A Path Integral Preliminary Approach to the FKG Inequality for $Yukawa_2$ Quantum Field Theory (Applications of Renormalization Group Methods in Mathematical Sciences)

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A Path Integral Preliminary Approach to the FKG Inequality for Yukawa$_2$ Quantum Field Theory*

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1. By the method used in our previous paper [I1], we construct a countably additive path space measure for the 2-D Euclidean Dirac equation in the polar coordinates to give a path integral representation to its Green's function (For a brief survey, see [I2]). This is a report of trying a preliminary approach with use of the result to give an alternative proof of the FKG inequality for Yukawa$_2$ quantum field theory obtained by Battle–Rosen [BR], though not yet incomplete.

G.A.Battle and L.Rosen used Vekua–Bers theory of generalized analytic functions to show the FKG inequality for $Y_2$ QFT. The $Y_2$ measure is formally given by

$$\nu := \frac{1}{Z} e^{W(\phi)} \prod_{x \in \mathbb{R}^2} d\phi(x)$$

$$W(x) := \frac{1}{2}(\phi, (-\Delta + m_b^2)\phi) + \text{Tr } K - \frac{1}{2} \text{Tr } K^* K : + \text{Tr } \ln(1 - K) K,$$

with $Z$ is a normalized constant, where

$$K(x, y) := S(x, y)\phi(y)\chi_{\Lambda}(y), \quad \phi: \text{Boson field (mass } m_b),$$

$$\chi_{\Lambda}: \text{indicator function of a square } \Lambda \subset \mathbb{R}^2,$$

and

$$S(x, y) := (-\beta \partial_x + m_f)^{-1}\Gamma, \quad \beta \partial_x = \beta_0 \partial_0 + \beta_1 \partial_1, x = (x_0, x_1),$$

$$\beta_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \beta_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3,$$

with $m_f \geq 0$ the Fermi mass. They considered the two models

a) $\Gamma := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ (scalar $Y_2$),

b) $\Gamma := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2$ (pseudo-scalar $Y_2$).

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Then FKG inequality (like $\langle fg \rangle \geq \langle f \rangle \langle g \rangle$) holds: $\frac{\delta^2 W}{\delta \phi(x) \delta \phi(y)} \geq 0$, $x \neq y$.

By some heuristic arguments, this is equivalent to showing

$$\text{tr} S'(x, y) S'(y, x) \leq 0, \quad x \neq y.$$ \hspace{1cm}

where $S' := (1 - K)^{-1} S$ is the Green's function (vanishing at $\infty$) for 2D-Euclidean Dirac equation

$$[\Gamma^{-1}(-\beta \partial_x + m_f) - \phi(x) \chi_{\Lambda}(x)] S'(x, y) = \delta(x - y).$$

Battle and Rosen proved the above inequality for $m_f \geq 0$ in the case a) and for $m_f = 0$ in the case b).

So, the first thing to do is to construct this Green's function.

In [11], we constructed a countably additive path space measure to give a path integral representation for the Green's function for 3D-Dirac equation in the radial coordinate.

The aim of this talk is to give a preliminary approach to ask whether this method can apply to get the Green's function for the above 2D-Euclidean Dirac equation to show the desired inequality.

Put the 2D-Euclidian operator $L^2(\mathbb{R}^2)^2 \equiv L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ as:

$$T_{\Gamma} := \Gamma^{-1}(-\beta \partial_x + m_f) - V(x), \quad V(x) := \phi(x) \chi_{\Lambda}(x),$$

$$= \Gamma^{-1} \left[ - \sigma_1 \frac{\partial}{\partial x_0} - \sigma_3 \frac{\partial}{\partial x_1} + m_f \right] - V(x), \quad x = (x_0, x_1) \in \mathbb{R}^2,$$

$$\beta = (\beta_0, \beta_1), \quad \beta_0 = \sigma_1, \quad \beta_1 = \sigma_3.$$

They considered the two models: a) scalar $Y_2$ model: $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

b) pseudoscalar $Y_2$ model: $\Gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

In this note let us consider only a) the scalar $Y_2$ model.

2. Since $\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we have by the polar coordinates $x_0 = r \cos \theta$, $x_1 = r \sin \theta (0 \leq r < \infty, \ 0 \leq \theta < 2\pi)$,

$$T_{\Gamma} = -C(\theta) \frac{\partial}{\partial r} - \frac{1}{r} D(\theta) \frac{\partial}{\partial \theta} + m_f - V,$$

where

$$C(\theta) := \sigma_1 \cos \theta + \sigma_3 \sin \theta = \begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix},$$

$$D(\theta) := - (\sigma_1 \sin \theta - \sigma_3 \cos \theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & -\cos \theta \end{pmatrix}. $$
We write $\mathbb{R}_+ = (0, \infty)$ and $\overline{\mathbb{R}_+} = [0, \infty)$.

Making the unitary transformation

$$U(\theta) := \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{1 + \sin \theta} & \cos \theta \\ \frac{\cos \theta}{\sqrt{1 + \sin \theta}} & \sqrt{1 + \sin \theta} \end{pmatrix},$$

we have

$$U(\theta)T_{\Gamma}U(\theta)^{-1} = \left[ - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial r} + \frac{1}{2r} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \frac{1}{r} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \right] + m_f - V$$

in $L^2(\mathbb{R}^2) = L^2(\overline{\mathbb{R}_+} \times [0, 2\pi); r dr d\theta)^2$.

We make one more unitary transformation $W$ of the $r dr$-measure space to the $dr$-measure space:

$$W : L^2(\mathbb{R})^2 \equiv L^2(\overline{\mathbb{R}_+} \times [0, 2\pi); r dr d\theta)^2 \ni f \mapsto r^{1/2}f \in L^2(\overline{\mathbb{R}_+} \times [0, 2\pi); dr d\theta)^2$$

to get

$$WU(\theta)T_{\Gamma}U(\theta)^{-1}W^{-1} = \left[ - i r^{1/2} \frac{\partial}{\partial r} r^{1/2} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \right] + i(m_f - V)r.$$

Then we multiply $r^{1/2}$ from the left and the right and then multiply the factor $i$ to put

$$H_{sc}(rV) := ir^{1/2}WU(\theta)T_{\Gamma}U(\theta)^{-1}W^{-1}r^{1/2}$$

$$= \left[ - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} - i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \theta} \right] + i(m_f - V)r.$$

Since the operator $-i \frac{\partial}{\partial \theta}$ is a selfadjoint operator in $L^2([0, 2\pi); d\theta)$ having as the spectrum consisting of only the eigenvalues $\{k\}_{k \in \mathbb{Z}}$ with eigenfunctions $\{ e^{ik\theta}/\sqrt{2\pi} \}_{k \in \mathbb{Z}}$, our $L^2$ space $L^2(\overline{\mathbb{R}_+} \times [0, 2\pi); dr d\theta)^2$ admits the direct sum decomposition:

$$L^2(\overline{\mathbb{R}_+} \times [0, 2\pi); dr d\theta)^2 = \sum_{k \in \mathbb{Z}} \bigoplus \left( L^2(\overline{\mathbb{R}_+}; dr)^2 \otimes \frac{e^{ik\theta}}{\sqrt{2\pi}} \right).$$

Then we have

$$H_{sc}(rV) = \sum_{k \in \mathbb{Z}} \bigoplus H_{sc}(k),$$

$$H_{sc}(k) := \left[ - i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + i(m_f - V)r.$$

We want to find a path integral representation for the Green's function for this operator having a singularity at $r = 0$. 
For each fixed $k \in \mathbb{Z}$, put the free part of $H_{sc}(k)$ to be equal to 

$$H_0(k) := -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} r^{1/2} \frac{\partial}{\partial r} r^{1/2} + k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is an operator in $L^2(\mathbb{R}_+; dr)^2$. We can show that $H_0(k)$ is essentially selfadjoint on $C^\infty_c(\mathbb{R}_+)^2$, which is a non-trivial result. Therefore the Cauchy problem for it:

$$\frac{\partial}{\partial t} \psi(r, t) = -iH_0(k)\psi(r, t), \quad t \in \mathbb{R},$$
$$\psi(r, 0) = g(r), \quad t = 0,$$

is $L^2$ well-posed. In other words, we can solve it in the space $L^2(\mathbb{R}_+; dr)^2$.

Crucial is that this Cauchy problem is even $L^\infty$ well-posed. Namely, we have the following lemma.

**Lemma.** There exists a unique solution $\psi(r, t) = (e^{-itH_0(k)}g)(r)$ which satisfies

$$||\psi(\cdot, t)||_{\infty} = ||e^{-itH_0(k)}g||_{\infty} \leq e^{|t|(|k|+1/2)}||g||_{\infty}.$$

By the method in [11] based on this lemma, we can construct a $2 \times 2$-matrix-distribution-valued countably additive path space measure $\mu_{t,0}^{k}$ on the space $C([0, t] \rightarrow \mathbb{R}_+)$ of the continuous paths $R : [0, t] \rightarrow \mathbb{R}_+$ which represents the solution of the above Cauchy problem: for every pair of $f$ and $g$ in $C^\infty_c(\mathbb{R}_+)^2$,

$$(f, \psi(\cdot, t)) = \int_{0}^{\infty} \int_{C([0,t] \rightarrow \mathbb{R}_+, R(0)=\rho, R(t)=r)} \overline{f(r)} (e^{-itH_{sc}(k)}g)(r) \, dr \, d\mu_{t,0}^{k}(R).$$

Hence, supposing that we can get the inverse of the operator $H_{sc}(k)$ as $H_{sc}(k)^{-1} = i \int_{0}^{\infty} e^{-itH_{sc}(k)} \, dt$ by the Laplace transform, we have the following path integral representation for its Green's function, which is a little formally expressed, suppressing the use of test functions:

$$H_{sc}(k)^{-1}(r, \rho) = i \int_{0}^{\infty} dt \int_{C([0,t] \rightarrow \mathbb{R}_+, R(0)=\rho, R(t)=r)} r^{1/2} \rho^{1/2} e^{\int_{0}^{t}(m_f-V(R(s))R(s)) \, ds} \, d\mu_{t,0}^{k}(R).$$

3. We have

$$T_{\Gamma}^{-1} = ir^{1/2}WU(\theta)H_{sc}(rV)^{-1}U(\theta)^{-1}W^{-1}r^{-1/2}.$$

Here, if we use the polar coordinates for $x = (x_0, x_1), \, y = (y_0, y_1) \in \mathbb{R}^2$

$$x_0 = r \cos \theta, \quad x_1 = r \sin \theta \ (0 \leq r < \infty, \ 0 \leq \theta < 2\pi),$$
$$y_0 = r' \cos \theta', \quad y_1 = r' \sin \theta' \ (0 \leq r' < \infty, \ 0 \leq \theta' < 2\pi),$$
we may write the integral kernel of the operator $H_{sc}(rV)^{-1}$ as

$$H_{sc}(rV)^{-1}(r, \theta; r', \theta')$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} H_{sc}(k)^{-1}(r, r') e^{-ik(\theta-\theta')}$$

$$= \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} e^{-ik(\theta-\theta')} i \int_{R(0)=r', R(t)=r} \frac{1}{2^{1/2} \int_{R(0)-V(R(s))} R(s) ds} d\mu_{v,0}^{(k)}(R).$$

Then

$$\text{tr} \left[ T_{\Gamma}^{-1}(r, \theta; r', \theta') T_{\Gamma}^{-1}(r', \theta' ; r, \theta) \right]$$

$$= - \text{tr} \left[ r^{-1/2} W U(\theta) H_{sc}(rV)^{-1}(r, \theta; r', \theta') U(\theta')^{-1} W^{-1} r^{-1/2} \right]$$

$$\times r^{-1/2} W U(\theta') H_{sc}(rV)^{-1}(r', \theta' ; r, \theta) U(\theta)^{-1} W^{-1} r^{-1/2} \right]$$

$$= - \text{tr} \left[ r r' H_{sc}(rV)^{-1}(r, \theta; r', \theta') H_{sc}(rV)^{-1}(r', \theta' ; r, \theta) \right]$$

$$= - r r' \text{tr} \left[ \left( \sum_{k \in \mathbb{Z}} H_{sc}(k)^{-1}(r, r') \frac{e^{-ik(\theta-\theta')}}{2\pi} \right) \left( \sum_{\ell \in \mathbb{Z}} H_{sc}(\ell)^{-1}(r', r) \frac{e^{-itu(\theta'-\theta')}}{2\pi} \right) \right]$$

$$= - \frac{r r'}{(2\pi)^2} \text{tr} \sum_{k, \ell \in \mathbb{Z}} a_{k\ell} e^{-i(k-\ell)(\theta-\theta')}.$$

Here we seem to have

$$a_{k\ell} := \int_{0}^{\infty} e^{-itH_{sc}(k)}(r, r') dt (-i) \int_{0}^{\infty} e^{iuH_{sc}(\ell)}(r', r) du$$

$$= \int_{0}^{\infty} dt \int_{0}^{\infty} du$$

$$\times \int_{C([0, t] \to \mathbb{R}^{+}), R_{1}(0)=r', R_{1}(t)=r} \int_{C([0, u] \to \mathbb{R}^{+}), R_{2}(0)=r', R_{2}(u)=r} e^{i t \int_{0}^{t}(m_{f}-V(R_{1}(\epsilon))) R_{1}(\epsilon) d\epsilon} d\mu_{t,0}^{(k)}(R_{1})$$

$$\times e^{i u \int_{0}^{u}(m_{f}-V(R_{2}(s))) R_{2}(s) ds} d\mu_{0,u}^{(\ell)}(R_{2})$$

$$= \int_{0}^{\infty} dt \int_{0}^{\infty} du \int_{C([0, t] \to \mathbb{R}^{+}), R_{1}(0)=r', R_{1}(t)=r} \int_{C([0, u] \to \mathbb{R}^{+}), R_{2}(0)=r', R_{2}(u)=r} e^{t \int_{0}^{t}(m_{f}-V(R_{1}(\epsilon))) R_{1}(\epsilon) d\epsilon} d\mu_{t,0}^{(k)}(R_{1})$$

$$\times e^{u \int_{0}^{u}(m_{f}-V(R_{2}(s))) R_{2}(s) ds} d\mu_{0,u}^{(\ell)}(R_{2}) \times t \mu_{0,0}^{\ell}$$

where $t \mu_{0,0}^{\ell}$ is the transposed of the $2 \times 2$-matrix-distribution valued-measure $\mu_{0,0}^{\ell}$.

Then the problem is to show in the case a) that

$$\text{tr} \sum_{k, \ell \in \mathbb{Z}} a_{k\ell} e^{-i(k-\ell)(\theta-\theta')} \geq 0.$$
But our the argument is stopped here, and will be discussed elsewhere.

References

