Fourier Transformation of 2D $O(N)$ Spin Model and Anderson Localization

K. R. Ito *
Department of Mathematics and Physics, Setsunan University,
Neyagawa, Osaka 572-8508, Japan

F. Hiroshima †
Department of Mathematics, Kyushu University,
Hakozaki 6-10-1, Fukuoka, 812-8581, Japan

H. Tamura ‡
Department of Mathematics, Kanazawa University,
Kanazawa 920-1192, Japan

February 10, 2006

Abstract

We Fourier transform the 2D $O(N)$ spin model $N > 2$, and start with a representation of the correlation functions in terms of integrals by complex random fields. Since this integral is complicated, we use the idea of the Anderson localization to discard non-local terms which make the integrals difficult. Through this approximation, we obtain the correlation functions which decay exponentially fast for all $\beta > 0$ if $N >> 3$.

1 Introduction: Result and Motivation

It is a longstanding problem to prove or disprove non-existence of phase transitions in 4 dimensional non-Abelian lattice gauge theories. In many points, this is similar to the same problem in the two-dimensional $O(N)$ symmetric spin models (Heisenberg or $\sigma$ model) with $N \geq 3$.

In models such as $O(N)$ spin modes and $SU(N)$ lattice gauge models [13, 17], the field variables form compact manifolds and the block spin transformations break the original

*Email: ito@mpg.setsunan.ac.jp
†Email: hiroshima@math.kyushu-u.ac.jp
‡Email: tamurah@kenroku.kanazawa-u.ac.jp
structures. In some cases, this can be avoided by introducing an auxiliary field $\psi$ [3] which can be regarded as complex random field. The $\nu$ dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$<f> \equiv \frac{1}{Z_{\Lambda}(\beta)} \int f(\phi) \exp[-H_{\Lambda}(\phi)] \prod_{i \in \Lambda} \delta(\phi_{i}^{2} - N\beta) d\phi_{i} \quad (1.1)$$

Here $\Lambda$ is an arbitrarily large square with the center at the origin, $\phi(x) = (\phi(x)^{(1)}, \ldots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$ and $Z_{\Lambda}$ is the partition function defined so that $<1> = 1$

The Hamiltonian $H_{\Lambda}$ is given by

$$H_{\Lambda} = -\frac{1}{2} \sum_{|x-y|=1} \phi(x)\phi(y), \quad (1.2)$$

where $|x|_{1} = \sum_{i=1}^{\nu} |x_{i}|$.

We substitute the identity $\delta(\phi^{2} - N\beta) = \int \exp[-ia(\phi^{2} - N\beta)] da/2\pi$ into eq.(1.1) with the condition that $\text{Im} a_{i} < -\nu$ [3], and set

$$\text{Im} a_{i} = -(\nu + \frac{m^{2}}{2}), \quad \text{Re} a_{i} = \frac{1}{\sqrt{N}} \psi_{i} \quad (1.3)$$

where $m > 0$ will be determined soon. Thus we have

$$Z_{\Lambda} = c^{|\Lambda|} \int \cdots \int \exp[-\frac{1}{2} \phi_{0}(m^{2} - \Delta + \frac{2i}{\sqrt{N}} \psi)\phi_{0} + \sum_{j} i\sqrt{N} \beta \psi_{j}] \prod_{j} \frac{d\phi_{j} d\psi_{j}}{2\pi} \quad (1.4)$$

where $c$'s are constants being different on lines, $\Delta_{ij} = -2\nu \delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian,

$$F(\psi) = \det(1 + i\kappa G \psi)^{-N/2} \exp[i\sqrt{N} \beta \sum_{j} \psi_{j}], \quad (1.5)$$

and $\kappa = 2/\sqrt{N}$. Moreover $G = (m^{2} - \Delta)^{-1}$ is the covariant matrix discussed later. In the same way, the two-point function is given by

$$<\phi_{0}\phi_{x}> = \frac{1}{Z} \int \cdots \int \left(\frac{1}{m^{2} - \Delta + i\kappa \psi}\right) (0,x) F(\psi) \prod_{j} \frac{d\psi_{j}}{2\pi} \quad (1.6)$$

namely by an average of the Green's function which includes complex fields $\psi(x)$, $x \in Z^{2}$, where the constant $Z$ is chosen so that $<\phi_{0}^{2}> = N\beta$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where

$$G(x) = \int \frac{e^{ipx}}{m^{2} + 2 \sum_{i=1}^{\nu} (1 - \cos p_{i})} \prod_{i=1}^{\nu} \frac{dp_{i}}{2\pi} \quad (1.7)$$
This is possible for any $\beta$ if $\nu \leq 2$, and we easily find that
\[ m^2 \sim 32e^{-4\pi\beta} \text{ for } \nu = 2 \] (1.8)
as $\beta \to \infty$. Thus for $\nu = 2$, we can rewrite
\[ F(\psi) = \det_3^{-N/2}(1 + i\kappa G\psi) \exp[- <\psi, G^{\nu^2}\psi>], \] (1.9)
\[ \det_3(1 + A) \equiv \det[(1 + A)e^{-A + A^2/2}] \] (1.10)
where $G^{\nu^2}(x, y) = G(x, y)^2$ so that $\text{Tr}(G\psi)^2 = <\psi, G^{\nu^2}\psi>$. For any two matrices $A$ and $B$ of equal size, the Hadamard product [18] $A \circ B$ is defined by $(A \circ B)_{ij} = A_{ij}B_{ij}$ and we denote $G \circ G$ by $G^{\nu^2}$.

Decompose $\Lambda \subset Z^2$ into small blocks $\Delta_i$, and define $G_{\Lambda} = \chi_{\Lambda}G\chi_{\Lambda}$:
\[ \Lambda = \bigcup_{i=1}^{n} \Delta_i, \]

Then we use the Feshbach-Krein formula (blockwise diagonalizations of matrices), to decompose $\det(1 + i\kappa G_{\Lambda}\psi_{\Lambda})$ into a product of $\det(1 + i\kappa G_{\Delta}\psi_{\Delta})$ as follows:
\[ \det^{-N/2}(1 + i\kappa G_{\Lambda}\psi_{\Lambda}) = \prod_{i=1}^{n-1} \det^{-N/2}(1 + W(\Delta_i, \Lambda_i)) \prod_{i=1}^{n} \det^{-N/2}(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}). \] (1.11)
where $\kappa = 2/\sqrt{N}$, $\Lambda_i = \bigcup_{k=1+i\kappa}^{n} \Delta_i$,
\[ W(\Delta_i, \Lambda_i) = -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}} \frac{1}{1 + i\kappa G_{\Lambda_i}\psi_{\Lambda_i}} G_{\Lambda_i, \Delta_i}\psi_{\Delta_i} \] (1.12)
\[ = - (i\kappa)^2 \frac{1}{[G_{\Delta_i}]^{-1} + i\kappa G_{\Lambda_i}[G_{\Delta_i}]^{-1} + i\kappa G_{\Lambda_i}} [G_{\Delta_i}]^{-1} G_{\Lambda_i, \Delta_i}\psi_{\Delta_i} \] (1.13)
and $[G_{\Delta}]^{-1}$ is the Laplacian with free boundary condition and almost equal to the Laplacian restricted to the square $\Delta$ with no boundary. Thus $\inf \text{spec } [G_{\Delta}]^{-1} \sim 0$ and we can prove that $([G_{\Lambda}]^{-1} + i\kappa G_{\Lambda_i})^{-1}$ behaves like a massive Green's function which decreases fast since $\psi$ behaves like a Gaussian random variable of zero mean and covariance $[G^{\nu^2}]^{-1}$.

Let us consider the measure localized on each block $\Delta$:
\[ d\mu_{\Delta} = \frac{1}{Z_{\Delta}} \det_3^{-N/2}(1 + i\kappa G_{\Delta}\psi_{\Delta}) \exp[-(\psi_{\Delta}, G^{\nu^2}_{\Delta}\psi_{\Delta})] \prod_{x \in \Delta} d\psi(x) \] (1.14)
where $Z_{\Delta}$ is chosen so that $\int d\mu_{\Delta} = 1$. Since the norm of $G_{\Delta}$ is of order $O(|\Delta|\beta) >> 1$, one may think that it is still impossible to expand the determinant. However, this comes with the factor $\exp[-(\psi_{\Delta}, G^{\nu^2}_{\Delta}\psi_{\Delta})]$, which makes the norm of $\frac{\Delta}{\sqrt{N}} G_{\Delta}\psi_{\Delta}$ small. To see if this is the case, we introduce new variables $\tilde{\psi}_{\Delta}(x)$ by
\[ \psi_{\Delta}(x) = \frac{1}{\sqrt{2}} \sum_{y \in \Delta} \hat{G}_{\Delta}^{-1}(x, y) \tilde{\psi}(y), \quad \hat{G}_{\Delta} = [G^{\nu^2}_{\Delta}]^{1/2} \] (1.15)
so that $d\mu_\Delta$ is rewritten

$$
d\mu_\Delta = \frac{1}{Z_\Delta} \det^{-N/2}_3(1 + i\kappa K_\Delta) \prod_{x \in \Delta} \exp[-i^\Delta_\psi(x)^2] \frac{d\tilde{\psi}(x)}{\sqrt{2\pi}},
$$

(1.16)

$$
K_\Delta = \frac{1}{\sqrt{2}} G_\Delta^{1/2}(\hat{G}_\Delta^{-1}\tilde{\psi})G_\Delta^{1/2}
$$

(1.17)

Put

$$
d\nu_\Delta^{(0)} = \prod \exp[-\frac{1}{2}\tilde{\psi}^2(x)] \frac{d\tilde{\psi}(x)}{\sqrt{2\pi}}
$$

(1.18)

$$
= \det^{-1/2}[G_\Delta^{02}] \exp[-<\psi_\Delta, G_\Delta^{02}\psi_\Delta>] \prod_{x \in \Delta} \frac{d\psi(x)}{\sqrt{2\pi}}
$$

(1.19)

and define

$$
||K||_p = \left( \int \text{Tr}(K^*K)^{p/2}d\nu_\Delta \right)^{1/p}
$$

(1.20)

**Lemma 1** It holds that

$$
\int \text{Tr}K_\Delta^2 d\nu_\Delta = \frac{1}{2}|\Delta|,
$$

(1.21)

$$
||K_\Delta||_p \leq (p-1)||K_\Delta||_2, \quad \text{for all } p \geq 2
$$

(1.22)

**Proof.** The first equation is immediate. See [16] for the second inequality. Q.E.D.

Thus we see that $\kappa K_\Delta$ are a.e. bounded with respect to $d\nu_\Delta$, and converges to 0 as $N \to \infty$. To see to what extent $K_\Delta$ is diagonal, we estimate

$$
\int \text{Tr}K_\Delta^4 d\nu_\Delta = \sum_{x_i \in \Delta} \frac{1}{4 \beta} G_\Delta(x_i, x_{i+1})
\times \left[ 2[G^{02}]^{-1}(x_1, x_2)[G^{02}]^{-1}(x_3, x_4) + [G^{02}]^{-1}(x_1, x_3)[G^{02}]^{-1}(x_2, x_4) \right]
$$

where $x_5 = x_1$. As is proved in [8]

$$
[G^{02}_\Delta]^{-1}(x, y) = \frac{1}{2\beta} G_\Delta^{-1} - \hat{B}_\Delta, \quad \hat{B}_\Delta(x, y) = O(\beta^{-2})
$$

(1.23)

The main contribution comes from the term containing $2[G^{02}]^{-1}(x_1, x_2) \cdots$. To bound this, set $G_\Delta(x_i, x_{i+1}) = \beta - \delta G(x_i, x_{i+1}).$ Then $\delta G(x, x) = 0$, $\delta G(x, x + e_\mu) = 0.25 - O(\beta m^2)$, $(-\Delta)_{xy} = 0$ unless $|x - y| \leq 1$, and we have

$$
\int \text{Tr}K_\Delta^4 d\nu_\Delta \geq \text{const.} \sum_{x_i \in \Delta} \frac{1}{4\beta^2} \left\{ \beta^2 \sum_{x_1, x_2} \delta_{x_1, x_4} + \sum_{x_4} G^2(x_1, x_4) \right\}
$$

$$
\geq \text{const.}(|\Delta| + |\Delta|^2)
$$
which means that $K$ is approximately diagonal but off-diagonal parts are still considerably large. However, there is a reason to believe that $W$ functions are of short range and small. In fact we know that

$$ \left| \frac{1}{[G_{\Lambda}]^{-1} + i\kappa\psi_{\Lambda}}(x, y) \right| \leq \frac{1}{[G_{\Lambda}]^{-1} + c(N\beta)^{-1}}(x, y) $$

for almost all $\psi$. Then $(G_{\Lambda}^{-1} + m^2 + i\kappa\psi_{\Lambda})^{-1}(x, y)$ is negligible if $|x - y| > \sqrt{N\beta}$. Moreover it is shown in two dimension that

$$ \int \frac{1}{[G_{\Lambda}]^{-1} + i\kappa\psi_{\Lambda}}(x, y) d\mu \leq \frac{1}{[G_{\Lambda}]^{-1} + m_{eff}^2}(x, y), $$

$$ m_{eff}^2 = c \frac{\log(N\beta)}{N\beta} $$

if $d\mu(\psi)$ is Gaussian of mean zero and covariance $[G^{02}]^{-1}$. This logarithmic correction comes from the two-dimensionality. This implies that

$$ \lim_{N\beta \to \infty} \frac{1}{N\beta} \sum_{x} \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(0, x) d\mu = 0 $$

Furthermore $\psi$ in the numerators of $W$ acts as a differential operators since

$$ \psi = \frac{1}{\sqrt{2}} [G_{\Delta}^{02}]^{-1/2} \tilde{\psi} \sim \frac{1}{2\sqrt{\beta}} [G_{\Delta}]^{-1/2} \tilde{\psi} $$

Thus $W(\Delta, \Lambda)$ seems to be small as $N\beta \to \infty$.

We choose $N$ larger than $|\Delta| = L^2$, i.e.,

$$ N^{1/3-\epsilon} \geq |\Delta| = L^2 \quad (1.24) $$

This assumption is artificial and its role is to simplify the large field problem to bound the integrals in the region where $|\psi_\lambda|$ are large. So more elaborate idea may remove this condition (it is natural to think that $N \geq 3$ is enough).

To imagine that the non-diagonal terms $W$ are small, we perhaps choose $L$ larger than some power of $\beta$, say $L > (\beta)^{1+\delta}$, $\delta > 0$, but we do not know how to determine it yet though it is now under investigation, see [9].

**Assumption:** We take $N$ larger than $|\Delta| = L^2$ as above, and for sufficiently large $\Delta$, non-local terms $W$ are negligible in this case.

Once $W$ is neglected and $N$ is chosen larger than $|\Delta|$, we can prove the following result uniformly in $\beta$:

**Main Theorem:** Assume $N$ is sufficiently large: $N^{1/3-\epsilon} \geq |\Delta|$. Neglect non-local terms $W(\Delta, \Lambda)$. Then the two point correlation function

$$ \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(x, y) \prod_{\Delta} d\mu_{\Delta} $$

decays exponentially fast for all $\beta \geq 0$. 

209
2 Averaged Green’s Function by the measure $d\mu_0$

Let us come back to the present case where $\Delta_i$ are boxes of equal size $L \times L$ ($L \geq 2$) such that $\cup_i \Delta_i = \mathbb{Z}^2$ and $\Delta_i \cap \Delta_j = \emptyset$, $i \neq j$. Let us estimate

$$G^{(\text{ave})}(x, y) \equiv \int G^{(\psi)}(x, y) d\mu(\psi)$$

(2.1)

where

$$G^{(\psi)}(x, y) \equiv \left( \frac{1}{G^{-1} + i\kappa\psi} \right)(x, y),$$

(2.2a)

$$d\mu(\psi) \equiv \prod \frac{1}{Z_{\Delta}} \det_{3}^{-N/2}(1 + i\kappa G_{\Delta}\psi_{\Delta}) d\nu_{\Delta},$$

(2.2b)

$$d\nu_{\Delta} = \frac{1}{\det(G^{\Delta}2)} \exp[-(\psi_{\Delta}, G_{\Delta}^{0}\psi_{\Delta})] \prod_{x \in \Delta} \frac{d\psi(x)}{\sqrt{2\pi}}.$$  

(2.2c)

and $\kappa = 2/\sqrt{N}$. Expanding $G^{(\psi)}$ by random walk, we have

$$G^{(\psi)}(x, y) = \sum_{\omega:x \to y} \prod_{\zeta \in \omega} \frac{1}{(4 + m^2 + i\kappa\psi_{\zeta})^{n_{\zeta}}}$$

(2.3)

where $n_{\zeta} \in \mathbb{N}$ is the visiting number of $\omega$ at $\zeta \in \mathbb{Z}^2$. We set

$$d\nu = \prod_{\Delta \subset \mathbb{Z}^2} d\nu_{\Delta}$$

(2.4)

We first prove our assertion for the Gaussian case:

**Theorem 2** The following bound holds:

$$\int G^{(\psi)}(x, y) d\nu \leq \frac{1}{-\Delta + m_{\text{eff}}^2}(x, y)$$

(2.5)

where

$$m_{\text{eff}}^2 = m^2 + \frac{c}{N\beta}$$

(2.6)

with a constant $c > 1$.

**Proof.** Let $\Delta$ be the square of size $L \times L$ centered at the origin, and let $n_x \in \{0, 1, 2, \cdots \}$, $x \in \Delta$. We estimate

$$D_{\Delta}(\{n\}) \equiv \int \prod_{x \in \Delta} \frac{1}{4 + m^2 + i\kappa\psi(x)n_x} d\nu_{\Delta}(\psi)$$

(2.7)

For large $\sum_{x \in \Delta} n_x$ such that

$$\sum_{x \in \Delta} n_x \geq \beta(\#\{x \in \Delta; n_x \neq 0\})^2$$

(2.8)
the bound follows by the complex translation estimate by putting \( \psi_x \to \psi_x - ih_x \), where

\[
h_x = \frac{c_x}{\beta \sqrt{N}}, \quad c_x = \begin{cases} 
    c > 0 & \text{if } n_x \geq 1 \\
    0 & \text{if } n_x = 0
\end{cases}
\]

(2.9)

In fact we have:

\[
D_\Delta(\{n\}) \leq \frac{e^{<h, G_\Delta^2 h>}}{\prod_{x \in \Delta} (4 + m^2 + \kappa h_x)^{n_x}} \leq \frac{e^{\beta^2 (\sum_{x \in \Delta} h_x)^2}}{(4 + m^2 + c(\beta N)^{-1})^{\sum n_x}} \leq \left( \frac{1}{4 + m^2 + c' (\beta N)^{-1}} \right)^{\sum n_x}
\]

(2.10)

with a constant \( 0 < c' < c \).

For small \( \{n_x; x \in \Delta\} \), we start with the new expression of \( D_\Delta(\{n\}) \):

\[
\prod_{x \in \Delta} \frac{1}{(n_x - 1)!} \int_0^\infty \prod s_x^{n_x - 1} \exp\left[-(4 + m^2) \sum s_x - \frac{\kappa^2}{4} < s_\Delta, [G_\Delta^2]^{-1} s_\Delta >\right] \prod ds_x = \prod_{x \in \Delta} T^{-n(x)} \int \exp\left[-\frac{1}{NT^2} < s_\Delta, [G_\Delta^0]^{-1} s_\Delta >\right] \prod_x d\nu_{n_x}(s_x)
\]

(2.11)

where \( T = 4 + m^2 \) and

\[
d\nu_n(s) = \frac{s^{n-1}e^{-s}}{(n-1)!}ds
\]

(2.12)

Since \( \int d\nu_n(s) = 1 \) and \( n \log s - s \) takes its maximum at \( s = n \), we set \( s_x = n_x + \sqrt{n_x} \delta_x \) \((x \in \Delta)\) and note that

\[
d\nu_n(s) = \exp\left[-\frac{1}{2} \bar{s}^2\right] \frac{e^{\delta_n(\bar{s})}}{\sqrt{2\pi}}, \quad \delta_n(\bar{s}) = -\sqrt{n\bar{s}} + (n - 1) \log(1 + \frac{\bar{s}}{\sqrt{n}}) + \frac{1}{2} \bar{s}^2 \quad \text{and} \quad \delta_n(\bar{s}) = -\log(1 + \frac{\bar{s}}{\sqrt{n}}) + O\left(\bar{s}^3/\sqrt{n}\right),
\]

(2.13)

\[
< e^{\delta_n(\bar{s})} > = \int_{-\infty}^{\infty} e^{\delta_n(\bar{s})} e^{-\bar{s}^2/2} \frac{d\bar{s}}{\sqrt{2\pi}} = 1 + O(1/n) > 1
\]

(2.14)

(2.15)

Put

\[
\alpha^2 = \frac{1}{N\beta}
\]

(2.16)

Then if \( \alpha^2 n(x) < 1 \) and \( N^{-1} < n, [G_\Delta^2]^{-1} n > \) is small, the integral (2.11) is carried out by perturbative calculations.
For large $\alpha^2 n(x) \geqq 1$ or for non-smooth $n$ such that $N^{-1} < n, [G_{\Delta}^2]^{-1} n >> 1$ we use a priori bound. See [8].

In the case where $d\mu$ is Gaussian, we can obtain $G^{(\text{ave})}$ in a closed form. See [8] where $m^2_{\text{eff}} \sim \log(N\beta)/N\beta$ is obtained.

**Remark 1** We note that this is similar to the pinch singularity encountered in the study of the Anderson localization [5], where

$$
\int G(E + i\epsilon, v)(x, y)dP(v)
$$

has a convergent random walk expansion, and

$$
\int |G(E + i\epsilon, v)(x, y)|^2 dP(v)
$$

does not have.

## 3 Averaged Green's Function by the measure $d\mu(\psi)$

It remains to discuss the effects of the determinants $\det^{-N/2}_3(1 + \cdots)$. Set

$$
S_{\Delta} = \{\psi_x; x \in \Delta, \text{Tr}K_{\Delta}^2 < N^{1-2\varepsilon}\},
$$

$$
K_{\Delta} = G_{\Delta}^{1/2}\psi_{\Delta}G_{\Delta}^{1/2}
$$

(3.1)

(3.2)

Since

$$
\exp[-\text{Tr}K_{\Delta}^2] \leq \left|\det_2^{-N/2}(1 + i\kappa K_{\Delta})\right| \leq \left(1 + \frac{4}{N}\text{Tr}K_{\Delta}^2\right)^{-N/4}
$$

(3.3)

and $\text{Tr}K_{\Delta}^2 = \sum \tilde{\psi}_x^2/2$, we have

$$
\int \exp[-\text{Tr}K_{\Delta}^2] \prod_x \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} = \int \exp[-\sum_x \frac{1}{2}\tilde{\psi}(x)^2] \prod_x \frac{d\tilde{\psi}_x}{\sqrt{2\pi}} = 1
$$

(3.4)

and $\int (1 + \frac{2}{N}\sum \tilde{\psi}^2(x))^{-N/4} \prod_{x \in \Delta} d\tilde{\psi}_x$ is convergent for $2|\Delta| < N$. Even so, it is obvious that $|\det^{-N/2}(1 + i\kappa G_{\Delta}\psi)|$ is integrable if and only if $N > 2$ since

$$
\det^{-N/2}(1 + i\kappa G_{\Delta}\psi) = \det^{-N/2}(G_{\Delta}) \det^{-N/2}(G_{\Delta}^{-1} + i\kappa\psi)
$$

$$
\sim \det^{-N/2}(G_{\Delta}) \prod_{x \in \Delta} \left(\frac{1}{4 + m^2 + i\kappa\psi(x)}\right)^{N/2}
$$

(3.5)

(3.6)

holds for $\psi$ such that $|\psi/\sqrt{N}| > O(1)$
3.1 Small Fields, Large Fields and Complex Displacements

Let us estimate

\[
D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} \psi_{x}}}{\prod_{x \in \Delta} (4 + m^2 + i\kappa \psi_{x})^{n_{x}} \cdot \det N/2(1 + i\kappa G_{\Delta} \psi_{\Delta})} \prod_{x \in \Delta} d\psi_{x},
\]

(3.7)

\[
Z_{\Delta} = \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} \psi_{x}}}{\det N/2(1 + i\kappa G_{\Delta} \psi_{\Delta})} \prod_{x \in \Delta} d\psi_{x}
\]

(3.8)

by putting \( \psi_{x} \to \psi_{x} - ih_{x}, \psi_{x} \in \mathbb{R} \). Then

\[
D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} (i\psi_{x} + h_{x})}}{\prod_{x \in \Delta} (4 + m^2 + \kappa G_{\Delta}^{1/2}(i\psi_{x} + h_{x})G_{\Delta}^{1/2})} \prod_{x \in \Delta} d\psi_{x}
\]

(3.9)

where

\[
K_{\Delta}(\psi_{\Delta}) \equiv G_{\Delta}^{1/2} \psi_{\Delta} G_{\Delta}^{1/2} \quad \text{and} \quad K_{\Delta}(h_{\Delta}) \equiv G_{\Delta}^{1/2} h_{\Delta} G_{\Delta}^{1/2}
\]

and \( K_{\Delta}(h_{\Delta}) \geq 0 \) since \( h_{x} \geq 0 \). We again put \( h_{x} = c_{x}/(\sqrt{N}\beta) \) and then

\[
\kappa K_{\Delta}(h_{\Delta}) \leq \frac{c|\Delta|}{N}, \quad c = O(1) > 0.
\]

(3.10)

We repeat the previous arguments by using \((n-1)!x^{-n} = \int_{0}^{\infty} s^{n-1}e^{-sx}ds\). Define \( I_{n}^{(k)} = \{ s; k\sqrt{n} < |s-n| < (k+1)\sqrt{n}, s \geq 0 \}, k = 0, 1, 2, \ldots \), and let \( \chi_{x}^{(k)}(s_{x}) \) be the characteristic function of the interval \( I_{n_{x}}^{(k)} \). Then

\[
D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int_{0}^{\infty} \prod_{x \in \Delta} \left( \sum_{k} \chi_{x}^{(k)}(s_{x}) \right) \frac{s_{x}^{n_{x}-1}ds_{x}}{(n_{x}-1)!} \int_{-\infty}^{\infty} \prod_{x \in \Delta} d\psi_{x}
\]

\[
\times \exp \left\{ -\sum_{x} (4 + m^2 + \kappa(i\psi(x) + h(x)))s(x) \right\} \frac{e^{-<\psi_{\Delta} - ih_{\Delta}, G_{\Delta}^{g2}(\psi_{\Delta} - ih_{\Delta})>}}{\det \frac{3}{N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))}
\]

(3.11)

\[
= \frac{1}{Z_{\Delta}^{(0)}} \frac{1}{\prod_{x} T_{x}^{n_{x}}} \int_{0}^{\infty} \prod_{x \in \Delta} d\nu_{n_{x}}(s_{x}) \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left( \sum_{k} \chi_{x}^{(k)}(s_{x}) \right) \prod_{x \in \Delta} d\psi_{x}
\]

\[
\times \exp \left[ -<\psi - ih + i\zeta, G_{\Delta}^{g2}(\psi - ih + i\zeta) > - \frac{1}{N} s, [G_{\Delta}^{g2}]^{-1} \frac{1}{T}s > + \kappa < h, \frac{s}{T} > \right]
\]

\[
\times \det \frac{3}{N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))
\]

(3.12)
where $T_x = 4 + m^2 + \kappa h_x$, $(s/T)_x = s_x/T_x$

$$d\nu_n(s) = \frac{1}{(n-1)!}e^{-\epsilon}s^{n-1}ds, \quad \zeta_x = \frac{\kappa}{2}([G_{\Delta}^{02}]^{-1} \frac{1}{T}s)(x)$$  \hspace{1cm} (3.13)

and

$$Z_{\Delta}^{(0)} = \int \exp[-<\psi - ih, G_{\Delta}^{02}(\psi - ih)>] \times \det_{3}^{-N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta})) \prod_{x\in\Delta}d\psi_x$$  \hspace{1cm} (3.14)

We then change the contour of $\psi_x$ by replacing $\psi_x + i\zeta_x$ by $\psi$ (namely we put $\psi_x \rightarrow \psi - i\zeta_x$). The contours depend on $\{s_x; x \in \Delta\}$. This yields

$$D_{\Delta}(n) = \frac{1}{Z_{\Delta}^{(0)}T^{n_x}_x} \int_{0}^{\infty} \prod_{x \in \Delta} d\nu_n(s_x) \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left( \sum_{k} \chi_{x}^{(k)}(s_x) \right) d\psi_x \times \exp\left[-<\psi - ih, G_{\Delta}^{02}(\psi - ih)> - \frac{1}{N} < \frac{1}{T}s, [G_{\Delta}^{02}]^{-1} \frac{1}{T}s > + \kappa < h, \frac{s}{T} > \right] \times \det_{3}^{-N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}) + \kappa K_{\Delta}(\zeta_{\Delta}))$$

$$= \frac{1}{\prod_{x}T_{x}^{n_{x}}} \int_{0}^{\infty} \prod_{x \in \Delta} d\nu_{n_{x}}(s_{x}) \exp\left[-\frac{1}{N} < \frac{1}{T}s, [G_{\Delta}^{02}]^{-1} \frac{1}{T}s > + \kappa < h, \frac{s}{T} > \right] \times \frac{-1}{Z_{\Delta}^{(0)}T^{n_x}_x} \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left( \sum_{k} \chi_{x_{x}}^{(k)}(s_{x}) \right) d\psi_x \det_{3}^{-N/2}(1 + i\kappa K_{\Delta}(\psi_{\Delta})) \exp\left[-<\psi, G_{\Delta}^{02}\psi>\right]$$

$$\times \det_{3}^{-N/2}(1 + \kappa J_{\Delta}(\zeta_{\Delta})) \times \exp[R_3]$$  \hspace{1cm} (3.15)

where

$$K_{\Delta}(\zeta_{\Delta})(x, y) = \frac{1}{\sqrt{N}} \sum_{\xi} G_{\Delta}^{1/2}(x, \xi) \left([G_{\Delta}^{02}]^{-1} \frac{1}{T}s\right)(\xi)G_{\Delta}^{1/2}(\xi, y),$$

$$J_{\Delta}(\zeta_{\Delta}) = \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi_{\Delta})}} \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi_{\Delta})}}$$

and

$$R_3 = \frac{N}{2} \text{Tr} \left[ \left( \frac{1}{1 + i\kappa K_{\Delta}(\psi_{\Delta})} - 1 + i\kappa K_{\Delta}(\psi_{\Delta}) \right) \kappa K_{\Delta}(\zeta_{\Delta}) + \kappa^2 \left( J_{\Delta}^{2} - K_{\Delta}(\zeta_{\Delta})^{2} \right) \right]  \hspace{1cm} (3.16)$$
3.2 \(K_{\Delta}(\psi_{\Delta}), K_{\Delta}(\zeta_{\Delta})\) and \(R_{3}\)

Let

\[
G_{\Delta} = \sum_{i=0}^{|\Delta|-1} e_{i} P_{i}, \quad G_{\Delta}^{\circ2} = \sum_{i=0}^{|\Delta|-1} \hat{e}_{i} \hat{P}_{i}
\]

be the spectral resolutions of the positive matrices \(G_{\Delta}\) and \(G_{\Delta}^{\circ2}\) respectively, where \(e_{0} \geq e_{1} \geq \cdots \geq e_{|\Delta|-1}, \ \hat{e}_{0} \geq \hat{e}_{1} \geq \cdots \geq \hat{e}_{|\Delta|-1}, \ \ P_{i} P_{j} = \delta_{i,j} P_{i}\) and so on. Then

\[
G_{\Delta}^{1/2} = \sum_{i=0}^{|\Delta|-1} \sqrt{e_{i}} P_{i}, \quad [G_{\Delta}^{\circ2}]^{-1} = \sum_{i=0}^{|\Delta|-1} \frac{1}{\hat{e}_{i}} \hat{P}_{i}
\]

It is convenient to introduce the abbreviation for the Green’s function with the largest eigenvalue part extracted:

\[
G_{\Delta}^{(0)} = \sum_{k \neq 0} e_{k} P_{k} = G_{\Delta} - e_{0}^{-1} P_{0}
\]

We let \(\{u_{i}\}_{i=0}^{|\Delta|-1}\) and \(\{\hat{u}_{i}\}_{i=0}^{|\Delta|-1}\) be the normalized eigenvectors such that

\[
G_{\Delta} u_{i} = e_{i} u_{i}, \quad G_{\Delta}^{\circ2} \hat{u}_{i} = \hat{e}_{i} \hat{u}_{i}
\]

Then

\[
P_{i} = |u_{i} > < u_{i}|, \quad \hat{P}_{i} = |\hat{u}_{i} > < \hat{u}_{i}|
\]

and for small \(\Delta\), we have

\[
e_{0} = |\Delta| \beta - O(1), \quad e_{i} = O(1) > 0 \quad (i \neq 0)
\]

\[
\hat{e}_{0} = |\Delta| \beta^{2} - O(\beta), \quad \hat{e}_{i} = 2 \beta e_{i} + O(1)
\]

\(i \neq 0\) and

\[
P_{0} \sim \hat{P}_{0} \sim \frac{1}{|\Delta|} |U > < U| = \frac{1}{|\Delta|} \begin{pmatrix}
1 & 1 & \cdots & 1 \\
1 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1
\end{pmatrix},
\]

where \(U = ^{t}(1,1,\cdots,1) \sim \sqrt{|\Delta|} u_{0}\). Moreover we can symbolically write

\[
P_{i} \sim \hat{P}_{i} \sim \frac{1}{2} \begin{pmatrix}
1 & -1 & 0 & \cdots & 0 \\
-1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]

(3.24)
namely $P_i (i \neq 0)$ is a matrix which represents a lattice differentiation since $<u_i, u_0> = 0$.  
Note that $e_i \leq O(\log|\Delta|), e_0 = \beta|\Delta| - O(|\Delta|\log|\Delta|)$ and

$$
(P_0 \zeta P_0)_{x,y} = \sum_\zeta \frac{1}{|\Delta|^2} \zeta \zeta = \left( \frac{1}{|\Delta|} \sum \zeta \zeta \right) P_0, \quad P_0 (\hat{P}_i \zeta) P_0 = O(\beta^{-1}) \quad (3.25)
$$

We insert $\psi = \hat{G}^{-1} \tilde{\psi}/\sqrt{2}$ into $K_\Delta$ and use $\hat{e}_i = 2\beta e_i + O(1) (i \neq 0), P_i = \hat{P}_i + O(\beta^{-1})$ and $\sum_{i\neq 0} P_i = 1 - P_0$ to find that

$$
K_\Delta = \frac{\sum \tilde{\psi}(x) P_0 + \sqrt{|\Delta|}}{\sqrt{2\Delta} (\sum_{i\neq 0} P_0(\hat{P}_i \tilde{\psi} + \tilde{\psi} P_0)) + O(\beta^{-1})}
$$

$$
K_\Delta^2 = \left[ \frac{1}{4} X + (1 - \sqrt{2}) Y^2 \right] P_0 + \frac{\sqrt{2} - 1}{4} \sqrt{|\Delta|} (\tilde{\psi} P_0 + P_0 \tilde{\psi}) + O(\beta^{-1})
$$

where

$$
X = \sum_{x \in \Delta} \tilde{\psi}_x^2, \quad Y = \frac{1}{\sqrt{|\Delta|}} \sum_{x \in \Delta} \tilde{\psi}_x
$$

(3.26)

Note that $\text{Tr} K_\Delta^2 = \sum \tilde{\psi}_x^2 / 2$ as expected. Just in the same way, we have

$$
K_\Delta(\zeta) = G_\Delta^{1/2} \zeta^2 G_\Delta^{1/2} = G_\Delta^{1/2} \left( \frac{1}{\sqrt{N}} \sum_k \frac{1}{\hat{e}_k} \hat{P}_k \frac{s}{T} \right) G_\Delta^{1/2}
$$

$$
= (\sum \zeta_x) \beta P_0 + \frac{1}{2} \left( \frac{|\Delta|}{\beta N} \right)^{1/2} \left( \left( G_\Delta^{(0)} \right)^{-1/2} \frac{s}{T} P_0 + P_0 \left( G_\Delta^{(0)} \right)^{-1/2} \frac{s}{T} \right) + \left( G_\Delta^{(0)} \right)^{1/2} \zeta^2 \left( G_\Delta^{(0)} \right)^{1/2}
$$

and

$$
K_\Delta(\zeta)^2
$$

$$
= \left[ \beta^2 (\sum \zeta_x)^2 + \frac{1}{4 \beta N} < \frac{s}{T}, G_\Delta^{-1/2} \frac{s}{T} > + \left( \frac{\beta}{N|\Delta|} \right)^{1/2} (\sum \zeta_x) (\sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) \right] P_0
$$

$$
+ \left[ \left( \frac{|\Delta|}{N} \right)^{1/2} (\sum \zeta_x) + \frac{1}{4 \beta N} (\sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) \right] \left( P_0 \left( G_\Delta^{(0)} \right)^{-1/2} \frac{s}{T} P_0 + \left( G_\Delta^{(0)} \right)^{-1/2} \frac{s}{T} P_0 \right)
$$

$$
+ \frac{\sqrt{|\Delta|}}{2 \sqrt{N}} \left( P_0 \left( \frac{s}{T} \circ \zeta \right) [G_\Delta^{(0)}]^{1/2} + [G_\Delta^{(0)}]^{1/2} \left( \frac{s}{T} \circ \zeta \right) P_0 \right)
$$

$$
+ \frac{|\Delta|}{4N\beta} \left( [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} P_0 \left( G_\Delta^{(0)} \right)^{-1/2} \frac{s}{T} \right) + \left( G_\Delta^{(0)} \right)^{1/2} \zeta^2 \left( G_\Delta^{(0)} \right)^{1/2}
$$
Here \( \zeta_x = N^{-1/2}(G^{(0)})^{-1/2}(s/T)\zeta_x \equiv x_k y_k \) for two vectors \( x \) and \( y \), and we have used \( P_0 P_i = 0 \) (\( i \neq 0 \)) and

\[
P_0(G^{1/2}_\Delta - \frac{s}{T})[G^{(0)}_\Delta]^{-1/2} = P_0(S \circ \zeta)[G^{(0)}_\Delta]^{-1/2}
\]

We can obtain similar expressions for \( K(\psi)^n \) etc., and \( R_3 \) is represented by these functions of \( \psi \) and \( \zeta \). We decompose our set \( \{s_x; s_x \geq 0, x \in \Delta\} \) into 2 regions:

1. small \( s \) region
2. large \( s \) region

and each region is also decomposed into large \( \psi \) region and small \( \psi \) region, where the small \( \psi \) field \( S_\Delta(\psi) \) means the set of \( \psi \) such that

\[
S_\Delta(\psi) = \{\psi_x = 1 \sqrt{2} \hat{G}_\Delta^{-1} \tilde{\psi}(x), \sum_{x \in \Delta} \tilde{\psi}_x^2 \leq N^{1-2\varepsilon}\}
\]

and small \( s \) field \( S_\Delta(s) \) means the set of \( s_x \) such that

\[
S_\Delta(s) = \left\{ s_x = n(x) + \sqrt{n(x)} \tilde{s}(x) \geq 0, \quad \frac{1}{N\beta} \sum_{n.n} (\frac{s_x}{T_x} - \frac{s_y}{T_y})^2 \leq O(1) \right\}
\]

(3.27)

### 3.3 Small field Region of \( s_x \)

For small smooth \( \{s_x\} \), we see that \( \det^{-N/2}_3(1 - \kappa J_\Delta(\psi)) \) yields a convergent small factor uniformly in \( \psi_x \). Put

\[
\det^{-N/2}_3(1 + \kappa J_\Delta(\psi)) = \exp[\mathcal{E}_3]
\]

Then

\[
|\mathcal{E}_3| = \left| \frac{4}{3\sqrt{N}} \text{Tr} J_\Delta^3 + \cdots \right| = o(1) \text{Tr} K(\zeta_\Delta)^2
\]

\[
= o(1) \frac{1}{N} < \frac{s}{T}, [G^{(0)}_\Delta]^{-1} \frac{s}{T} >
\]

Contrary to the above, we must be careful about \( R_3 \) which depend on \( \psi \) sensitively.

#### 3.3.1 small \( \psi \) region

We first assume \( \psi \) are small. Let us begin our calculation

\[
I = \frac{1}{Z_\Delta} \int \exp[\mathcal{E}_3 + R_3] \det^{-N/2}_3(1 + i\kappa K_\Delta) \exp[-<\psi, [G^{(0)}_\Delta] \psi>] \prod \frac{d\psi_x}{\sqrt{2\pi}}
\]

\[
= \frac{1}{Z_\Delta} \int \exp[\mathcal{E}_3 + R_3] \det^{-N/2}_3(1 + i\kappa K_\Delta) \nu_\Delta
\]

(3.28)

\[
d
\nu_\Delta \equiv \exp[-\frac{1}{2} \sum \tilde{\psi}_x^2] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}
\]

(3.29)
by decomposing \( \{ \tilde{\psi}_x \in R; x \in \Delta \} \) into small field region

\[
S_\Delta = \left\{ \sum_x \tilde{\psi}_x^2 < |\Delta||N^\varepsilon| \right\}, \quad \varepsilon \in (0, 1)
\]  

(3.30)

and its compliment \( S^c \), where the normalization constants \( Z^{(0)}_\Delta \) and \( \tilde{Z}^{(0)}_\Delta \) are defined in the obvious way. Thus we evaluate

\[ I = I_S + I_{S^c} \]

(3.31)

where

\[
I_S = \frac{1}{Z_\Delta^{(0)}} \int_S \left( 1 + \mathcal{E}_3 + R_3 + O(R_3^2) \right) \det_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta
\]

(3.32)

\[
I_{S^c} = \frac{1}{Z_\Delta^{(0)}} \int_{S^c} \det_1^{-N/2}(1 + i\kappa K_\Delta(\psi - ih + i\zeta)) e^{i\sqrt{N}\beta \sum_x(\psi_x - ih_x + i\zeta_x)}
\]

\[
\times \exp \left[ - \frac{2}{N} < \frac{s}{T}, [G^{(s)}_\Delta]^{-1} \frac{s}{T} > + 2\kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}
\]

(3.33)

We first calculate the small field contribution \( I_S \) given by

\[
I_S = \frac{<\chi_S D_\Delta>}{<D_\Delta>} \left\{ 1 + <\chi_S \mathcal{E}_3> + <\chi_S R_3> + <\chi_S O(R_3^2)> + \frac{<\chi_S D_\Delta; \chi_S \mathcal{E}_3>}{<\chi_S D_\Delta>} \frac{<\chi_S D_\Delta; \chi_S R_3>}{<\chi_S D_\Delta>} + \frac{<\chi_S D_\Delta; \chi_S O(R_3^2)>}{<\chi_S D_\Delta>} \right\}
\]

(3.34)

where

\[
D_\Delta \equiv \det_3^{-N/2}(1 + i\kappa K_\Delta), \quad <A> = \int A d\nu_\Delta
\]

and

\[
< A; B > = \int A B d\nu - (\int A d\nu)(\int B d\nu)
\]

We calculate \( <D> \) and \( <\chi_S D> \) first. We assumed that

\[
\frac{|\Delta| - 2}{2} \leq N^{1/3 - 2\varepsilon}, \quad 0 < \varepsilon << 1
\]

(3.35)

Then

\[
<\chi_S D> = \int_{x_S} \det_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta = 1 - O(N^{-1/3})
\]

(3.36)

To bound \( <\chi_{S^c} D> \), we use the bounds (3.3), and set \( r^2 = 2\text{Tr}K^2_\Delta = \sum \tilde{\psi}_x^2 \). Then for \( R^2 > \rho_0 = (|\Delta| - 2)/2 \), we have that

\[
\frac{(|\Delta| - 2)!!}{(2\pi)^{|\Delta|/2}} \int_{r > R} \left( \frac{1}{1 + \frac{2}{N}r^2} \right)^{N/4} r^{|\Delta| - 1} dr \leq O(\exp[-N^{1/3}])
\]
This means that
\[
\frac{\langle \chi_S D \Delta \rangle}{\langle D \rangle} = \frac{\langle \chi_S D \Delta \rangle}{\langle \chi_S D \rangle + \langle \chi_S^c D \rangle} = 1 - O(\exp[-cN^{1/3}])
\] (3.37)

Estimates are straightforward and we see that the most significant contribution is from $\text{Tr}K^2(\psi)K(\zeta)$ in $R_3$ and we have:
\[
\langle \chi_S R_3 \rangle = -\frac{|\Delta|}{\sqrt{N}} \left( c_1 \beta \left( \sum_x \zeta_x \right) + \frac{c_2}{\sqrt{\beta |\Delta|}} \left( \sum \left[ G^{(0)}_\Delta \right]^{-1/2} \frac{s}{T} \right) \right)
\]
\[-\frac{1}{\sqrt{2N}} \text{Tr}G^{(0)}_\Delta \zeta + \text{(smaller terms)}
\] (3.38)

where $c_i = 1 + O(|\Delta|^{-1})$ ($i = 1, 2$) are positive constants. Moreover we have (see [8]):
\[
\sum_{x \in \Delta} \zeta_x = \frac{1}{\sqrt{N}} \sum_{x \in \Delta} \left( [G^{(0)}_\Delta]^{-1/2} \frac{s}{T} \right) (x) = \frac{1}{\sqrt{N}} \left( \sum \sum_{x \in \partial \Delta} \frac{1}{\beta} \delta_{\partial \Delta}(x) \frac{s_x}{T_x} + O(\beta^{-3}) \right)
\] (3.39)
\[
\delta_{\partial \Delta}(x) = O\left( \frac{1}{\beta \sqrt{|\Delta|}} \right) \geq 0
\] (3.40)

and
\[
\text{Tr}G^{(0)}_\Delta \zeta = \beta \sum \zeta_x - T_0 P_0 \zeta
\]
\[= O \left( \frac{\log |\Delta|}{|\Delta|} \right) \left( \sum \zeta_x \right) - (\beta - \frac{\sigma_0}{|\Delta|}) \left( \sum \zeta_x \right) + \frac{1}{2\beta \sqrt{N|\Delta|}} \sum \frac{s_x}{T_x}
\]

Then the largest contribution comes from $\langle \chi_S R_3 \rangle$ and is negative, and other contributions can be made less than $\frac{1}{N \beta} \sum s_x/T_x$

3.3.2 large $\psi$ region

For $\{\tilde{\psi}\} \notin S_\Delta$, we start with
\[
I_{S^c} = \frac{1}{Z_D^0} \int_{S^c} \det^{-N/2}(1 + i\kappa K_\Delta(\psi - ih + i\zeta))e^{i\sqrt{N}\beta \sum_{x}(\psi_x - ih_x + i\zeta)}
\]
\[\times \exp \left[ -\frac{2}{N} \left( \sum \left[ G^{(0)}_\Delta \right]^{-1} \frac{s}{T} \right) \right] \prod \frac{d\psi_x}{\sqrt{2\pi}}
\]
\[= \frac{1}{Z_D^0} \int_{S^c} \det^{-N/2}(1 + i\kappa K_\Delta(\psi))e^{i\sqrt{N}\beta \sum_{x} \psi_x}
\]
\[\times \det^{-N/2}(1 + \kappa J_\Delta(h - \zeta))e^{i\sqrt{N}\beta \sum(h_x - \zeta_x)}
\]
\[\times \exp \left[ -\frac{2}{N} \left( \sum \left[ G^{(0)}_\Delta \right]^{-1} \frac{s}{T} \right) \right] \prod \frac{d\psi_x}{\sqrt{2\pi}}
\] (3.41)
\[
= \frac{1}{Z_D^0} \int_{S^c} \det^{-N/2}(1 + i\kappa K_\Delta(\psi))e^{i\sqrt{N}\beta \sum_{x} \psi_x}
\]
\[\times \det^{-N/2}(1 + \kappa J_\Delta(h - \zeta))e^{i\sqrt{N}\beta \sum(h_x - \zeta_x)}
\]
\[\times \exp \left[ -\frac{2}{N} \left( \sum \left[ G^{(0)}_\Delta \right]^{-1} \frac{s}{T} \right) \right] \prod \frac{d\psi_x}{\sqrt{2\pi}}
\] (3.42)
where
\[ J_{\Delta}(h - \zeta) = \frac{1}{1 + i\kappa K_{\Delta}(\psi)} K_{\Delta}(h - \zeta) \frac{1}{1 + i\kappa K_{\Delta}(\psi)} \] (3.43)

and then
\[
\det^{-N/2}(1 + \kappa J_{\Delta}(h - \zeta))e^{\sqrt{N}\beta\sum_{x}(h_{x} - \zeta_{x})} \times \exp \left[ -\frac{2}{N} < \frac{s}{T}, [G_{\Delta}^{2}]^{-1} \frac{s}{T} > + 2\kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > \right] \\
= \det^{-N/2}_{3}(1 + \kappa J_{\Delta}(h - \zeta)) \\
\times \exp \left[ -\frac{1}{N} < \frac{s}{T}, [G_{\Delta}^{2}]^{-1} \frac{s}{T} > + \kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > + < h, [G_{\Delta}^{2}]h > \right] \\
\times \exp \left[ \text{Tr} \left( \frac{2iK_{\Delta}(\psi)}{1 + i\kappa K_{\Delta}(\psi)} \right) K_{\Delta}(h - \zeta) + \text{Tr}(J_{\Delta}^{2}(h - \zeta) - K_{\Delta}^{2}(h - \zeta)) \right]
\]

and
\[
\text{Re} \text{Tr}J_{\Delta}^{2}(h - \zeta) \leq \text{Tr}K_{\Delta}^{2}(h - \zeta) \\
= \frac{1}{N} < \frac{s}{T}, [G_{\Delta}^{2}]^{-1} \frac{s}{T} > - \kappa < h, \frac{s}{T} > + < h, [G_{\Delta}^{2}]h > 
\]

Then putting \( S_{\Delta}^{c} = \bigcup_{k=1}^{\infty} S_{k} \) where
\[ S_{k} = \{ \{ \tilde{\psi}_{x} \}; kN^{1-2\epsilon} \leq \sum \tilde{\psi}^{2} \leq (k+1)N^{1-2\epsilon} \} \]
we estimate the integral on each shell of \( S_{c} \):
\[
\int_{S_{k}} \left| \det^{-N/2}(1 + i\kappa K_{\Delta}(\psi)) \right| \prod \frac{d\tilde{\psi}}{\sqrt{2\pi}} \leq \frac{(|\Delta| - 2)!!}{(2\pi)^{\Delta/2}} \frac{(kN^{1-2\epsilon})^{(\Delta-1)/2}}{(1+2kN^{-2\epsilon})^{N/4}}
\]

3.3.3 integration over small-smooth \( s_{x} \)

It remains to integrate over \( \{ s_{x} = n_{x} + \sqrt{n_{x}}\tilde{s}_{x} \} \) such that \( 0 < s_{x} < N\beta \) and \( |s_{x} - s_{x+\mu}| < \sqrt{N}\beta \). Since the contribution from \( I_{S_{c}} \) is negligible, we can apply the previous methods of analysis: we set
\[
d\nu_{n}(s) = \frac{1}{(n-1)!} e^{-n/\sqrt{n}} (n + \sqrt{n}\tilde{s})^{n-1} \sqrt{n}d\tilde{s} \\
= \frac{e^{-n}n^{n}}{(n-1)!} \sqrt{n} \exp[-\sqrt{n}\tilde{s} + (n-1) \log(1 + \tilde{s}/\sqrt{n})]d\tilde{s} \\
= \exp\left[ -\frac{1}{2}s^{2} + O(\tilde{s}/\sqrt{n}) \right] \frac{d\tilde{s}}{\sqrt{\pi}}
\]

We note that \( K_{\Delta}(x, y) \sim \frac{1}{\sqrt{2|\Delta|}}(\tilde{\psi}(x) + \tilde{\psi}(y)) \) is not of short range, though \( K_{\Delta}(x, y) = O(|\Delta|^{-1/2}) \). This long range nature of the interaction is expected compensated by the Anderson localization like phenomena.
3.4 Large field Region of \( s_x \)

For \( \{s_x; s_x > N\beta, \exists x \in \Delta \} \) or \( \{s_x; |s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1 \} \), we need an a priori bound to estimate the two-point function. Continuing the argument in the previous section (3.1), we start from

\[
D_\Delta(n)[C_0] = \frac{1}{Z_\Delta} \int \frac{\exp[-<\psi - ih, G_{\Delta}^{\chi}(\psi - ih)>]}{\prod_{x \in \Delta}(4 + m^2 + \kappa(\psi_x - h_x))^{n_x}} \det_3^{N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta)) \prod_{x \in \Delta} d\psi_x
\]

(3.44)

\[
\begin{align*}
&= \frac{1}{Z_\Delta^{(0)}} \frac{1}{\prod_{x} T_{x}^{n_{x}}} \int_{0}^{\infty} \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left( \sum_{k} \chi_{x}^{(k)}(s_x) \right) d\psi_x \\
&\times \exp\left[ -<\psi - ih + i\zeta, G_{\Delta}^{\chi}(\psi - ih + i\zeta)> - \frac{1}{N} <\frac{1}{T}s, [G_{\Delta}^{\chi}]^{-1} \frac{1}{T}s > + \kappa <h, \frac{s}{T}> \right] \\
&\times \det_3^{-N/2}(1 + i\kappa K_\Delta(\psi_\Delta) + \kappa K_\Delta(h_\Delta))
\end{align*}
\]

(3.45)

We choose \( h_x = c_x / (\beta \sqrt{N}) \). Then

\[
<h, G_{\Delta}^{\chi} h> \leq \frac{(\sum c_x)^2}{N} \leq \frac{\Delta^2}{N}
\]

(3.46)

and we see

\[
\left| \prod_{x \in \Delta} \frac{1}{(4 + m^2 + ic(\psi_x - ih_x))^{n_x}} \right| \leq \left( \frac{1}{4 + m^2 + \frac{c}{\beta N}} \right)^{\sum n_x}
\]

(3.47)

Then if \( \sum n(x) \) is so large that \( \sum n(x) h(x) / \sqrt{N} > |\Delta|^2 / N \), namely if \( \sum n_x > \beta|\Delta|^2 \), we easily see that the following a priori bound holds

\[
D_\Delta(n) \leq \left( \frac{1}{4 + m_{\text{eff}}^2} \right)^{\sum_{x \in \Delta} n(x)}
\]

(3.48)

\[
m_{\text{eff}}^2 = m^2 + \alpha^2, \quad \alpha^2 \equiv \frac{c}{N\beta}
\]

(3.49)

Therefore in the following discussion, we assume that \( \sum_{x \in \Delta} n_x \leq \beta|\Delta|^2 \) and \( \{s_x = n_x + \sqrt{n_x} \tilde{n}_x, x \in \Delta \} \) satisfy

(1) \( s_x \geq N\beta, \exists x \in \Delta \), or

(2) \( |s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1 \)

If (1) occurs, then the factor

\[
d_n(s) = \frac{s^n}{(n-1)!} e^{-s} ds
\]
restricted on this region yields a small coefficient less than
\[
\exp[-\frac{1}{2}N\beta] \leq \exp[-\frac{\sum n_x}{N\beta}]
\]
If (1) does not take place and (2) happens, then we can implement the complex deformation\[\psi_x \rightarrow \psi_x + i\tau \zeta_x,\] where \[\zeta = (N)^{-1/2}[G_{\Delta}^{02}]^{-1}(s/T)\] and \[0 < \tau \leq 1,\] and we see that the following factor arises from the complex deformation:
\[
\exp[-\frac{1 - (1 - \tau)^2}{N} < \frac{s}{T}, [G_{\Delta}^{02}]^{-1}\frac{s}{T}>] \leq \exp[-\frac{1 - (1 - \tau)^2}{2N\beta} < \frac{s}{T}, (-\Delta)\frac{s}{T}>] = \exp[-\frac{1 - (1 - \tau)^2}{2NT\beta}\sum_{nn}(s_x - s_{x'})^2] \quad (3.50)
\]
On the other hand, since
\[
||\kappa K_{\Delta}(\tau\zeta)||^2 = \frac{4\tau^2}{N^2} < \frac{s}{T}, [G_{\Delta}^{02}]^{-1}\frac{s}{T}> \quad (3.51)
\]
we have the bound
\[
|\det_{3}^{-N/2}(1 + \kappa K_{\Delta}(\tau\zeta))| \leq \exp \left[ O \left( \frac{1}{\sqrt{N}} \right) ||K_{\Delta}(\tau\zeta)||^2 \right] \quad (3.52)
\]
which is close to 1 and has no effects on the bound (3.50) if \[N\] is large.

4 Conclusions and Discussions

We have shown that if the non-local factor
\[
\prod_i \det^{-N/2}(1 + W(\Delta_i, \Lambda_i))
\]
are discarded, then the resultant system exhibits exponential clustering for all \[\beta\] if \[N\] is large enough:
\[
<s_0s_x> \sim \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(0, x) \prod d\mu_{\Delta}(\psi_{\Delta}), \quad (4.1)
\]
\[
\leq \exp[-m_{eff}|x|] \quad (4.2)
\]
where \[m_{eff}^2 = m^2 + c(N\beta)^{-1}\] and
\[
d\mu_{\Delta}(\psi_{\Delta}) = \det_{3}^{-N/2}(1 + i\kappa K_{\Delta})d\nu_{\Delta} \quad (4.3)
\]
is the complex measure localized to each block \[\Delta\] of size \[L \times L\] in \[Z^2\]. The assumption \[N >> 1\] is to simplify the large field problem and could be removed by additional efforts. The smallness of \[W(\Delta, \Lambda)\] is due to the Anderson localization type arguments which remains to be justified \[8, 9\].
Acknowledgments

K.R.I.'s and F.H.'s works are partially supported by the Grant-in-Aid for Scientific Research, No.13640227 and No.1554019 respectively, the Ministry of Education, Science and Culture, Japanese Government.

References


