit formula). Here h_{n-1} and h_{n-2} in the rhs are suitably embedded in $\mathbb{C}^{2n}$, and Z_n is a certain linear map. From the recursion equation we see that the conjecture is true, and we can calculate the rational function f in principle by solving the recursion equation repeatedly.

In the rest of this article we give the explicit formula of the recursion equation. We omit the proof of our result in this article. See [1] for the proof.

2. Main result

Let us define some ingredients of the recursion formula.

2.1. L operators and R matrix. We denote the generators of sl_2 by E, F and H:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Introduce the two-dimensional space $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$, and consider the tensor product

$$V_1 \otimes \cdots \otimes V_n \otimes V_{\bar n} \otimes \cdots \otimes V_{\bar 1}$$

of $2n$ copies of V labeled by $1, \dots, n, \bar n, \dots, \bar 1$. We consider that the function h_n (1.5) takes values in this space.

The R matrix of the XXX model is given by

$$R(\lambda) = \rho(\lambda)\frac{\lambda + P}{\lambda + 1} \in \mathrm{End}(V \otimes V),$$

where the function $\rho(\lambda)$ is defined by

$$\rho(\lambda) = -\frac{\Gamma\left(\frac{\lambda}{2}\right)\Gamma\left(-\frac{\lambda}{2}+\frac{1}{2}\right)}{\Gamma\left(-\frac{\lambda}{2}\right)\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)},$$

and P is the permutation operator, $P(u \otimes v) = v \otimes u$.

Define the L operator by

$$L(\lambda) = \begin{bmatrix} \lambda + \frac{1+H}{2} & F \\ E & \lambda + \frac{1-H}{2} \end{bmatrix} \quad \in U(sl_2) \otimes \mathrm{End}(V).$$

Here we used the identification that $v_+ = {}^t(1,0)$ and $v_- = {}^t(0,1)$.

2.2. **Trace function.** First we introduce the trace function Tr_x. By definition it is the unique $\mathbb{C}[x]$-linear map

$$\mathrm{Tr}_x : \; U(sl_2) \otimes \mathbb{C}[x] \longrightarrow \mathbb{C}[x]$$

such that for any non-negative integer k we have

$$\mathrm{Tr}_{k+1}(A) = \mathrm{tr}\, \pi^{(k)}(A) \qquad (A \in U(sl_2)),$$

where $\pi^{(k)}$ is the $(k+1)$-dimensional irreducible representation of sl_2, and tr is the usual trace. We can calculate the value of trace function by using the following properties repeatedly:

$$\begin{aligned}
&\mathrm{Tr}_x(AB) = \mathrm{Tr}_x(BA), \quad \mathrm{Tr}_x(1) = x, \\
&\mathrm{Tr}_x(A) = 0 \text{ if } A \text{ has non-zero weight}, \\
&\mathrm{Tr}_x(e^{zH}) = \frac{\sinh(xz)}{\sinh z}, \\
&\mathrm{Tr}_x\left((\frac{H^2}{2} + H + 2FE)A \right) = \frac{x^2-1}{2} \mathrm{Tr}_x(A) \qquad (A \in U(sl_2) \otimes \mathbb{C}[x]).
\end{aligned}$$

2.3. **Operators $X_n^{[1,j]}$.** Define the monodromy matrix

$$T(\lambda) = L_{\bar{2}}(\lambda - \lambda_2 - 1) \cdots L_{\bar{n}}(\lambda - \lambda_n - 1) L_n(\lambda - \lambda_n) \cdots L_2(\lambda - \lambda_2).$$

This is an element of $U(sl_2) \otimes \mathrm{End}(V_2 \otimes \cdots \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \otimes V_{\bar{2}})$.

Now we define the operators $X^{[1,j]}$ $(j = 2, \ldots n)$ by

$$X_n^{[1,j]}(\lambda_1, \ldots, \lambda_n) = \frac{1}{\lambda_{1,j} \prod_{p \neq 1,j} \lambda_{1,p} \lambda_{j,p}} \mathrm{Tr}_{\lambda_{1,j}} \left(T(\frac{\lambda_1 + \lambda_j}{2}) \right) \tag{2.1}$$
$$\times R_{j,j-1}(\lambda_{j,j-1}) \cdots R_{j,2}(\lambda_{j,2}) R_{\overline{j-1},\bar{j}}(\lambda_{j-1,j}) \cdots R_{\bar{2},\bar{j}}(\lambda_{2,j}),$$

where $\lambda_{i,j} = \lambda_i - \lambda_j$. This operator acts on the space $V_2 \otimes \cdots \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \otimes V_{\bar{2}}$.

2.4. **Singlet vectors.** Define the singlet vectors in $V^{\otimes 2}$ and $V^{\otimes 4}$ by

$$\begin{aligned}
s^{(1)} &= v_+ \otimes v_- - v_- \otimes v_+ \qquad \text{and} \\
s^{(2)} &= v_+ \otimes v_+ \otimes v_- \otimes v_- + v_- \otimes v_- \otimes v_+ \otimes v_+ \\
&\quad - \frac{1}{2}(v_+ \otimes v_- + v_- \otimes v_+) \otimes (v_+ \otimes v_- + v_- \otimes v_+),
\end{aligned}$$

respectively.

2.5. **Recursion equation.** To state the main theorem we introduce a convention of tensor products of vectors. For $w \in V^{\otimes(2n-2)}$, we set

$$\begin{aligned} s^{(1)}_{1,\bar{1}} \cdot w_{2,\ldots,n,\bar{n},\ldots,\bar{2}} &= v_+ \otimes w \otimes v_- - v_- \otimes w \otimes v_+ \\ &\in V_1 \otimes \cdots \otimes V_n \otimes V_{\bar{n}} \otimes \cdots \otimes V_{\bar{1}}. \end{aligned}$$

Thus we use the symbol $\cdot$ to show the tensor product of vectors embedded in the components of the space $V_1 \otimes \cdots \otimes V_{\bar{1}}$ specified by the indices.

Now we can state the main theorem:

Theorem 2.1. *We have the following recursion formula.*

$$\begin{aligned} (2.2)\quad & h_n(\lambda_1, \cdots, \lambda_n) \\ &= \frac{1}{2} s^{(1)}_{1,\bar{1}} \cdot h_{n-1}(\lambda_2, \cdots, \lambda_n)_{2,\cdots,n,\bar{n},\cdots,\bar{2}} \\ &- \sum_{j=2}^{n} Z^{[1,j]}_n(\lambda_1, \cdots, \lambda_n)\, s^{(2)}_{1,j,\bar{j},\bar{1}} \cdot h_{n-2}(\lambda_2, \cdots, \widehat{\lambda_j}, \cdots, \lambda_n)_{2,\cdots,\hat{j},\cdots,n,\bar{n},\cdots,\hat{\bar{j}},\cdots,\bar{2}}. \end{aligned}$$

Here

$$\begin{aligned} Z^{[1,j]}_n(\lambda_1, \cdots, \lambda_n) &= \frac{\omega(\lambda_{1,j})}{\lambda_{1,j}^2 - 1} X^{[1,j]}_n(\lambda_1, \lambda_2, \cdots, \lambda_n) \\ &+ \sum_{p(\neq 1,j)} \frac{\omega(\lambda_{p,j})}{\lambda_{p,1}(\lambda_{p,j}^2 - 1)} \mathrm{res}_{\sigma=\lambda_p} X^{[1,j]}_n(\sigma, \lambda_2, \cdots, \lambda_n), \end{aligned}$$

where $\omega(\lambda)$ is given by (1.4), and $X^{[1,j]}_n(\lambda_1, \cdots, \lambda_n)$ is defined in (2.1). The poles at $\lambda_{i,j} = \pm 1$ in the rhs of (2.2) are spurious.

From the theorem and the initial conditions

$$(2.3)\qquad h_0 = 1, \quad h_1 = \frac{1}{2}(v_+ \otimes v_- - v_- \otimes v_+) = \frac{1}{2} s^{(1)},$$

we see that the conjecture (1.3) is true:

Theorem 2.2. *The function $h_n(\lambda_1, \cdots, \lambda_n)$ has the structure*

$$h_n(\lambda_1, \cdots, \lambda_n) = \sum_{m=0}^{[n/2]} \sum_{I,J} \prod_{p=1}^{m} \omega(\lambda_{i_p} - \lambda_{j_p})\, f_{n,I,J}(\lambda_1, \cdots, \lambda_n),$$

where $f_{n,I,J}(\lambda_1, \cdots, \lambda_n) \in V^{\otimes 2n}$ are rational functions with only simple poles along the diagonal $\lambda_i = \lambda_j$, and $I = (i_1, \ldots, i_m)$, $J = (j_1, \ldots, j_m)$ run over sequences satisfying $I \cap J = \emptyset$, $i_1 < \cdots < i_m$, $1 \leq i_p < j_p \leq n$ $(1 \leq p \leq m)$. The representation of h_n in the above form is unique.

2.6. **Example** ($n = 2$). Here let us calculate h_2 by using the recursion equation. From (2.3) and the recursion equation we have

$$h_2(\lambda_1, \lambda_2) = \frac{1}{4} s^{(1)}_{1,\bar{1}} \cdot s^{(1)}_{2,\bar{2}} - \frac{\omega(\lambda_{1,2})}{\lambda_{1,2}^2 - 1} X^{[1,2]}(\lambda_1, \lambda_2) s^{(2)}_{1,2,\bar{2},\bar{1}}.$$

The operator $X^{[1,2]}(\lambda_1, \lambda_2)$ is given by

$$X^{[1,2]}(\lambda_1,\lambda_2) = \frac{1}{\lambda_{1,2}}\mathrm{Tr}_{\lambda_{1,2}}\left(L_{\bar{2}}(\frac{\lambda_{1,2}}{2}-1)L_2(\frac{\lambda_{1,2}}{2})\right).$$

Let us calculate the trace function. Rewrite the product of L operators in the matrix form:

$$L_{\bar{2}}(\frac{\lambda_{1,2}}{2}-1)L_2(\frac{\lambda_{1,2}}{2})$$
$$= \begin{bmatrix} \frac{\lambda_{1,2}+H-1}{2} & F & 0 & 0 \\ E & \frac{\lambda_{1,2}-H-1}{2} & 0 & 0 \\ 0 & 0 & \frac{\lambda_{1,2}+H-1}{2} & F \\ 0 & 0 & E & \frac{\lambda_{1,2}-H-1}{2} \end{bmatrix}$$
$$\times \begin{bmatrix} \frac{\lambda_{1,2}+H+1}{2} & 0 & F & 0 \\ 0 & \frac{\lambda_{1,2}+H+1}{2} & 0 & F \\ E & 0 & \frac{\lambda_{1,2}-H+1}{2} & 0 \\ 0 & E & 0 & \frac{\lambda_{1,2}-H+1}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{(\lambda_{1,2}+H)^2-1}{4} & F\frac{\lambda_{1,2}+H+1}{2} & \frac{\lambda_{1,2}+H-1}{2}F & F^2 \\ E\frac{\lambda_{1,2}+H+1}{2} & \frac{\lambda_{1,2}^2-(1+H)^2}{4} & EF & \frac{\lambda_{1,2}-H-1}{2}F \\ \frac{\lambda_{1,2}+H-1}{2}E & FE & \frac{\lambda_{1,2}^2-(H-1)^2}{4} & F\frac{\lambda_{1,2}-H+1}{2} \\ E^2 & \frac{\lambda_{1,2}-H-1}{2}E & E\frac{\lambda_{1,2}-H+1}{2} & \frac{(\lambda_{1,2}-H)^2-1}{4} \end{bmatrix}$$

Here we arranged the elements of $U(sl_2)$ with respect to the basis $\{v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_+, v_- \otimes v_-\}$ of $V_2 \otimes V_{\bar{2}}$. By taking the trace $\mathrm{Tr}_{\lambda_{1,2}}$ we have

$$X^{[1,2]}(\lambda_1,\lambda_2) = \frac{\lambda_{1,2}^2-1}{3}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the above formula we find

$$h_2(\lambda_1,\lambda_2) = \frac{1}{4}s^{(1)}_{1,\bar{1}} \cdot s^{(1)}_{2,\bar{2}} - \frac{1}{3}\omega(\lambda_1-\lambda_2)s^{(2)}_{1,\bar{1},\bar{2},2}.$$

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