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Fourier Transformation of 2D $O(N)$ Spin Model and Anderson Localization

K. R. Ito *
Department of Mathematics and Physics, Setsunan University,
Neyagawa, Osaka 572-8508, Japan
F. Hiroshima †
Department of Mathematics, Kyushu University,
Hakozaki 6-10-1, Fukuoka, 812-8581, Japan
H. Tamura ‡
Department of Mathematics,Kanazawa University,
Kanazawa 920-1192, Japan

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Abstract

We Fourier transform the 2D $O(N)$ spin model $N > 2$, and start with a representation of the correlation functions in terms of integrals by complex random fields. Since this integral is complicated, we use the idea of the Anderson localization to discard non-local terms which make the integrals difficult. Through this approximation, we obtain the correlation functions which decay exponentially fast for all $\beta > 0$ if $N >> 3$.

1 Introduction: Result and Motivation

It is a longstanding problem to prove or disprove non-existence of phase transitions in 4 dimensional non-Abelian lattice gauge theories. In many points, this is similar to the same problem in the two-dimensional $O(N)$ symmetric spin models (Heisenberg or σ model) with $N \geq 3$.

In models such as $O(N)$ spin modes and $SU(N)$ lattice gauge models [13, 17], the field variables form compact manifolds and the block spin transformations break the original

*Email: ito@mpg.setsunan.ac.jp
†Email: hiroshima@math.kyushu-u.ac.jp
‡Email: tamurah@kenroku.kanazawa-u.ac.jp

structures. In some cases, this can be avoided by introducing an auxiliary field ψ [3] which can be regarded as complex random field. The ν dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$< f >\equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_{i\in\Lambda} \delta(\phi_i^2 - N\beta) d\phi_i \tag{1.1}$$

Here Λ is an arbitrarily large square with the center at the origin, $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$ and Z_Λ is the partition function defined so that $< 1 >= 1$. The Hamiltonian H_Λ is given by

$$H_\Lambda \equiv -\frac{1}{2} \sum_{|x-y|_1=1} \phi(x)\phi(y), \tag{1.2}$$

where $|x|_1 = \sum_{i=1}^{\nu} |x_i|$.

We substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)]da/2\pi$ into eq.(1.1) with the condition that $\mathrm{Im} a_i < -\nu$ [3], and set

$$\mathrm{Im}\, a_i = -(\nu + \frac{m^2}{2}), \quad \mathrm{Re}\, a_i = \frac{1}{\sqrt{N}}\psi_i \tag{1.3}$$

where $m > 0$ will be determined soon. Thus we have

$$\begin{aligned} Z_\Lambda &= c^{|\Lambda|} \int \cdots \int \exp[-\frac{1}{2} < \phi, (m^2 - \Delta + \frac{2i}{\sqrt{N}}\psi)\phi > + \sum_j i\sqrt{N}\beta\psi_j] \prod \frac{d\phi_j d\psi_j}{2\pi} \\ &= c^{|\Lambda|} \det(m^2 - \Delta)^{-N/2} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \end{aligned} \tag{1.4}$$

where c's are constants being different on lines, $\Delta_{ij} = -2\nu\delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian,

$$F(\psi) = \det(1 + i\kappa G\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j], \tag{1.5}$$

and $\kappa = 2/\sqrt{N}$. Moreover $G = (m^2 - \Delta)^{-1}$ is the covariant matrix discussed later. In the same way, the two-point function is given by

$$< \phi_0\phi_x > = \frac{1}{\tilde{Z}} \int \cdots \int \left(\frac{1}{m^2 - \Delta + i\kappa\psi} \right) (0, x) F(\psi) \prod \frac{d\psi_j}{2\pi} \tag{1.6}$$

namely by an average of the Green's function which includes complex fields $\psi(x)$, $x \in Z^2$, where the constant $\tilde{Z}$ is chosen so that $< \phi_0^2 >= N\beta$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where

$$G(x) = \int \frac{e^{ipx}}{m^2 + 2\sum(1 - \cos p_i)} \prod_{i=1}^{\nu} \frac{dp_i}{2\pi} \tag{1.7}$$

This is possible for any β if $\nu \leq 2$, and we easily find that

$$m^2 \sim 32e^{-4\pi\beta} \text{ for } \nu = 2 \tag{1.8}$$

as $\beta \to \infty$. Thus for $\nu = 2$, we can rewrite

$$F(\psi) = \det{}_3^{-N/2}(1 + i\kappa G\psi)\exp[- < \psi, G^{\circ 2}\psi >], \tag{1.9}$$

$$\det{}_3(1 + A) \equiv \det[(1 + A)e^{-A+A^2/2}] \tag{1.10}$$

where $G^{\circ 2}(x,y) = G(x,y)^2$ so that $\mathrm{Tr}(G\psi)^2 =< \psi, G^{\circ 2}\psi >$. For any two matrices A and B of equal size, the Hadamard product [18] $A \circ B$ is defined by $(A \circ B)_{ij} = A_{ij}B_{ij}$ and we denote $G \circ G$ by $G^{\circ 2}$.

Decompose $\Lambda \subset Z^2$ into small blocks Δ_i, and define $G_\Lambda = \chi_\Lambda G \chi_\Lambda$:

$$\Lambda = \cup_{i=1}^n \Delta_i,$$

Then we use the Feshbach-Krein formula (blockwise diagonlizations of matrices), to decompose $\det(1 + i\kappa G_\Lambda \psi_\Lambda)$ into a product of $\det(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i})$ as follows:

$$\begin{aligned} &\det{}^{-N/2}(1 + i\kappa G_\Lambda \psi_\Lambda) \\ &= \left[\prod_{i=1}^{n-1} \det{}^{-N/2}(1 + W(\Delta_i, \Lambda_i))\right] \prod_{i=1}^{n} \det{}^{-N/2}(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}). \end{aligned} \tag{1.11}$$

where $\kappa = 2/\sqrt{N}$, $\Lambda_k = \cup_{i=k+1}^n \Delta_i$,

$$W(\Delta_i, \Lambda_i) = -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}} G_{\Delta_i,\Lambda_i}\psi_{\Lambda_i} \frac{1}{1 + i\kappa G_{\Lambda_i}\psi_{\Lambda_i}} G_{\Lambda_i,\Delta_i}\psi_{\Delta_i} \tag{1.12}$$

$$= -(i\kappa)^2 \frac{1}{[G_{\Delta_i}]^{-1} + i\kappa\psi_{\Delta_i}} [G_{\Delta_i}]^{-1} G_{\Delta_i,\Lambda_i}\psi_{\Lambda_i} \frac{1}{[G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i}} [G_{\Lambda_i}]^{-1} G_{\Lambda_i,\Delta_i}\psi_{\Delta_i} \tag{1.13}$$

and $[G_\Delta]^{-1}$ is the Laplacian with free boundary condition and almost equal to the Laplacian restricted to the square Δ with no boundary. Thus inf spec $[G_\Delta]^{-1} \sim 0$ and we can prove that $([G_{\Lambda_i}]^{-1} + i\kappa\psi_{\Lambda_i})^{-1}$ behaves like a massive Green's function which decreases fast since ψ behaves like a Gaussian random variable of zero mean and covariance $[G^{\circ 2}]^{-1}$.

Let us consider the measure localized on each block Δ:

$$d\mu_\Delta = \frac{1}{Z_\Delta} \det{}_3^{-N/2}(1 + i\kappa G_\Delta \psi_\Delta)\exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta)] \prod_{x\in\Delta} d\psi(x) \tag{1.14}$$

where Z_Δ is chosen so that $\int d\mu_\Delta = 1$. Since the norm of G_Δ is of order $O(|\Delta|\beta) >> 1$, one may think that it is still impossible to expand the determinant. However, this comes with the factor $\exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta)]$, which makes the norm of $\frac{2i}{\sqrt{N}} G_\Delta \psi_\Delta$ small. To see if this is the case, we introduce new variables $\tilde{\psi}_\Delta(x)$ by

$$\psi_\Delta(x) = \frac{1}{\sqrt{2}} \sum_{y\in\Delta} \hat{G}_\Delta^{-1}(x,y)\tilde{\psi}(y), \quad \hat{G}_\Delta \equiv [G_\Delta^{\circ 2}]^{1/2} \tag{1.15}$$

so that $d\mu_\Delta$ is rewritten

$$d\mu_\Delta = \frac{1}{Z_\Delta} \det{}_3^{-N/2}(1 + i\kappa K_\Delta) \prod_{x\in\Delta} \exp[-\frac{1}{2}\tilde{\psi}(x)^2]\frac{d\tilde{\psi}(x)}{\sqrt{2\pi}}, \tag{1.16}$$

$$K_\Delta = \frac{1}{\sqrt{2}} G_\Delta^{1/2}(\hat{G}_\Delta^{-1}\tilde{\psi})G_\Delta^{1/2} \tag{1.17}$$

Put

$$d\nu_\Delta^{(0)} = \prod \exp[-\frac{1}{2}\tilde{\psi}^2(x)]\frac{d\tilde{\psi}(x)}{\sqrt{2\pi}} \tag{1.18}$$

$$= \det{}^{-1/2}[G_\Delta^{\circ 2}] \exp[- < \psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta >] \prod_{x\in\Delta} \frac{d\psi(x)}{\sqrt{2\pi}} \tag{1.19}$$

and define

$$||K||_p = \left(\int \mathrm{Tr}(K^*K)^{p/2} d\nu_\Delta\right)^{1/p} \tag{1.20}$$

Lemma 1 *It holds that*

$$\int \mathrm{Tr} K_\Delta^2 d\nu_\Delta = \frac{1}{2}|\Delta|, \tag{1.21}$$

$$||K_\Delta||_p \le (p-1)||K_\Delta||_2, \quad \textit{for all } p \ge 2 \tag{1.22}$$

Proof. The first equation is immediate. See [16] for the second inequality. Q.E.D.

Thus we see that κK_{Δ_i} are a.e. bounded with respect to $d\nu_\Delta$, and converges to 0 as $N \to \infty$. To see to what extent K_Δ is diagonal, we estimate

$$\int \mathrm{Tr} K_\Delta^4 d\nu_\Delta = \sum_{x_i\in\Delta} \frac{1}{4} \prod_{i=1}^{4} G_\Delta(x_i, x_{i+1})$$
$$\times \left[2[G^{\circ 2}]^{-1}(x_1,x_2)[G^{\circ 2}]^{-1}(x_3,x_4) + [G^{\circ 2}]^{-1}(x_1,x_3)[G^{\circ 2}]^{-1}(x_2,x_4)\right]$$

where $x_5 = x_1$. As is proved in [8]

$$[G_\Delta^{\circ 2}]^{-1}(x,y) = \frac{1}{2\beta}G_\Delta^{-1} - \hat{B}_\Delta, \quad \hat{B}_\Delta(x,y) = O(\beta^{-2}) \tag{1.23}$$

The main contribution comes from the term containing $2[G^{\circ 2}]^{-1}(x_1,x_2)\cdots$. To bound this, set $G_\Delta(x_i,x_{i+1}) = \beta - \delta G(x_i,x_{i+1})$. Then $\delta G(x,x) = 0$, $\delta G(x, x+e_\mu) = 0.25 - O(\beta m^2)$, $(-\Delta)_{xy} = 0$ unless $|x-y| \le 1$, and we have

$$\int \mathrm{Tr} K_\Delta^4 d\nu_\Delta \ge \text{const.} \sum_{x_1\in\Delta} \frac{1}{4\beta^2}\left\{\beta^2 \sum_{x_4} \delta_{x_1,x_4} + \sum_{x_4} G^2(x_1,x_4)\right\}$$
$$\ge \text{const.}(|\Delta| + |\Delta|^2)$$

which means that K_Δ is approximately diagonal but off-diagonal parts are still considerably large. However, there is a reason to believe that W functions are of short range and small. In fact we know that

$$\left|\frac{1}{[G_{\Lambda_i}]^{-1}+i\kappa\psi_{\Lambda_i}}(x,y)\right| \le \frac{1}{[G_{\Lambda_i}]^{-1}+c(N\beta)^{-1}}(x,y)$$

for almost all ψ. Then $(G_{\Lambda_i}^{-1}+m^2+i\kappa\psi_{\Lambda_i})^{-1}(x,y)$ is negligible if $|x-y|>\sqrt{N\beta}$. Moreover it is shown in two dimension that

$$\begin{aligned}\int\frac{1}{[G_{\Lambda_i}]^{-1}+i\kappa\psi_{\Lambda_i}}(x,y)d\mu &\le \frac{1}{[G_{\Lambda_i}]^{-1}+m_{eff}^2}(x,y),\\ m_{eff}^2 &= c\frac{\log(N\beta)}{N\beta}\end{aligned}$$

if $d\mu(\psi)$ is Gaussian of mean zero and covariance $[G^{\circ 2}]^{-1}$. This logarithmic correction comes from the two-dimensionality. This implies that

$$\lim_{N\beta\to\infty}\frac{1}{N\beta}\sum_x\int\frac{1}{-\Delta+m^2+i\kappa\psi}(0,x)d\mu=0$$

Furthermore ψ in the numerators of W acts as a differential operators since

$$\psi = \frac{1}{\sqrt{2}}[G_\Delta^{\circ 2}]^{-1/2}\tilde{\psi}\sim\frac{1}{2\sqrt{\beta}}[G_\Delta]^{-1/2}\tilde{\psi}$$

Thus $W(\Delta_i,\Lambda_i)$ seems to be small as $N\beta\to\infty$.

We choose N larger than $|\Delta|=L^2$, i.e.,

$$N^{1/3-\varepsilon}\geqq|\Delta|=L^2 \tag{1.24}$$

This assumption is artificial and its role is to simplify the large field problem to bound the integrals in the region where $|\psi_x|$ are large. So more elaborate idea may remove this condition (it is natural to think that $N\geqq 3$ is enough).

To imagine that the non-diagonal terms W are small, we perhaps choose L larger than some power of β, say $L>(\beta)^{1+\delta}$, $\delta>0$, but we do not know how to determine it yet though it is now under investigation, see [9].

Assumption: *We take N larger than $|\Delta|=L^2$ as above, and for sufficiently large Δ, non-local terms W are negligible in this case.*

Once W is neglected and N is chosen larger than $|\Delta|$, we can prove the following result uniformly in β :

Main Theorem: *Assume N is sufficiently large: $N^{1/3-\varepsilon}\geqq|\Delta|$. Neglect non-local terms $W(\Delta,\Lambda)$. Then the two point correlation function*

$$\int\frac{1}{-\Delta+m^2+i\kappa\psi}(x,y)\prod_\Delta d\mu_\Delta \tag{1.25}$$

decays exponentially fast for all $\beta\ge 0$.

2 Averaged Green's Function by the measure $d\mu_0$

Let us come back to the present case where Δ_i are boxes of equal size $L \times L$ ($L \geqq 2$)such that $\cup_i \Delta_i = Z^2$ and $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$. Let us estimate

$$G^{(ave)}(x,y) \equiv \int G^{(\psi)}(x,y) d\mu(\psi) \tag{2.1}$$

where

$$G^{(\psi)}(x,y) \equiv \left(\frac{1}{G^{-1} + i\kappa\psi}\right)(x,y), \tag{2.2a}$$

$$d\mu(\psi) \equiv \prod \frac{1}{Z_\Delta} \det{}_3^{-N/2}(1 + i\kappa G_\Delta \psi_\Delta) d\nu_\Delta, \tag{2.2b}$$

$$d\nu_\Delta = \frac{1}{\det^{1/2}(G_\Delta^{\circ 2})} \exp[-(\psi_\Delta, G_\Delta^{\circ 2}\psi_\Delta)] \prod_{x\in\Delta} \frac{d\psi(x)}{\sqrt{2\pi}}. \tag{2.2c}$$

and $\kappa = 2/\sqrt{N}$. Expanding $G^{(\psi)}$ by random walk, we have

$$G^{(\psi)}(x,y) = \sum_{\omega: x\to y} \prod_{\zeta\in\omega} \frac{1}{(4 + m^2 + i\kappa\psi_\zeta)^{n_\zeta}} \tag{2.3}$$

where $n_\zeta \in N$ is the visiting number of ω at $\zeta \in Z^2$. We set

$$d\nu = \prod_{\Delta\subset Z^2} d\nu_\Delta \tag{2.4}$$

We first prove our assertion for the Gaussian case:

Theorem 2 *The following bound holds:*

$$\int G^{(\psi)}(x,y) d\nu \leq \frac{1}{-\Delta + m_{eff}^2}(x,y) \tag{2.5}$$

where

$$m_{eff}^2 = m^2 + \frac{c}{N\beta} \tag{2.6}$$

with a constant $c > 1$.

Proof. Let Δ be the square of size $L \times L$ centered at the origin, and let $n_x \in \{0, 1, 2, \cdots\}, x \in \Delta$. We estimate

$$D_\Delta(\{n\}) \equiv \int \prod_{x\in\Delta} \frac{1}{(4 + m^2 + i\kappa\psi(x))^{n_x}} d\nu_\Delta(\psi) \tag{2.7}$$

For large $\sum_{x\in\Delta} n_x$ such that

$$\sum_{x\in\Delta} n_x \geq \beta(\#\{x \in \Delta; n_x \neq 0\})^2 \tag{2.8}$$

the bound follows by the complex translation estimate by putting $\psi_x \to \psi_x - ih_x$, where

$$h_x = \frac{c_x}{\beta\sqrt{N}}, \quad c_x = \begin{cases} c > 0 & \text{if } n_x \geq 1 \\ 0 & \text{if } n_x = 0 \end{cases} \tag{2.9}$$

In fact we have :

$$\begin{aligned} D_\Delta(\{n\}) &\leq \frac{e^{<h, G_\Delta^{\circ 2} h>}}{\prod_{x\in\Delta}(4+m^2+\kappa h_x)^{n_x}} \leq \frac{e^{\beta^2(\sum_{x\in\Delta} h_x)^2}}{(4+m^2+c(\beta N)^{-1})^{\sum n_x}} \\ &\leq \left(\frac{1}{4+m^2+c'(\beta N)^{-1}}\right)^{\sum n_x} \end{aligned} \tag{2.10}$$

with a constant $0 < c' < c$.

For small $\{n_x; x \in \Delta\}$, we start with the new expression of $D_\Delta(\{n\})$:

$$\begin{aligned} &\prod_{x\in\Delta} \frac{1}{(n_x-1)!} \int_0^\infty \prod s_x^{n_x-1} \exp[-(4+m^2)\sum s_x - \frac{\kappa^2}{4} < s_\Delta, [G_\Delta^{\circ 2}]^{-1} s_\Delta >] \prod ds_x \\ &= \prod_{x\in\Delta} T^{-n(x)} \int \exp[-\frac{1}{NT^2} < s_\Delta, [G_\Delta^{\circ 2}]^{-1} s_\Delta >] \prod_x d\nu_{n_x}(s_x) \end{aligned} \tag{2.11}$$

where $T = 4 + m^2$ and

$$d\nu_n(s) = \frac{s^{n-1}e^{-s}}{(n-1)!} ds \tag{2.12}$$

Since $\int d\nu_n(s) = 1$ and $n \log s - s$ takes its maximum at $s = n$, we set $s_x = n_x + \sqrt{n_x}\tilde{s}_x$ ($x \in \Delta$) and note that

$$d\nu_n(s) = \exp[-\frac{1}{2}\tilde{s}^2] \frac{e^{\delta_n(\tilde{s})}}{< e^{\delta_n(\tilde{s})} >} \frac{d\tilde{s}}{\sqrt{2\pi}}, \tag{2.13}$$

$$\begin{aligned} \delta_n(\tilde{s}) &= -\sqrt{n}\tilde{s} + (n-1)\log(1 + \frac{\tilde{s}}{\sqrt{n}}) + \frac{1}{2}\tilde{s}^2 \\ &= -\log(1 + \frac{\tilde{s}}{\sqrt{n}}) + O\left(\frac{\tilde{s}^3}{\sqrt{n}}\right), \end{aligned} \tag{2.14}$$

$$\begin{aligned} < e^{\delta_n(\tilde{s})} > &= \int_{-\sqrt{n}}^\infty e^{\delta_n(\tilde{s})} e^{-\tilde{s}^2/2} \frac{d\tilde{s}}{\sqrt{2\pi}} \\ &= 1 + O(1/n) > 1 \end{aligned} \tag{2.15}$$

Put

$$\alpha^2 = \frac{1}{N\beta} \tag{2.16}$$

Then if $\alpha^2 n(x) < 1$ and $N^{-1} < n, [G_\Delta^{\circ 2}]^{-1} n >$ is small, the integral (2.11) is carried out by perturbative calculations.

For large $\alpha^2 n(x) \geqq 1$ or for non-smooth n such that $N^{-1} < n, [G^{\circ 2}_\Delta]^{-1} n >> 1$ we use a priori bound. See [8]. Q.E.D.

In the case where $d\mu$ is Gaussian, we can obtain $G^{(ave)}$ in a closed form. See [8] where $m^2_{eff} \sim \log(N\beta)/N\beta$ is obtained.

Remark 1 *We note that this is similar to the pinch singularity encountered in the study of the Anderson localization [5], where*

$$\int G(E+i\varepsilon, v)(x,y)dP(v)$$

has a convergent random walk expansion, and

$$\int |G(E+i\varepsilon, v)(x,y)|^2 dP(v)$$

does not have.

3 Averaged Green's Function by the measure $d\mu(\psi)$

It remains to discuss the effects of the determinants $\det_3^{-N/2}(1+\cdots)$. Set

$$\begin{aligned} S_\Delta &= \{\psi_x; x \in \Delta, \ \mathrm{Tr} K^2_\Delta < N^{1-2\varepsilon}\}, &(3.1)\\ K_\Delta &= G^{1/2}_\Delta \psi_\Delta G^{1/2}_\Delta &(3.2)\end{aligned}$$

Since

$$\exp[-\mathrm{Tr}K^2_\Delta] \le \left|\det{}_2^{-N/2}(1+i\kappa K_\Delta)\right| \le \left(1+\frac{4}{N}\mathrm{Tr}K^2_\Delta\right)^{-N/4} \tag{3.3}$$

and $\mathrm{Tr}K^2_\Delta = \sum \tilde{\psi}^2_x/2$, we have

$$\int \exp[-\mathrm{Tr}K^2_\Delta]\prod_x \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} = \int \exp[-\sum_x \frac{1}{2}\tilde{\psi}(x)^2]\prod_x \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} = 1 \tag{3.4}$$

and $\int (1+\frac{2}{N}\sum\tilde{\psi}^2(x))^{-N/4}\prod_{x\in\Delta} d\tilde{\psi}_x$ is convergent for $2|\Delta| < N$. Even so, it is obvious that $|\det^{-N/2}(1+i\kappa G_\Delta\psi)|$ is integrable if and only if $N > 2$ since

$$\begin{aligned} \det{}^{-N/2}(1+i\kappa G_\Delta\psi) &= \det{}^{-N/2}(G_\Delta)\det{}^{-N/2}(G^{-1}_\Delta + i\kappa\psi) &(3.5)\\ &\sim \det{}^{-N/2}(G_\Delta)\prod_{x\in\Delta}\left(\frac{1}{4+m^2+i\kappa\psi(x)}\right)^{N/2} &(3.6)\end{aligned}$$

holds for ψ such that $|\psi/\sqrt{N}| > O(1)$

3.1 Small Fields, Large Fields and Complex Displacements

Let us estimate

$$D_\Delta(n) = \frac{1}{Z_\Delta}\int \frac{e^{i\sqrt{N}\beta\sum_{x\in\Delta}\psi_x}}{\left[\prod_{x\in\Delta}(4+m^2+i\kappa\psi_x)^{n_x}\right]\det{}^{N/2}(1+i\kappa G_\Delta\psi_\Delta)}\prod_{x\in\Delta}d\psi_x, \quad (3.7)$$

$$Z_\Delta = \int \frac{e^{i\sqrt{N}\beta\sum_{x\in\Delta}\psi_x}}{\det{}^{N/2}(1+i\kappa G_\Delta\psi_\Delta)}\prod_{x\in\Delta}d\psi_x \quad (3.8)$$

by putting $\psi_x \to \psi_x - ih_x$, $\psi_x \in R$. Then

$$\begin{aligned}
&D_\Delta(n)\\
&= \frac{1}{Z_\Delta}\int \frac{e^{\sqrt{N}\beta\sum_{x\in\Delta}(i\psi_x+h_x)}}{\left[\prod_{x\in\Delta}(4+m^2+\kappa(i\psi_x+h_x))^{n_x}\right]\det{}^{N/2}(1+\kappa G_\Delta^{1/2}(i\psi_\Delta+h_\Delta)G_\Delta^{1/2})}\prod_{x\in\Delta}d\psi_x\\
&= \frac{1}{Z_\Delta}\int \frac{\exp[-<\psi-ih, G_\Delta^{o2}(\psi-ih)>]}{\left[\prod_{x\in\Delta}(4+m^2+\kappa(i\psi_x+h_x))^{n_x}\right]\det{}_3^{N/2}(1+i\kappa K_\Delta(\psi_\Delta)+\kappa K_\Delta(h_\Delta))}\prod_{x\in\Delta}d\psi_x
\end{aligned}$$

where

$$K_\Delta(\psi_\Delta) \equiv G_\Delta^{1/2}\psi_\Delta G_\Delta^{1/2}, \quad K_\Delta(h_\Delta) \equiv G_\Delta^{1/2}h_\Delta G_\Delta^{1/2} \quad (3.9)$$

and $K_\Delta(h_\Delta) \geq 0$ since $h_x \geq 0$. We again put $h_x = c_x/(\sqrt{N}\beta)$ and then

$$\kappa K_\Delta(h_\Delta) \leq \frac{c|\Delta|}{N}, \quad c = O(1) > 0. \quad (3.10)$$

We repeat the previous arguments by using $(n-1)!x^{-n} = \int_0^\infty s^{n-1}e^{-sx}ds$. Define $I_n^{(k)} = \{s; k\sqrt{n} < |s-n| < (k+1)\sqrt{n}, s \geq 0\}$, $k = 0,1,2,\cdots$, and let $\chi_x^{(k)}(s_x)$ be the characteristic function of the interval $I_{n_x}^{(k)}$. Then

$$\begin{aligned}
&D_\Delta(n)\\
&= \frac{1}{Z_\Delta}\int_0^\infty \prod_{x\in\Delta}\left(\sum_k \chi_x^{(k)}(s_x)\right)\frac{s_x^{n(x)-1}ds_x}{(n(x)-1)!}\int_{-\infty}^\infty \prod_{x\in\Delta}d\psi_x\\
&\times \exp[-\sum_x(4+m^2+\kappa(i\psi(x)+h(x))s(x)]\frac{e^{-<(\psi_\Delta-ih_\Delta),G^{o2}(\psi_\Delta-ih_\Delta)>}}{\det{}_3^{N/2}(1+i\kappa K_\Delta(\psi_\Delta)+\kappa K_\Delta(h_\Delta))} \quad (3.11)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{Z_\Delta^{(0)}}\frac{1}{\prod_x T_x^{n_x}}\int_0^\infty \prod_{x\in\Delta}d\nu_{n_x}(s_x)\int_{-\infty}^\infty \prod_{x\in\Delta}\left(\sum_k \chi_x^{(k)}(s_x)\right)d\psi_x\\
&\times \exp\left[-<(\psi-ih+i\zeta), G_\Delta^{o2}(\psi-ih+i\zeta)> -\frac{1}{N}<\frac{1}{T}s,[G_\Delta^{o2}]^{-1}\frac{1}{T}s> +\kappa<h,\frac{s}{T}>\right]\\
&\times \det{}_3^{-N/2}(1+i\kappa K_\Delta(\psi_\Delta)+\kappa K_\Delta(h_\Delta)) \quad (3.12)
\end{aligned}$$

where $T_x = 4 + m^2 + \kappa h_x$, $(s/T)_x = s_x/T_x$

$$d\nu_n(s) = \frac{1}{(n-1)!} e^{-s} s^{n-1} ds, \quad \zeta_x = \frac{\kappa}{2}([G_\Delta^{\circ 2}]^{-1}\frac{1}{T}s)(x) \tag{3.13}$$

and

$$\begin{aligned} Z_\Delta^{(0)} &= \int \exp[- < (\psi - ih), G_\Delta^{\circ 2}(\psi - ih) >] \\ &\quad \times \det{}_3^{-N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta)) \prod_{x\in\Delta} d\psi_x \end{aligned} \tag{3.14}$$

We then change the contour of ψ_x by replacing $\psi_x + i\zeta_x$ by ψ (namely we put $\psi_x \to \psi - i\zeta_x$). The contours depend on $\{s_x; x \in \Delta\}$. This yields

$$\begin{aligned} &D_\Delta(n) \\ &= \frac{1}{Z_\Delta^{(0)}} \frac{1}{\prod_x T_x^{n_x}} \int_0^\infty \prod_{x\in\Delta} d\nu_{n_x}(s_x) \int_{-\infty}^\infty \prod_{x\in\Delta} \left(\sum_k \chi_x^{(k)}(s_x)\right) d\psi_x \\ &\quad \times \exp\left[- < (\psi - ih), G_\Delta^{\circ 2}(\psi - ih) > - \frac{1}{N} < \frac{1}{T}s, [G_\Delta^{\circ 2}]^{-1}\frac{1}{T}s > + \kappa < h, \frac{s}{T} >\right] \\ &\quad \times \det{}_3^{-N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta) + \kappa K_\Delta(\zeta_\Delta)) \\ &= \frac{1}{\prod_x T_x^{n_x}} \int_0^\infty \prod_{x\in\Delta} d\nu_{n_x}(s_x) \exp\left[-\frac{1}{N} < \frac{1}{T}s, [G_\Delta^{\circ 2}]^{-1}\frac{1}{T}s > + \kappa < h, \frac{s}{T} >\right] \\ &\quad \times \frac{1}{Z_\Delta^{(0)}} \int_{-\infty}^\infty \prod_{x\in\Delta} \left(\sum_k \chi_x^{(k)}(s_x)\right) d\psi_x \det{}_3^{-N/2}(1 + i\kappa K_\Delta(\psi_\Delta)) \exp\left[- < \psi, G_\Delta^{\circ 2}\psi >\right] \\ &\quad \times \det{}_3^{-N/2}(1 + \kappa J_\Delta(\zeta_\Delta)) \times \exp[R_3] \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} K_\Delta(\zeta_\Delta)(x,y) &= \frac{1}{\sqrt{N}} \sum_\xi G_\Delta^{1/2}(x,\xi)\left([G_\Delta^{\circ 2}]^{-1}\frac{s}{T}\right)(\xi) G_\Delta^{1/2}(\xi,y), \\ J_\Delta(\zeta_\Delta) &= \frac{1}{\sqrt{1 + i\kappa K_\Delta(\psi_\Delta)}} K(\zeta_\Delta) \frac{1}{\sqrt{1 + i\kappa K_\Delta(\psi_\Delta)}} \end{aligned}$$

and

$$R_3 = \frac{N}{2}\mathrm{Tr}\left[\left(\frac{1}{1 + i\kappa K_\Delta(\psi_\Delta)} - 1 + i\kappa K_\Delta(\psi_\Delta)\right)\kappa K_\Delta(\zeta_\Delta) + \kappa^2(J_\Delta^2 - K_\Delta(\zeta_\Delta)^2)\right] \tag{3.16}$$

3.2 $K_\Delta(\psi_\Delta)$, $K_\Delta(\zeta_\Delta)$ and R_3

Let

$$G_\Delta = \sum_{i=0}^{|\Delta|-1} e_i P_i, \quad G_\Delta^{\circ 2} = \sum_{i=0}^{|\Delta|-1} \hat{e}_i \hat{P}_i \tag{3.17}$$

be the spectral resolutions of the positive matrices G_Δ and $G_\Delta^{\circ 2}$ respectively, where $e_0 \geq e_1 \geq \cdots \geq e_{|\Delta|-1}$, $\hat{e}_0 \geq \hat{e}_1 \geq \cdots \geq \hat{e}_{|\Delta|-1}$, $P_i P_j = \delta_{i,j} P_i$ and so on. Then

$$G_\Delta^{1/2} = \sum_{i=0}^{|\Delta|-1} \sqrt{e_i} P_i, \quad [G_\Delta^{\circ 2}]^{-1} = \sum_{i=0}^{|\Delta|-1} \frac{1}{\hat{e}_i} \hat{P}_i \tag{3.18}$$

It is convenient to introduce the abbreviation for the Green's function with the largest eigenvalue part extracted:

$$G_\Delta^{(0)} = \sum_{k \neq 0} e_k P_k = G_\Delta - e_0^{-1} P_0$$

We let $\{u_i\}_{i=0}^{|\Delta|-1}$ and $\{\hat{u}_i\}_{i=0}^{|\Delta|-1}$ be the normalized eigenvectors such that

$$G_\Delta u_i = e_i u_i, \quad G_\Delta^{\circ 2} \hat{u}_i = \hat{e}_i \hat{u}_i \tag{3.19}$$

Then

$$P_i = |u_i >< u_i|, \quad \hat{P}_i = |\hat{u}_i >< \hat{u}_i| \tag{3.20}$$

and for small Δ, we have

$$\begin{aligned} e_0 &= |\Delta|\beta - O(1), \quad e_i = O(1) > 0 & (3.21) \\ \hat{e}_0 &= |\Delta|\beta^2 - O(\beta), \quad \hat{e}_i = 2\beta e_i + O(1) & (3.22) \end{aligned}$$

$(i \neq 0)$ and

$$P_0 \sim \hat{P}_0 \sim \frac{1}{|\Delta|} |U >< U| = \frac{1}{|\Delta|} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \tag{3.23}$$

where $U = {}^t(1, 1, \cdots, 1) \sim \sqrt{|\Delta|} u_0$. Moreover we can symbolically write

$$P_i \sim \hat{P}_i \sim \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \tag{3.24}$$

namely P_i $(i \neq 0)$ is a matrix which represents a lattice differentiation since $< u_i, u_0 >= 0$. Note that $e_i \leqq O(\log|\Delta|)$, $e_0 = \beta|\Delta| - O(|\Delta|\log|\Delta|)$ and

$$(P_0\zeta P_0)_{x,y} = \sum_{\xi} \frac{1}{|\Delta|^2}\zeta_\xi = (\frac{1}{|\Delta|}\sum \zeta_\xi)P_0, \quad P_0(\hat{P}_i\zeta)P_0 = O(\beta^{-1}) \tag{3.25}$$

We insert $\psi = \hat{G}^{-1}\tilde{\psi}/\sqrt{2}$ into K_Δ and use $\hat{e}_i = 2\beta e_i + O(1)$ $(i \neq 0)$, $P_i = \hat{P}_i + O(\beta^{-1})$ and $\sum_{i\neq 0} P_i = 1 - P_0$ to find that

$$\begin{aligned}
K_\Delta &= \frac{\sum\tilde{\psi}(x)}{\sqrt{2|\Delta|}}P_0 + \frac{\sqrt{|\Delta|}}{2}\left(\sum_{i\neq 0} P_0(P_i\tilde{\psi}) + \sum_{i\neq 0}(\tilde{\psi}P_i)P_0\right) + O(\beta^{-1}) \\
&= \frac{(\sum\tilde{\psi}(x))}{\sqrt{|2\Delta|}}(1-\sqrt{2})P_0 + \frac{\sqrt{|\Delta|}}{2}\left(P_0\tilde{\psi} + \tilde{\psi}P_0\right) + O(\beta^{-1}) \\
K_\Delta^2 &= \left[\frac{1}{4}X + (1-\sqrt{2})Y^2\right]P_0 + \frac{\sqrt{2}-1}{4}\sqrt{|\Delta|}Y\left(P_0\tilde{\psi} + \tilde{\psi}P_0\right) \\
&\quad + \frac{|\Delta|}{4}\tilde{\psi}P_0\tilde{\psi} + O(\beta^{-1})
\end{aligned}$$

where

$$X = \sum_{x\in\Delta}\tilde{\psi}_x^2, \quad Y = \frac{1}{\sqrt{|\Delta|}}\sum_{x\in\Delta}\tilde{\psi}_x \tag{3.26}$$

Note that $\mathrm{Tr}K_\Delta^2 = \sum\tilde{\psi}_x^2/2$ as expected. Just in the same way, we have

$$\begin{aligned}
K_\Delta(\zeta) &= G_\Delta^{1/2}\zeta G_\Delta^{1/2} = G_\Delta^{1/2}\left(\frac{1}{\sqrt{N}}\sum_k \frac{1}{\hat{e}_k}\hat{P}_k\frac{s}{T}\right)G_\Delta^{1/2} \\
&= (\sum_x \zeta_x)\beta P_0 + \frac{1}{2}\left(\frac{|\Delta|}{\beta N}\right)^{1/2}\left[([G_\Delta^{(0)}]^{-1/2}\frac{s}{T})P_0 + P_0([G_\Delta^{(0)}]^{-1/2}\frac{s}{T})\right] + [G_\Delta^{(0)}]^{1/2}\zeta[G_\Delta^{(0)}]^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&K_\Delta(\zeta)^2 \\
&= \left[\beta^2(\sum\zeta)^2 + \frac{1}{4\beta N} < \frac{s}{T}, G_\Delta^{-1}\frac{s}{T} > + \left(\frac{\beta}{N|\Delta|}\right)^{1/2}(\sum\zeta)(\sum[G_\Delta^{(0)}]^{-1/2}\frac{s}{T})\right]P_0 \\
&\quad + \left[\left(\frac{\beta|\Delta|}{N}\right)^{1/2}(\sum\zeta) + \frac{1}{4\beta N}(\sum[G_\Delta^{(0)}]^{-1/2}\frac{s}{T})\right]\left(P_0([G_\Delta^{(0)}]^{-1/2}\frac{s}{T}) + ([G_\Delta^{(0)}]^{-1/2}\frac{s}{T})P_0\right) \\
&\quad + \frac{\sqrt{|\Delta|}}{2\sqrt{\beta N}}\left(P_0(\frac{s}{T}\circ\zeta)[G_\Delta^{(0)}]^{1/2} + [G_\Delta^{(0)}]^{1/2}(\frac{s}{T}\circ\zeta)P_0\right) \\
&\quad + \frac{|\Delta|}{4N\beta}([G_\Delta^{(0)}]^{-1/2}\frac{s}{T})P_0([G_\Delta^{(0)}]^{-1/2}\frac{s}{T}) + [G_\Delta^{(0)}]^{1/2}\zeta[G_\Delta^{(0)}]\zeta[G_\Delta^{(0)}]^{1/2}
\end{aligned}$$

Here $\zeta_x = N^{-1/2}(G_\Delta^{\circ 2})^{-1}(s/T)(x)$, $(x \circ y)_k \equiv x_k y_k$ for two vectors x and y, $e_0 = \beta|\Delta| - O(|\Delta|\log|\Delta|)$ and we have used $P_0 P_i = 0$ $(i \neq 0)$ and

$$P_0(G_\Delta^{-1/2}\frac{s}{T})[G_\Delta^{(0)}]^{1/2}\zeta[G_\Delta^{(0)}]^{1/2} = P_0(\frac{s}{T}\circ\zeta)[G_\Delta^{(0)}]^{1/2}$$

We can obtain similar expressions for $K(\psi)^n$ etc., and R_3 is represented by these functions of ψ and ζ. We decompose our set $\{s_x; s_x \geqq 0, x \in \Delta\}$ into 2 regions:

(1) small s region

(2) large s region

and each region is also decomposed into large ψ region and small ψ region, where the small ψ field $\mathcal{S}_\Delta(\psi)$ means the set of ψ such that

$$\mathcal{S}_\Delta(\psi) = \{\psi_x = \frac{1}{\sqrt{2}}(\hat{G}_\Delta^{-1}\tilde{\psi})(x), \sum_{x\in\Delta}\tilde{\psi}_x^2 \leq N^{1-2\epsilon}\}$$

and small s field $\mathcal{S}_\Delta(s)$ means the set of s_x such that

$$\begin{aligned}\mathcal{S}_\Delta(s) &= \left\{ s_x = n(x) + \sqrt{n(x)}\tilde{s}(x) \geq 0,\ \frac{1}{N\beta}\sum_{n.n.}\left(\frac{s_x}{T_x} - \frac{s_y}{T_y}\right)^2 \leqq O(1)\right.\\ &\left.\frac{1}{N^2\beta^2}\sum_{x\in\partial\Delta}\frac{s_x^2}{T_x^2} \leqq O(1)\right\}\end{aligned} \tag{3.27}$$

3.3 Small field Region of s_x

For small smooth $\{s_x\}$, we see that $\det_3^{-N/2}(1 - \kappa J_\Delta(\psi))$ yields a convergent small factor uniformly in ψ_x. Put

$$\det{}_3^{-N/2}(1 + \kappa J_\Delta(\psi)) = \exp[\mathcal{E}_3]$$

Then

$$\begin{aligned}|\mathcal{E}_3| &= \left|\frac{4}{3\sqrt{N}}\mathrm{Tr}J_\Delta^3 + \cdots\right| = o(1)\mathrm{Tr}K(\zeta_\Delta)^2\\ &= o(1)\frac{1}{N} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1}\frac{s}{T} >\end{aligned}$$

Contrary to the above, we must be careful about R_3 which depend on ψ sensitively.

3.3.1 small ψ region

We first assume ψ are small. Let us begin our calculation

$$\begin{aligned} I &= \frac{1}{Z_\Delta^0}\int \exp[\mathcal{E}_3 + R_3]\det{}_3^{-N/2}(1 + i\kappa K_\Delta)\exp[- < \psi, [G_\Delta^{\circ 2}]\psi >]\prod\frac{d\psi_x}{\sqrt{2\pi}}\\ &= \frac{1}{\tilde{Z}_\Delta^0}\int \exp[\mathcal{E}_3 + R_3]\det{}_3^{-N/2}(1 + i\kappa K_\Delta)d\nu_\Delta \end{aligned} \tag{3.28}$$

$$d\nu_\Delta \equiv \exp[-\frac{1}{2}\sum\tilde{\psi}_x^2]\prod\frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \tag{3.29}$$

by decomposing $\{\tilde{\psi}_x \in R; x \in \Delta\}$ into small field region

$$\mathcal{S}_\Delta = \{\sum_x \tilde{\psi}_x^2 < |\Delta| N^\varepsilon\}, \quad \varepsilon \in (0,1) \tag{3.30}$$

and its compliment $\mathcal{S}^c$, where the normalization constants $Z_\Delta^{(0)}$ and $\tilde{Z}_\Delta^{(0)}$ are defined in the obvious way. Thus we evaluate

$$I = I_S + I_{S^c} \tag{3.31}$$

where

$$I_S = \frac{1}{\tilde{Z}_\Delta^0} \int_S \left(1 + \mathcal{E}_3 + R_3 + O(R_3^2)\right) \det{}_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta \tag{3.32}$$

$$\begin{aligned} I_{S^c} &= \frac{1}{\tilde{Z}_\Delta^0} \int_{S^c} \det{}_1^{-N/2}(1 + i\kappa K_\Delta(\psi - ih + i\zeta)) e^{i\sqrt{N}\beta \sum_x (\psi_x - ih_x + i\zeta_x)} \\ &\times \exp\left[-\frac{2}{N} < \frac{s}{T}, [G_\Delta^{o2}]^{-1} \frac{s}{T} > +2\kappa < h, \frac{s}{T} > +i\kappa < \psi, \frac{s}{T} >\right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \end{aligned} \tag{3.33}$$

We first calculate the small field contribution I_S given by

$$\begin{aligned} I_S = \frac{< \chi_S \mathcal{D}_\Delta >}{< \mathcal{D}_\Delta >} \{ 1 + &< \chi_S \mathcal{E}_3 > + < \chi_S R_3 > + < \chi_S O(R_3^2) > \\ &+ \frac{< \chi_S \mathcal{D}_\Delta; \chi_S \mathcal{E}_3 >}{< \chi_S \mathcal{D}_\Delta >} \frac{< \chi_S \mathcal{D}_\Delta; \chi_S R_3 >}{< \chi_S \mathcal{D}_\Delta >} + \frac{< \chi_S \mathcal{D}_\Delta; \chi_S O(R_3^2) >}{< \chi_S \mathcal{D}_\Delta >} \Bigg\} \end{aligned} \tag{3.34}$$

where

$$\mathcal{D}_\Delta \equiv \det{}_3^{-N/2}(1 + i\kappa K_\Delta), \quad < A > = \int A d\nu_\Delta$$

and

$$< A; B > = \int AB d\nu - (\int A d\nu)(\int B d\nu)$$

We calculate $< \mathcal{D} >$ and $< \chi_S \mathcal{D} >$ first. We assumed that

$$\frac{|\Delta| - 2}{2} \le N^{1/3 - 2\varepsilon}, \quad 0 < \varepsilon << 1 \tag{3.35}$$

Then

$$< \chi_S \mathcal{D} > = \int_{\chi_S} \det{}_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta = 1 - O(N^{-1/3}) \tag{3.36}$$

To bound $< \chi_{S^c} \mathcal{D} >$, we use the bounds (3.3), and set $r^2 = 2\mathrm{Tr} K_\Delta^2 = \sum \tilde{\psi}_x^2$. Then for $R^2 > \rho_0 = (|\Delta| - 2)/2$, we have that

$$\frac{(|\Delta| - 2)!!}{(2\pi)^{|\Delta|/2}} \int_{r > R} \left(\frac{1}{1 + \frac{2}{N} r^2}\right)^{N/4} r^{|\Delta| - 1} dr \le O(\exp[-N^{1/3}])$$

This means that

$$\frac{<\chi_S \mathcal{D}_\Delta>}{<\mathcal{D}>} = \frac{<\chi_S \mathcal{D}_\Delta>}{<\chi_S \mathcal{D}> + <\chi_{S^c}\mathcal{D}>} = 1 - O(\exp[-cN^{1/3}]) \tag{3.37}$$

Estimates are straightforward and we see that the most significant contribution is from $\mathrm{Tr}K_\Delta^2(\psi)K_\Delta(\zeta)$ in R_3 and we have:

$$\begin{aligned} <\chi_S R_3> &= -\frac{|\Delta|}{\sqrt{N}}| \left(c_1\beta(\sum_x \zeta_x) + \frac{c_2}{\sqrt{\beta|\Delta|}}(\sum [G_\Delta^{(0)}]^{-1/2}\frac{s}{T}) \right) \\ &\quad -\frac{\sqrt{|\Delta|}}{\sqrt{2\beta}N}(\sum [G_\Delta^{(0)}]^{-1/2}\frac{s}{T}) - \frac{1}{\sqrt{N}}\mathrm{Tr}G_\Delta^{(0)}\zeta + (\text{smaller terms}) \end{aligned} \tag{3.38}$$

where $c_i = 1 + O(|\Delta|^{-1})$ $(i = 1, 2)$ are positive constants. Moreover we have (see [8]) :

$$\sum_{x\in\Delta} \zeta_x = \frac{1}{\sqrt{N}}\sum_{x\in\Delta}\left([G_\Delta^{o2}]^{-1}\frac{s}{T}\right)(x) = \frac{1}{\sqrt{N}}\left(\sum_{x\in\partial\Delta}\frac{1}{\beta}\delta_{\partial\Delta}(x)\frac{s_x}{T_x} + O(\beta^{-3})\right) \tag{3.39}$$

$$\delta_{\partial\Delta}(x) = O\left(\frac{1}{\beta\sqrt{\Delta}}\right) \geq 0 \tag{3.40}$$

and

$$\begin{aligned} &\mathrm{Tr}G_\Delta^{(0)}\zeta = \beta\sum \zeta_x - \mathrm{Tr}e_0P_0\zeta \\ &= O\left(\frac{\log\Delta|}{|\Delta|}\right)(\sum \zeta_x) - (\beta - \frac{\sigma_0}{|\Delta|})(\sum \zeta_x) + \frac{1}{2\beta\sqrt{N}|\Delta|}\sum \frac{s_x}{T_x} \end{aligned}$$

Then the largest contribution comes from $<\chi_S R_3>$ and is negative, and other contributions can be made less than $\frac{1}{N\beta}\sum s_x/T_x$

3.3.2 large ψ region

For $\{\tilde{\psi}\} \notin \mathcal{S}_\Delta$, we start with

$$\begin{aligned} I_{S^c} &= \frac{1}{\tilde{Z}_\Delta^0}\int_{S^c} \det{}^{-N/2}(1 + i\kappa K_\Delta(\psi - ih + i\zeta))e^{i\sqrt{N}\beta\sum_x(\psi_x - ih_x + i\zeta_x)} \\ &\quad \times \exp\left[-\frac{2}{N} < \frac{s}{T}, [G_\Delta^{o2}]^{-1}\frac{s}{T} > +2\kappa < h, \frac{s}{T} > +i\kappa < \psi, \frac{s}{T} >\right]\prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \\ &= \frac{1}{\tilde{Z}_\Delta^0}\int_{S^c} \det{}^{-N/2}(1 + i\kappa K_\Delta(\psi))e^{i\sqrt{N}\beta\sum_x \psi_x} \qquad (3.41) \\ &\quad \times \det{}^{-N/2}(1 + \kappa J_\Delta(h - \zeta))e^{\sqrt{N}\beta\sum_x(h_x - \zeta_x)} \\ &\quad \times \exp\left[-\frac{2}{N} < \frac{s}{T}, [G_\Delta^{o2}]^{-1}\frac{s}{T} > +2\kappa < h, \frac{s}{T} > +i\kappa < \psi, \frac{s}{T} >\right]\prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} \qquad (3.42) \end{aligned}$$

where

$$J_\Delta(h-\zeta) = \frac{1}{\sqrt{1+i\kappa K_\Delta(\psi)}} K_\Delta(h-\zeta) \frac{1}{\sqrt{1+i\kappa K_\Delta(\psi)}} \tag{3.43}$$

and then

$$\begin{aligned}
&\det{}^{-N/2}(1+\kappa J_\Delta(h-\zeta)) e^{\sqrt{N\beta}\sum_x (h_x - \zeta_x)} \\
&\times \exp\left[-\frac{2}{N} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1}\frac{s}{T} > +2\kappa < h, \frac{s}{T} > +i\kappa < \psi, \frac{s}{T} >\right] \\
&= \det{}_3^{-N/2}(1+\kappa J_\Delta(h-\zeta)) \\
&\times \exp\left[-\frac{1}{N} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1}\frac{s}{T} > +\kappa < h, \frac{s}{T} > +i\kappa < \psi, \frac{s}{T} > + < h, [G_\Delta^{\circ 2}]h >\right] \\
&\times \exp\left[\mathrm{Tr}\left(\frac{2iK_\Delta(\psi)}{1+i\kappa K_{\Delta(\psi)}}\right) K_\Delta(h-\zeta) + \mathrm{Tr}(J_\Delta^2(h-\zeta) - K_\Delta^2(h-\zeta))\right]
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\,\mathrm{Tr} J_\Delta^2(h-\zeta) &\leqq \mathrm{Tr} K_\Delta^2(h-\zeta) \\
&= \frac{1}{N} < \frac{s}{T}, [G_\Delta^{\circ 2}]^{-1}\frac{s}{T} > -\kappa < h, \frac{s}{T} > + < h, [G_\Delta^{\circ 2}]h >
\end{aligned}$$

Then putting $\mathcal{S}_\Delta^c = \cup_{k=1}^\infty S_k$ where

$$S_k = \{\{\tilde{\psi}_x\}; kN^{1-2\varepsilon} \leqq \sum \tilde{\psi}^2 \leq (k+1)N^{1-2\varepsilon}\}$$

we estimate the integral on each shell of $\mathcal{S}^c$:

$$\int_{S_k} \left|\det{}^{-N/2}(1+i\kappa K_\Delta(\psi))\right| \prod \frac{d\tilde{\psi}}{\sqrt{2\pi}} \leqq \frac{(|\Delta|-2)!!}{(2\pi)^{|\Delta|/2}} \frac{(kN^{1-2\varepsilon})^{(|\Delta|-1)/2}}{(1+2kN^{-2\varepsilon})^{N/4}}$$

3.3.3 integration over small-smooth s_x

It remains to integrate over $\{s_x = n_x + \sqrt{n_x}\tilde{s}_x\}$ such that $0 < s_x < N\beta$ and $|s_x - s_{x\pm\mu_i}| < \sqrt{N\beta}$. Sine the contribution from I_{S^c} is negligible, we can apply the previous methods of analysis: we set

$$\begin{aligned}
d\nu_n(s) &= \frac{1}{(n-1)!} e^{-n-\sqrt{n}\tilde{s}} (n+\sqrt{n}\tilde{s})^{n-1}\sqrt{n}d\tilde{s} \\
&= \frac{e^{-n}n^n}{(n-1)!\sqrt{n}} \exp[-\sqrt{n}\tilde{s} + (n-1)\log(1+\tilde{s}/\sqrt{n})]d\tilde{s} \\
&= \exp[-\frac{1}{2}\tilde{s}^2 + O(\tilde{s}/\sqrt{n})]\frac{d\tilde{s}}{\sqrt{\pi}}
\end{aligned}$$

We note that $K_\Delta(x,y) \sim \frac{1}{\sqrt{2|\Delta|}}(\tilde{\psi}(x) + \tilde{\psi}(y))$ is not of short range, though $K_\Delta(x,y) = O(|\Delta|^{-1/2})$. This long range nature of the interaction is expected compensated by the Anderson localization like phenomena.

3.4 Large field Region of s_x

For $\{s_x;\ s_x > N\beta, \exists x \in \Delta\}$ or for $\{s_x; |s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1\}$, we need a priori bound to estimate the two-point function. Continuing the argument in the previous section (3.1), we start from

$$D_\Delta(n)\lceil_{C_0}$$
$$= \frac{1}{Z_\Delta}\int \frac{\exp[- < \psi - ih, G_\Delta^{\circ 2}(\psi - ih) >]}{[\prod_{x\in\Delta}(4 + m^2 + \kappa(i\psi_x + h_x))^{n_x}]\det_3^{N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta))}\prod_{x\in\Delta} d\psi_x \tag{3.44}$$

$$= \frac{1}{Z_\Delta^{(0)}}\frac{1}{\prod_x T_x^{n_x}}\int_0^\infty \prod_{x\in\Delta} d\nu_{n_x}(s_x)\int_{-\infty}^\infty \prod_{x\in\Delta}\left(\sum_k \chi_x^{(k)}(s_x)\right) d\psi_x$$
$$\times \exp\left[- < (\psi - ih + i\zeta), G_\Delta^{\circ 2}(\psi - ih + i\zeta) > -\frac{1}{N} < \frac{1}{T}s, [G_\Delta^{\circ 2}]^{-1}\frac{1}{T}s > +\kappa < h, \frac{s}{T} >\right]$$
$$\times \det{}_3^{-N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta)) \tag{3.45}$$

We choose $h_x = c_x/(\beta\sqrt{N})$. Then

$$< h, G_\Delta^{\circ 2} h > \le \frac{(\sum c_x)^2}{N} \le \frac{|\Delta|^2}{N} \tag{3.46}$$

and we see

$$\left|\prod \frac{1}{(4 + m^2 + i\kappa(\psi_x - ih_x))^{n_x}}\right| \le \left(\frac{1}{4 + m^2 + \frac{c_x}{\beta N}}\right)^{\sum n_x} \tag{3.47}$$

Then if $\sum n(x)$ is so large that $\sum n(x)h(x)/\sqrt{N} > |\Delta|^2/N$, namely if $\sum n_x > \beta|\Delta|^2$, we easily see that the following a priori bound holds

$$D_\Delta(n) \le \left(\frac{1}{4 + m_{eff}^2}\right)^{\sum_{x\in\Delta} n(x)} \tag{3.48}$$

$$m_{eff}^2 = m^2 + \alpha^2, \quad \alpha^2 \equiv \frac{c}{N\beta} \tag{3.49}$$

Therefore in the following discussion, we assume that $\sum_{x\in\Delta} n_x \leqq \beta|\Delta|^2$ and $\{s_x = n_x + \sqrt{n_x}\tilde{n}_x, x \in \Delta\}$ satisfy

(1) $\quad s_x \geqq N\beta, \exists x \in \Delta,$ or

(2) $\quad |s_x - s_{x'}| > \sqrt{N\beta},\ \exists x \in \Delta,\ \exists x' \in \Delta,\ |x - x'| = 1$

If (1) occurs, then the factor

$$d_n(s) = \frac{s^n}{(n-1)!}e^{-s}ds$$

restricted on this region yields a small coefficient less than

$$\exp[-\frac{1}{2}N\beta] \leqq \exp[-\frac{\sum n_x}{N\beta}]$$

If (1) does not take place and (2) happens, then we can implement the complex deformation $\psi_x \rightarrow \psi_x + i\tau\zeta_x$, where $\zeta = (N)^{-1/2}[G_\Delta^{o2}]^{-1}(s/T)$ and $0 < \tau \leqq 1$, and we see that the following factor arises from the complex deformation:

$$\begin{aligned}\exp[-\frac{1-(1-\tau)^2}{N} < \frac{s}{T}, [G_\Delta^{o2}]^{-1}\frac{s}{T} >] &\leqq \exp[-\frac{1-(1-\tau)^2}{2N\beta} < \frac{s}{T}, (-\Delta)\frac{s}{T} >] \\ &= \exp[-\frac{1-(1-\tau)^2}{2NT\beta}\sum_{nn}(s_x - s_{x'})^2] \quad (3.50)\end{aligned}$$

On the other hand, since

$$||\kappa K_\Delta(\tau\zeta)||_2^2 = \frac{4\tau^2}{N^2} < \frac{s}{T}, [G_\Delta^{o2}]^{-1}\frac{s}{T} > \quad (3.51)$$

we have the bound

$$\left|\det{}_3^{-N/2}(1 + \kappa K_\Delta(\tau\zeta))\right| \leqq \exp\left[O\left(\frac{1}{\sqrt{N}}\right) ||K_\Delta(\tau\zeta)||_2^2\right] \quad (3.52)$$

which is close to 1 and has no effects on the bound (3.50) if N is large.

4 Conclusions and Discussions

We have shown that if the non-local factor

$$\prod_i \det{}^{-N/2}(1 + W(\Delta_i, \Lambda_i))$$

are discarded, then the resultant system exhibits exponential clustering for all β if N is large enough:

$$\begin{aligned}< s_0 s_x > &\sim \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(0, x)\prod d\mu_\Delta(\psi_\Delta), \quad (4.1) \\ &\leq \exp[-m_{eff}|x|] \quad (4.2)\end{aligned}$$

where $m_{eff}^2 = m^2 + c(N\beta)^{-1}$ and

$$d\mu_\Delta(\psi_\Delta) = \det{}_3^{-N/2}(1 + i\kappa K_\Delta)d\nu_\Delta \quad (4.3)$$

is the complex measure localized to each block Δ of size $L \times L$ in Z^2. The assumption $N >> 1$ is to simplify the large field problem and could be removed by additional efforts. The smallness of $W(\Delta, \Lambda)$ is due to the Anderson localization type arguments which remains to be justified [8, 9].

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