Homogenization of fully nonlinear PDEs and backward SDEs

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概要

In this note, we discuss a probabilistic approach to homogenization of fully nonlinear second-order PDEs of parabolic type. We also study the rate of convergence of solutions, which can be regarded as a byproduct of our stochastic representation of solutions based on backward stochastic differential equations.

1 Problem.

Let us consider the Cauchy problem with small parameter $\varepsilon > 0$ of the form

$$
\begin{aligned}
-u_t + H(\varepsilon^{-1}x, u, u_x, u_{xx}) &= 0, & \text{in } [0,T) \times \mathbb{R}^d, \\
u(T, x) &= h(x), & \text{on } \mathbb{R}^d,
\end{aligned}
$$

where $u_t$ stands for the partial derivative of $u$ with respect to $t$, and $u_x$ and $u_{xx}$ denote its first and second derivatives with respect to $x$, respectively. The continuous function $H : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d} \to \mathbb{R}$, called Hamiltonian, is assumed to be $\mathbb{Z}^d$-periodic with respect to its first variable. We also assume that $h(\cdot)$ is a bounded and uniformly continuous function. It is well known that (1.1) has a unique solution in the viscosity sense if $H$ is proper (possibly degenerate elliptic) and satisfies some other structure conditions (see [6]).

Our aim is to prove the following convergence theorem (homogenization) under certain conditions on $H$. 

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**Theorem 1.1.** Let \( \{u^\epsilon(t, x); \epsilon > 0\} \) be the family of viscosity solutions to (1.1). Then, as \( \epsilon \) goes to zero, it converges to a unique viscosity solution \( u^0(t, x) \) of the following PDE

\[
\begin{aligned}
-u_t + \overline{H}(u, u_x, u_{xx}) &= 0, & \text{in} & \quad [0, T) \times \mathbb{R}^d,
\end{aligned}
\]

\[
\begin{aligned}
u(T, x) &= h(x), & \text{on} & \quad \mathbb{R}^d.
\end{aligned}
\]

Here, the effective Hamiltonian \( \overline{H} = \overline{H}(y, p, X) \) is defined by the cell problem

\[
\overline{H} = H(\eta, y, p, X + v_{\eta\eta}(\eta)), \quad (v(\cdot), \overline{H}) : unknown.
\]

Such kind of homogenization problems have been largely studied by the so-called perturbed test function method based on the theory of viscosity solution (see [1], [2], [8], [9] for details). On the other hand, it seems to be worth studying (1.1)-(1.3) from probabilistic view point, for the class of fully nonlinear equations of this form contains important and interesting examples that are closely related to stochastic problems. Hamilton-Jacobi-Bellman equations (HJB equations, for short) are the most typical ones. There are also a number of literatures concerning homogenization of second-order PDEs treated by probabilistic methods. In particular, for the investigation of nonlinear PDEs, the notion of backward stochastic differential equation (BSDE) is useful (see [3], [4], [7], [10], [12] for the homogenization of semi-linear and quasi-linear equations by BSDE approaches, as well as [5] for that of fully nonlinear HJB equations). We remark that the literature [11], which this note is based on, also uses BSDE approach to prove the homogenization of fully nonlinear second-order PDEs.

The novelty of this note (and therefore that of [11]) is that under the assumption that \( H \) is uniformly elliptic and convex in the last variable, we obtain an estimate of convergence rate of solutions at the same time (Theorem 1.2 below). As far as fully nonlinear second-order equations concerned, to the best of our knowledge, such kind of rate of convergence have not been studied neither by the viscosity solution method nor by the probabilistic one.

**Theorem 1.2.** Let \( \delta \in (0, 1) \) be the Hölder exponent of the second derivatives of solution \( u^0 \) to (1.2), i.e. \( u^0 \in C^{1+\delta/2, 2+\delta}([0, T] \times \mathbb{R}^d) \). Then, for every compact subset \( Q \) of \([0, T] \times \mathbb{R}^d\), there exists a constant \( C > 0 \) independent of \( \epsilon > 0 \) such that the following holds:

\[
\sup_{(t, x) \in Q} |u^\epsilon(t, x) - u^0(t, x)| \leq C \epsilon^{2\delta/2+\delta}.
\]
**Remark 1.3.** Under Assumption 2.1 below, it is known that (1.2) has a unique classical solution in the Hölder space $C^{1+\delta/2,2+\delta}([0, T] \times \mathbb{R}^d)$.

This note is organized as follows. In the next section, we give the precise assumption on $H$ we suppose throughout this note. In Section 3, we discuss a stochastic representation of solutions by BSDEs. This interpretation makes us possible to treat homogenization of fully nonlinear equations in a probabilistic way. Section 4 is devoted to the proof of Theorem 1.2.

## 2 Assumption.

Throughout this note, the terminal function $h(\cdot)$ is assumed to be of $C^3_b$-class. Concerning the Hamiltonian $H$ in (1.1), we make the following assumption.

**Assumption 2.1.** There exist $K$ and $\nu > 0$ such that $H$ satisfies the following conditions.

(A1) $H$ is of $C^2$-class and all second derivatives are bounded.

(A2) $H$ is convex in $X$.

(A3) For every $(\eta, y, p, X)$ and $\xi \in \mathbb{R}^d$, 
\[
\nu |\xi|^2 \leq H(\eta, y, p, X) - H(\eta, y, p, X + \xi \otimes \xi) \leq \nu^{-1} |\xi|^2 ,
\]

where $\xi \otimes \xi$ stands for the $(d \times d)$-matrix defined by $(\xi \otimes \xi)_{ij} := \xi^i \xi^j$.

(A4) For every $(y, p, X)$, $(y', p', X')$ and $\eta$,
\[
|H(\eta, y, p, X) - H(\eta, y', p', X')| \leq K \{|y - y'| + |p - p'| + |X - X'|\}.
\]

(A5) For every $\eta, \eta'$ and $(y, p, X)$,
\[
|H(\eta, y, p, X) - H(\eta, y, p, X)| \leq K(1 + |p| + |X|)|\eta - \eta'|.
\]

## 3 Stochastic representation.

In this section, we introduce an appropriate family of controlled BSDEs in order to obtain a stochastic representation of solutions to (1.1). For this purpose, we prepare the following lemma.

**Lemma 3.1.** Let $E := \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$. Then, there exist a bounded continuous function $a$ on $\mathbb{R}^d \times E$ taking its values in the set of symmetric matrices $\mathbb{S}^d \subset \mathbb{R}^{d \times d}$.
and a continuous function $f : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times E \rightarrow \mathbb{R}$ such that $H$ can be written as follows:

$$H(x, y, p, X) = \max_{\zeta \in E} \left\{ -\sum_{i,j=1}^{d} a^{ij}(x, \zeta) X_{ij} - f(x, y, p, \zeta) \right\},$$

where the maximum of the right-hand side is attained when $\zeta = (-y, -p, -X)$. Moreover, we can take $a = (a^{ij})$ and $f$ such that $a^{ij}$ is Lipschitz continuous uniformly in $x$, and $f$ is Lipschitz continuous uniformly in $(y, p)$ and satisfies under the notation $\zeta = (\alpha, \beta, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$ the following inequalities:

$$-K(1 + \min\{|y|, |\alpha|\} + \min\{|p|, |\beta|\}) \leq f(x, y, p, \zeta) \leq \tilde{K}(1 + |y| + |p| + |\zeta|),$$

where $\tilde{K}$ is a constant depending only on $K$.

**Sketch of the proof.** We define $a^{ij}$ and $f$ by

$$a^{ij}(x, \zeta) := H_{X_{ij}}(x, \zeta),
\quad f(x, y, p, \zeta) := H_{X_{ij}}(x, \zeta) \gamma_{ij} - H(x, \zeta) + K|\alpha + y| + K|\beta + p|,$$

where $H(\eta, y., p, X) := H(\eta, -y, -p, -X)$. Then, by convexity and uniform Lipschitz continuity of $H$, we can easily check (3.1) as well as all properties of $a$ and $f$ stated in this lemma. \hfill $\square$

Now, let us take any complete probability space $(\Omega, \mathcal{F}, P)$ with $d$-dimensional Brownian motion $W = (W_t)_{0 \leq t \leq T}$ and set $W_{t,s} := W_s - W_t$, $\mathcal{F}_{t,s} := \sigma(W_{t,r}; t \leq r \leq s)$ \(\forall N\), where $N$ denotes the totality of all $P$-null sets. We fix an arbitrary point $(t, x) \in [0, T] \times \mathbb{R}^d$ and consider the following system of forward-backward stochastic differential equations (FBSDEs):

$$\begin{cases}
    dX_{s}^{\xi, \zeta} = \sigma(e^{-1}X_{s}^{\xi, \zeta}, \zeta_s) dW_{t,s}, \\
    -dY_{s}^{\xi, \zeta} = f(e^{-1}X_{s}^{\xi, \zeta}, Y_{s}^{\xi, \zeta}, Z_{s}^{\xi, \zeta}, \zeta_s) ds - \sigma^*(e^{-1}X_{s}^{\xi, \zeta}, \zeta_s) Z_{s}^{\xi, \zeta} dW_{t,s}, \\
    X_{t}^{\xi, \zeta} = x, \quad Y_{T}^{\xi, \zeta} = h(X_{T}^{\xi, \zeta}),
\end{cases}$$

where $\zeta : \Omega \times [t, T] \rightarrow E$ is a given $\mathcal{F}_{t,s}$-adapted control process satisfying the integrability condition $E \int_{t}^{T} |\zeta_s|^2 ds < \infty$. Notice that $\sigma = (\sigma^{ij}) : \mathbb{R}^d \times E \rightarrow \mathbb{R}^{d \times d}$ is a bounded and Lipschitz continuous function such that $\sum_{k=1}^{d} (\sigma^{ik} \sigma^{jk})(x, \zeta) = 2a^{ij}(x, \zeta)$. Then, we can show the following theorem (see [11], Theorem 1.3 for its proof).
Theorem 3.2. Let \( u^\epsilon(t, x) \) be a solution of (1.1), and let \( (X^\epsilon, Y^\epsilon, Z^\epsilon) \) be a unique pair of solutions to (3.3). Then, we have the following representation formula

\[
 u^\epsilon(t, x) = \inf_{\zeta} Y^\epsilon_{t}^\zeta,
\]

where the infimum is taken over all admissible control processes.

4 Probabilistic approach to homogenization.

The aim of this section is to give the sketch of proof of Theorem 1.2. To avoid heavy notation, we set

\[
v(\eta, s, x) := v(\eta, u^0(s, x), u^0_x(s, x), u^0_{xx}(s, x)), \quad (s, x) \in [0, T] \times \mathbb{R}^d,
\]

where \( v(\eta, y, p, X) \) is a solution to the cell problem (1.3) with \((y, p, X)\) frozen. Then, by applying Itô’s formula to \( Y^\epsilon_{t}^\zeta - u^0(s, X^\epsilon_{s}^\zeta) - \epsilon^2 v(\epsilon^{-1}X^\epsilon_{s}^\zeta, s, X^\epsilon_{s}^\zeta) \), we can expect the convergence of the form

\[
 \lim_{\epsilon \downarrow 0} \inf_{\zeta} E[Y^\epsilon_{t}^\zeta - u^0(s, X^\epsilon_{s}^\zeta) - \epsilon^2 v(\epsilon^{-1}X^\epsilon_{s}^\zeta, s, X^\epsilon_{s}^\zeta)] = 0.
\]

Unfortunately, the above observation cannot be justified since \( v \) is not differentiable with respect to \((s, x)\). Nevertheless, for each fixed \((s, x)\), \( v \) is twice differentiable in \( \eta \). So, we can prove the convergence by using local arguments (i.e. by freezing the slow variable \((s, X^\epsilon_{s}^\zeta)\)).

For this purpose, we first set \( \overline{Y}^\epsilon_{s}^\zeta := Y^\epsilon_{s}^\zeta - u^0(s, X^\epsilon_{s}^\zeta) \), \( \overline{Z}^\epsilon_{s} := Z^\epsilon_{s} - u_x^0(s, X^\epsilon_{s}^\zeta) \).

Then, \((\overline{Y}^\epsilon_{s}^\zeta, \overline{Z}^\epsilon_{s})\) satisfies the following linear BSDE:

\[
 \left\{
 \begin{aligned}
 -d\overline{Y}^\epsilon_{s} & = \{ \overline{\theta}(s, X^\epsilon_{s}^\zeta, \zeta_{s}) + \phi^\epsilon_{\theta} Y^\epsilon_{s}^\zeta + \psi^\epsilon_{\theta} Z^\epsilon_{s} \} \, ds \\
 & \quad -\sigma^*(\epsilon^{-1}X^\epsilon_{s}^\zeta, \zeta_{s}) \overline{Z}^\epsilon_{s} \, dW_{t,s}, \\
 \overline{Y}^\epsilon_{T} & = 0,
 \end{aligned}
 \right.
\]

where the function \( \overline{\theta} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times E \to \mathbb{R} \) and bounded processes \((\phi^\epsilon_{\theta}, \psi^\epsilon_{\theta})\) are defined as follows:

\[
 \overline{\theta}(s, X^\epsilon_{s}^\zeta, \zeta_{s}) := \overline{H}(u^0(s, x), u_x^0(s, x), u_{xx}^0(s, x)) + a^{ij}(\eta, \zeta) u_{x^{i}x^{j}}^0(s, x) + f(\eta, u^0(s, x), u_x^0(s, x), \zeta),
\]

\[
 \phi^\epsilon_{\theta} := \int_0^1 f_y(\epsilon^{-1}X^\epsilon_{s}^\zeta, \lambda Y^\epsilon_{s}^\zeta + (1 - \lambda)u^0(s, X^\epsilon_{s}^\zeta), u_x^0(s, X^\epsilon_{s}^\zeta), \zeta) \, d\lambda,
\]

\[
 \psi^\epsilon_{\theta} := \int_0^1 f_p(\epsilon^{-1}X^\epsilon_{s}^\zeta, Y^\epsilon_{s}^\zeta, \lambda Z^\epsilon_{s}^\zeta + (1 - \lambda)u_x^0(s, X^\epsilon_{s}^\zeta), \zeta) \, d\lambda.
\]
From the general theory of linear BSDEs, \( Y_t^{\epsilon, \zeta} \) can be written as

\[
Y_t^{\epsilon, \zeta} = E \int_t^T \Gamma_s^{\epsilon, \zeta} \overline{\theta}(s, X_s^{\epsilon, \zeta}, \epsilon^{-1}X_s^{\epsilon, \zeta}, \zeta_s) ds,
\]

(4.1)

where \( \Gamma_s^{\epsilon, \zeta} > 0 \) is an \( F_{t,s} \)-adapted process such that

\[
\sup_{\epsilon > 0} E \sup_{t \leq s \leq T} |\Gamma_s^{\epsilon, \zeta}|^q < \infty, \quad \forall q \geq 1.
\]

Note that it is possible to write down this process explicitly (see [11]).

Next, for any given \( N \in \mathbb{N} \) and \( n > 0 \), we consider the \( N \)-partition of the time duration

\[
(t, T] = \bigcup_{j=0}^{N-1} \Delta_j := \bigcup_{j=0}^{N-1} (s_j, s_{j+1}], \quad s_j = t + \frac{j(T-t)}{N}, \quad j = 0, 1, \ldots, N,
\]

and the disjoint decomposition of the ball \( B(n) := \{ x \in \mathbb{R}^d ; |x| \leq n \} = \bigcup_{k=1}^{N'} B_k \), where \( B_k \in B(\mathbb{R}^d) \) \( (k = 1, 2, \ldots, N') \) are constructed by a finite open covering of \( B(n) \) with radius less than \( 1/(2n) \). Then, we have the following lower estimate of \( \inf_{\zeta} Y_t^{\epsilon, \zeta} \).

**Proposition 4.1.** For every \( q > 1 \) and \( x_k \in B_k \) \( (k = 1, \ldots, N') \), we have

\[
\inf_{\zeta} Y_t^{\epsilon, \zeta} + C(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta}) > - \sup_{\zeta} \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{s_j}^{s_{j+1}} \Gamma_s^{\epsilon, \zeta} 1_{\{X_s^{\epsilon, \zeta} \in B_k\}} V(s, x, \epsilon^{-1}X_s^{\epsilon, \zeta}, \zeta_s) ds \right|
\]

(4.2)

where \( \delta > 0 \) is the exponent appearing in Theorem 1.2 and we have set \( V(s, x, \eta, \zeta) := \sum_{i,j=1}^{d} a^{ij}(\eta, \zeta) v_{\eta^{i} \eta^{j}}(\eta, s, x) \).

**Sketch of the proof.** We set

\[
A_n = \left\{ \sup_{t \leq s \leq T} |X_s^{\epsilon, \zeta}| \leq n \right\}, \quad B_{n,N} = \left\{ \max_{0 \leq j \leq N-1} \sup_{s \in \Delta_j} |X_s^{\epsilon, \zeta} - X_{s_j}^{\epsilon, \zeta}| \leq 1/n \right\}.
\]

Then, for each fixed \( q > 1 \), Chebyshev's inequality yields

\[
P(A_n^c) \leq \frac{C(1 + |x|)^{2q}}{n^{2q}}, \quad P(B_{n,N}^c) \leq \sum_{j=0}^{N-1} Cn^{2q} |s_{j+1} - s_j|^q = Cn^{2q}(T-t)^q \frac{n^{q-1}}{N^{q-1}},
\]

(4.3)

where \( C > 0 \) is a universal constant independent of \( n, N, \epsilon, \) etc. Since \( u^0 \in C^{1+\delta/2, 2+\delta}([0, T] \times \mathbb{R}^d) \), we can also show that

\[
|\bar{\theta}(s, x, \eta, \zeta) - \bar{\theta}(s', x', \eta, \zeta)| \leq C(|s - s'|^{\delta/2} + |x - x'|^{\delta}).
\]

(4.4)
Now, for each $k = 1, \ldots, N'$, we set $C_{j,k} := \{X_{s_{j}}^\varepsilon, \zeta \in B_{k}\}$ and fix $x_{k} \in B_{k}$ arbitrarily. Then, taking into account that $A_n \subset \bigcup_{k=1}^{N'} C_{j,k}$ and $C_{j,k} \cap C_{j,k'} = \emptyset$ (if $k \neq k'$), for every $s \in \Delta_{j}$, we have

$$
\overline{\theta}(s, X_{s}^\varepsilon, \zeta_{s}) = \sum_{k=1}^{N'} 1_{A_n \cap B_{n,N}} 1_{C_{j,k}} \{\overline{\theta}(s, X_{j}^\varepsilon, \zeta_{j}) - \overline{\theta}(s_{j}, x_{k}, e^{-1}X_{s}^\varepsilon, \zeta_{s})\} + \sum_{k=1}^{N'} 1_{(A_n \cap B_{n,N})^c} \overline{\theta}(s_{j}, x_{k}, e^{-1}X_{s}^\varepsilon, \zeta_{s}).
$$

Furthermore, since $\overline{\theta}(s, x, \eta, \zeta) \geq -V(s, x, \eta, \zeta)$,

$$
\overline{\theta}(s, X_{s}^\varepsilon, \zeta_{s}) \geq \sum_{j=0}^{N-1} E \int_{s_{j}}^{s_{j+1}} \Gamma_{s}^\varepsilon, \eta, \zeta \{\Psi_{1}^{j}(s) - \Psi_{2}^{j}(s) + \Psi_{3}^{j}(s) - \Psi_{4}^{j}(s)\} ds.
$$

We estimate the right-hand side one by one. Remark fist that on the event $A_n \cap B_{n,N} \cap C_{j,k}$,

$$
|X_{s}^\varepsilon - x_{k}| \leq |X_{s}^\varepsilon - X_{j}^\varepsilon| + |X_{j}^\varepsilon - x_{k}| \leq 2/n \quad \text{for all } s \in \Delta_{j}.
$$

Then, by (4.4), we have

$$
|E \int_{\Delta_{j}} \Gamma_{s}^\varepsilon, \eta, \zeta \Psi_{4}^{j}(s) ds| \leq K'E \left[ \int_{\Delta_{j}} \Gamma_{s}^\varepsilon, \eta, \zeta 1_{A_n \cap B_{n,N}} \sum_{k=1}^{N'} 1_{C_{j,k}} \{|s-s_{j}|^{\delta/2} + |X_{s}^\varepsilon - x_{k}|^{\delta}\} ds \right] 
\leq C(s_{j+1}-s_{j})(|s_{j+1}-s_{j}|^{\delta/2} + n^{-\delta}).
$$

By using (4.3), the inequalities

$$
|E \int_{\Delta_{j}} \Gamma_{s}^\varepsilon, \eta, \zeta \Psi_{4}^{j}(s) ds| \leq |V|_{L^\infty}(s_{j+1}-s_{j}) \sqrt{P((A_n \cap B_{n,N})^c)} \sqrt{E \sup_{t \leq s \leq T} |\Gamma_{s}^\varepsilon, \eta, \zeta|^{2}} 
\leq C |V|_{L^\infty}(s_{j+1}-s_{j}) \{n^{-q}(1+|x|)^{q} + n^{q}N^{(1-q)/2}\}.
$$
hold, from which we obtain
\[
\left| E \int_{\Delta_j} \Gamma_s^\epsilon \xi_3^j(s) ds \right| \leq C |V|_{L^\infty} (s_{j+i} - s_j) \{ n^{-q} (1 + |x|)^q + n^q N^{(1-q)/2} \}
\]
since \( \sum_{k=1}^N 1_{C_{j,k}} |V(s_j, x_k, \epsilon^{-1}X_s^\epsilon, \zeta_s)| \leq |V|_{L^\infty} < \infty \). Thus, we have
\[
(4.5) \quad \mathcal{V}_t^\epsilon \geq - \sum_{j=0}^{N-1} \frac{E}{\Delta_j} \Gamma_s^\epsilon \xi_3^j(s) ds - C(n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta}),
\]
where \( C > 0 \) depends only on \( |x|, \delta, K', T \) and \( |V|_{L^\infty} \). The above inequality doesn't depend on the choice of \( (\zeta_s) \). Hence, we have completed the proof. \( \square \)

We can also prove the inequality of the opposite direction in the same manner (the proof will by a little more complicated since we have to choose a "nice" control according to the parameter \( \epsilon > 0 \). See [11], Proposition 2.5).

**Proposition 4.2.** Let \( N, N' \in \mathbb{N}, n > 0, q > 1, \) etc. be the same parameters as in Proposition 4.1. Then,
\[
(4.6) \quad \inf_\zeta \mathcal{V}_t^\epsilon \geq C \{ n^{-q} + n^q N^{(1-q)/2} + N^{-\delta/2} + n^{-\delta} \}
\]
\[
< \sup_\zeta \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{\mathcal{B}_j^{s_j}} \Gamma_s^\epsilon \xi_3^{j,k}(s_j, x_k, \epsilon^{-1}X_s^\epsilon, \zeta_s) ds \right|.
\]

**Lemma 4.3.** For every \( N, N' \in \mathbb{N} \), we have
\[
\sup_\zeta \left| \sum_{j=0}^{N-1} \sum_{k=1}^{N'} E \int_{\Delta_j} 1_{C_{j,k}} \Gamma_s^\epsilon \xi_3^{j,k}(s_j, x_k, \epsilon^{-1}X_s^\epsilon, \zeta_s) ds \right| \leq (\epsilon + \epsilon^2) C + \epsilon^2 C N.
\]

**Sketch of the proof.** We set \( \bar{v}^{j,k}(\eta) = v(\eta, s_j, x_k) - v(0, s_j, x_k) \). Clearly, \( \bar{v}^{j,k}(\eta) = v_\eta(\eta, s_j, x_k) \). Thus, by Ito's formula,
\[
\Gamma_{s_{j+1}}^{\epsilon,\zeta} \bar{v}^{j,k}(\epsilon^{-1}X_{s_{j+1}}^{\epsilon,\zeta}) - \Gamma_{s_j}^{\epsilon,\zeta} \bar{v}^{j,k}(\epsilon^{-1}X_{s_j}^{\epsilon,\zeta})
\]
\[
= \frac{1}{\epsilon^2} \int_{\Delta_j} \Gamma_s^{\epsilon,\zeta} V(s_j, x_k, \epsilon^{-1}X_s^{\epsilon,\zeta}, \zeta_s) ds + \frac{1}{\epsilon} \int_{\Delta_j} \Gamma_s^{\epsilon,\zeta} (\sigma^\epsilon \bar{v}^{j,k}(\epsilon^{-1}X_s^{\epsilon,\zeta}, \zeta_s)) dW_t, \sigma
\]
\[
+ \frac{1}{\epsilon} \int_{\Delta_j} \Gamma_s^{\epsilon,\zeta} (\epsilon^{-1}X_s^{\epsilon,\zeta}, \zeta_s) \psi^{\epsilon,\zeta} \cdot \bar{v}^{j,k}(\epsilon^{-1}X_{s_j}^{\epsilon,\zeta}) ds
\]
\[
+ \int_{\Delta_j} \Gamma_s^{\epsilon,\zeta} \bar{v}^{j,k}(\epsilon^{-1}X_{s_j}^{\epsilon,\zeta}) \psi^{\epsilon,\zeta} dW_t, \psi
\]
\[
+ \int_{\Delta_j} \Gamma_s^{\epsilon,\zeta} \bar{v}^{j,k}(\epsilon^{-1}X_{s_j}^{\epsilon,\zeta}) \phi^{\epsilon,\zeta} ds.
\]
Remark here that each of stochastic integral terms appearing in the right-hand side is a $F_{t,s}$-martingale and $C_{j,k} \in F_{8_j}$. Taking expectation of both sides, we have

$$E\int_{\Delta_j} 1_{C_{j,k}} \Gamma_{s_j}^{\epsilon,\zeta} V(s_j, x_k, e^{-1}X_{s_j}^{\epsilon,\zeta}, \zeta_{s_j})\, ds$$

$$= -\epsilon E\left[ 1_{C_{j,k}} \int_{\Delta_j} \Gamma_{s_j}^{\epsilon,\zeta} \sigma(e^{-1}X_{s_j}^{\epsilon,\zeta}, \zeta_{s_j}) \psi_{\eta}^{j,k}(e^{-1}X_{s_j}^{\epsilon,\zeta})\, ds \right]$$

$$- \epsilon^2 E\left[ 1_{C_{j,k}} \int_{\Delta_j} \Gamma_{s_j}^{\epsilon,\zeta} \psi_{\eta}^{j,k}(e^{-1}X_{s_j}^{\epsilon,\zeta}) \phi_{\eta}^{j,k}(e^{-1}X_{s_j}^{\epsilon,\zeta})\, ds \right]$$

$$+ \epsilon^2 E1_{C_{j,k}} \{ \Gamma_{s_{j+1}}^{\epsilon,\zeta} \psi_{\eta}^{j,k}(e^{-1}X_{s_{j+1}}^{\epsilon,\zeta}) - \Gamma_{s_j}^{\epsilon,\zeta} \psi_{\eta}^{j,k}(e^{-1}X_{s_j}^{\epsilon,\zeta}) \}.$$ 

Thus, we can deduce the desired inequality by summing up over all $j, k$, and taking supremum over all controls.

\[\square\]

**The proof of Theorem 1.2.** From Propositions 4.1, 4.2 and Lemma 4.3, we obtain the following estimate:

$$|\inf_{\zeta} Y_t^{\epsilon,\zeta}| \leq C(n^{-q} + n^q N^{\frac{(1-q)}{2}} + N^{-\delta/2} + n^{-\delta} + \epsilon + \epsilon^2 + \epsilon^2 N),$$

where $C > 0$ may depend on $T > 0$ and $|x|$ but is independent of $N$, $n$, $q > 1$ and $\epsilon > 0$.

Fix arbitrarily $\gamma_1, \gamma_2 > 0$ and define $n \in \mathbb{R}_+$ and $N \in \mathbb{N}$ by

$$n := \epsilon^{-\gamma_1}, \quad N := [\epsilon^{-\gamma_2}] + 1.$$

Then,

$$(4.7) \quad |\inf_{\zeta} Y_t^{\epsilon,\zeta}| \leq C(\epsilon^{\gamma_1 q} + \epsilon^{\gamma_2 (q-1)/2 - \gamma_1 q} + \epsilon^{\delta \gamma_2/2} + \epsilon^{\delta \gamma_1} + \epsilon + \epsilon^2 + \epsilon^{2-\gamma_2}),$$

from which we get the following inequality:

$$|\inf_{\zeta} Y_t^{\epsilon,\zeta}| \leq C e^{F(\gamma_1, \gamma_2, q)},$$

where $F(\gamma_1, \gamma_2, q) := \min\{ \gamma_2(q-1)/2 - \gamma_1 q, \delta \gamma_1, 2 - \gamma_2 \}$. By straightforward computation, for each fixed $q > 1$,

$$F_{\max}(q) := \max\{ F(\gamma_1, \gamma_2, q) ; 0 < \gamma_1 < (q-1)\gamma_2/2q, \quad 0 < \gamma_2 < 2 \}$$

$$= \frac{2\delta(q-1)}{2q + \delta + \delta q}.$$
Since the last term is increasing with respect to $q$ and converges to $2\delta/(\delta + 2)$ as $q \to +\infty$, we finally obtain
\[
\left| \inf_{\xi} \overline{Y}_{t}^{\epsilon,\xi} \right| \leq \lim_{q \to +\infty} C \varepsilon^{F_{\max}(q)} \leq C \varepsilon^{\frac{2\delta}{2+\delta}}.
\]
We have completed the proof of Theorem 1.2. \qed

**Remark 4.4.** If $v$ and $u^0$ are sufficiently smooth (e.g. $v(\eta, y, p, X) \in C^2(\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d)$ and $u^0(t, x) \in C^{2,4}_b([0, T] \times \mathbb{R}^d)$), then the local argument we used above is not necessary and the rate of convergence can be improved. In fact, let us consider the case where the Hamiltonian $H$ is linear with respect to $(y, p, X)$:

\[
H(\eta, y, p, X) := -\sum_{i,j=1}^{d} a^{ij}(\eta)X_{ij} - \sum_{i=1}^{d} b^i(\eta)p_i - c(\eta)y.
\]

Then, the corresponding FBSDE can be written as
\[
\begin{cases}
    dX^\epsilon = b(\epsilon^{-1}X^\epsilon_s) \, ds + \sigma(\epsilon^{-1}X^\epsilon_s) \, dW_{t,s}, & X^\epsilon_0 = x, \\
    -dY^\epsilon = c(\epsilon^{-1}X^\epsilon_s)Y^\epsilon_s \, ds - \sigma^*(\epsilon^{-1}X^\epsilon_s)Z^\epsilon_s \, dW_{t,s}, & Y^\epsilon_T = h(X^\epsilon_T),
\end{cases}
\]
where we have set $\sigma\sigma^* = 2a$. Then, it is well known that the effective Hamiltonian $\overline{H}$ in (1.2) is characterized by

\[
\overline{H}(\eta, y, p, X) := -\sum_{i,j=1}^{d} \overline{a}^{ij}(\eta)X_{ij} - \sum_{i=1}^{d} \overline{b}^i p_i - \overline{c} y,
\]

\[
\overline{g} = \int_{(0,1)^d} g(\eta)m(\eta) \, d\eta, \quad g = a^{ij}, b^i, c,
\]

where $m(\eta) \, d\eta$ is the invariant measure on $(0,1)^d$ associated with the differential operator $L := a^{ij}(\eta)\partial_{x^i}\partial_{x^j}$.

Now let $v = v(\eta, y, p, X)$ be a unique solution of the cell problem (1.3) such that $v(0, y, p, X) = 0$. Then, $v$ satisfies
\[
v(\eta, \lambda_1 \Theta_1 + \lambda_2 \Theta_2) = \lambda_1 v(\eta, \Theta_1) + \lambda_2 v(\eta, \Theta_2), \quad \forall \lambda_i \in \mathbb{R}, \quad \Theta_i = (y_i, p_i, X_i), \quad i = 1, 2.
\]

In particular, $v$ is infinitely differentiable with respect to $(y, p, X)$.

Now, let $u^0$ be a solution to (1.2) and we assume that $u^0 \in C^{2,4}_b([0, T] \times \mathbb{R}^d)$. Then, by Itô's formula, we can easily see
\[
|Y^\epsilon_s - u^0(s, X^\epsilon_s) - \varepsilon^2 v(\epsilon^{-1}X^\epsilon_s, s, X^\epsilon_s)| \leq C(\varepsilon + \varepsilon^2),
\]

which is (formally) the case where $\delta = 2$ in Theorem 1.2.


