<table>
<thead>
<tr>
<th>Title</th>
<th>Fourier Transformation of 2D $O(N)$ Spin Model and Anderson Localization (Applications of Renormalization Group Methods in Mathematical Sciences)</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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</tbody>
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Kyoto University
Fourier Transformation of 2D $O(N)$ Spin Model and Anderson Localization

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Abstract

We Fourier transform the 2D $O(N)$ spin model $N > 2$, and start with a representation of the correlation functions in terms of integrals by complex random fields. Since this integral is complicated, we use the idea of the Anderson localization to discard non-local terms which make the integrals difficult. Through this approximation, we obtain the correlation functions which decay exponentially fast for all $\beta > 0$ if $N >> 3$.

1 Introduction: Result and Motivation

It is a longstanding problem to prove or disprove non-existence of phase transitions in 4 dimensional non-Abelian lattice gauge theories. In many points, this is similar to the same problem in the two-dimensional $O(N)$ symmetric spin models (Heisenberg or $\sigma$ model) with $N \geq 3$.

In models such as $O(N)$ spin modes and $SU(N)$ lattice gauge models [13, 17], the field variables form compact manifolds and the block spin transformations break the original

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structures. In some cases, this can be avoided by introducing an auxiliary field $\psi$ [3] which can be regarded as complex random field. The $\nu$ dimensional $O(N)$ spin (Heisenberg) model at the inverse temperature $N\beta$ is defined by the Gibbs expectation values

$$< f > \equiv \frac{1}{Z_\Lambda(\beta)} \int f(\phi) \exp[-H_\Lambda(\phi)] \prod_{i \in \Lambda} \delta(\phi_i^2 - N\beta) d\phi_i \quad (1.1)$$

Here $\Lambda$ is an arbitrarily large square with the center at the origin, $\phi(x) = (\phi(x)^{(1)}, \cdots, \phi(x)^{(N)})$ is the vector valued spin at $x \in \Lambda$ and $Z_\Lambda$ is the partition function defined so that $<1> = 1$. The Hamiltonian $H_\Lambda$ is given by

$$H_\Lambda = -\frac{1}{2} \sum_{|x-y|=1} \phi(x)\phi(y), \quad (1.2)$$

where $|x|_1 = \sum_{i=1}^{\nu} |x_i|$.

We substitute the identity $\delta(\phi^2 - N\beta) = \int \exp[-ia(\phi^2 - N\beta)] da/2\pi$ into eq.(1.1) with the condition that $\text{Im} a_i < -\nu [3]$, and set

$$\text{Im} a_i = -(\nu + \frac{m^2}{2}), \quad \text{Re} a_i = \frac{1}{\sqrt{N}} \psi_i \quad (1.3)$$

where $m > 0$ will be determined soon. Thus we have

$$Z_\Lambda = c|\Lambda| \int \cdots \int \exp[-\frac{1}{2} \phi_i (m^2 - \Delta + \frac{2i}{\sqrt{N}} \psi)\phi_i + \sum_j i\sqrt{N}\beta \psi_j] \prod \frac{d\phi_j d\psi_j}{2\pi}$$

$$= c|\Lambda| \det(m^2 - \Delta)^{-N/2} \int \cdots \int F(\psi) \prod \frac{d\psi_j}{2\pi} \quad (1.4)$$

where c's are constants being different on lines, $\Delta_{ij} = -2\nu \delta_{ij} + \delta_{|i-j|,1}$ is the lattice Laplacian,

$$F(\psi) = \det(1 + i\kappa G\psi)^{-N/2} \exp[i\sqrt{N}\beta \sum_j \psi_j], \quad (1.5)$$

and $\kappa = 2/\sqrt{N}$. Moreover $G = (m^2 - \Delta)^{-1}$ is the covariant matrix discussed later. In the same way, the two-point function is given by

$$< \phi_0 \phi_x > = \frac{1}{Z} \int \cdots \int \left( \frac{1}{m^2 - \Delta + i\kappa \psi} \right)(0, x) F(\psi) \prod \frac{d\psi_j}{2\pi} \quad (1.6)$$

namely by an average of the Green's function which includes complex fields $\psi(x), x \in Z^2$, where the constant $Z$ is chosen so that $<\phi_0^2> = N\beta$. We choose the mass parameter $m > 0$ so that $G(0) = \beta$, where

$$G(x) = \int \frac{e^{ipx}}{m^2 + 2\sum(1 - \cos p_i)} \prod_{i=1}^{\nu} \frac{dp_i}{2\pi} \quad (1.7)$$
This is possible for any $\beta$ if $\nu \leq 2$, and we easily find that

$$m^2 \sim 32e^{-4\pi\beta} \text{ for } \nu = 2$$

as $\beta \to \infty$. Thus for $\nu = 2$, we can rewrite

$$F(\psi) = \det_3^{-N/2}(1 + i\kappa G\psi) \exp[- <\psi, G^{02}\psi>],$$

$$\det_3(1 + A) = \det[(1 + A)e^{-A^2/2}]$$

where $G^{02}(x, y) = G(x, y)^2$ so that $\text{Tr}(G\psi)^2 = <\psi, G^{02}\psi>$. For any two matrices $A$ and $B$ of equal size, the Hadamard product [18] $A \circ B$ is defined by $(A \circ B)_{ij} = A_{ij}B_{ij}$ and we denote $G \circ G$ by $G^{02}$.

Decompose $\Lambda \subset Z^2$ into small blocks $\Delta_i$, and define $G_{\Lambda} = \chi_{\Lambda}G\chi_{\Lambda}$:

$$\Lambda = \bigcup_{i=1}^{n}\Delta_i,$$

Then we use the Feshbach-Krein formula (blockwise diagonalizations of matrices), to decompose $\det(1 + i\kappa G_{\Lambda}\psi_{\Lambda})$ into a product of $\det(1 + i\kappa G_{\Delta}\psi_{\Delta_i})$ as follows:

$$\det^{-N/2}(1 + i\kappa G_{\Lambda}\psi_{\Lambda}) = \prod_{i=1}^{n-1}\det^{-N/2}(1 + W(\Delta_i, \Lambda_i)) \prod_{i=1}^{n}\det^{-N/2}(1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}).$$

where $\kappa = 2/\sqrt{N}$, $\Lambda_k = \bigcup_{i=k+1}^{n}\Delta_i$,

$$W(\Delta_i, \Lambda_i) = -(i\kappa)^2 \frac{1}{1 + i\kappa G_{\Delta_i}\psi_{\Delta_i}}G_{\Delta_i, \Lambda_i}\psi_{\Lambda_i}$$

and $[G_{\Delta}]^{-1}$ is the Laplacian with free boundary condition and almost equal to the Laplacian restricted to the square $\Delta$ with no boundary. Thus $\inf \text{ spec } [G_{\Delta}]^{-1} \sim 0$ and we can prove that $([G_{\Lambda_i}]^{-1} + i\kappa \psi_{\Lambda_i})^{-1}$ behaves like a massive Green’s function which decreases fast since $\psi$ behaves like a Gaussian random variable of zero mean and covariance $[G^{02}]^{-1}$.

Let us consider the measure localized on each block $\Delta$:

$$d\mu_{\Delta} = \frac{1}{Z_{\Delta}} \det^{-N/2}(1 + i\kappa G_{\Delta}\psi_{\Delta}) \exp[-(\psi_{\Delta}, G^{02}_{\Delta}\psi_{\Delta})] \prod_{x \in \Delta} d\psi(x)$$

where $Z_{\Delta}$ is chosen so that $\int d\mu_{\Delta} = 1$. Since the norm of $G_{\Delta}$ is of order $O(|\Delta|\beta) >> 1$, one may think that it is still impossible to expand the determinant. However, this comes with the factor $\exp[-(\psi_{\Delta}, G^{02}_{\Delta}\psi_{\Delta})]$, which makes the norm of $\frac{2i}{\sqrt{N}}G_{\Delta}\psi_{\Delta}$ small. To see if this is the case, we introduce new variables $\tilde{\psi}_{\Delta}(x)$ by

$$\psi_{\Delta}(x) = \frac{1}{\sqrt{2}} \sum_{y \in \Delta} \tilde{G}_{\Delta}^{-1}(x, y)\tilde{\psi}(y), \quad \tilde{G}_{\Delta} = [G^{02}_{\Delta}]^{1/2}$$

Then $\tilde{G}_{\Delta}$ is chosen so that $\int d\mu_{\Delta} = 1$. Since the norm of $\tilde{G}_{\Delta}$ is of order $O(|\Delta|\beta) >> 1$, one may think that it is still impossible to expand the determinant. However, this comes with the factor $\exp[-(\psi_{\Delta}, G^{02}_{\Delta}\psi_{\Delta})]$, which makes the norm of $\frac{2i}{\sqrt{N}}G_{\Delta}\psi_{\Delta}$ small. To see if this is the case, we introduce new variables $\tilde{\psi}_{\Delta}(x)$ by

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so that \(d\mu_{\Delta}\) is rewritten
\[
d\mu_{\Delta} = \frac{1}{Z_{\Delta}} \det^{-N/2} 3(1 + i\kappa K_{\Delta}) \prod_{x \in \Delta} \exp\left[-\frac{1}{2} \tilde{\psi}(x)^2\right] \frac{d\tilde{\psi}(x)}{\sqrt{2\pi}},
\tag{1.16}
\]
\[
K_{\Delta} = \frac{1}{\sqrt{2}} G_{\Delta}^{1/2} (\hat{G}_{\Delta}^{-1} \psi) G_{\Delta}^{1/2}
\tag{1.17}
\]
Put
\[
d\nu_{\Delta}^{(0)} = \prod \exp\left[-\frac{1}{2} \tilde{\psi}^2(x)\right] \frac{d\tilde{\psi}(x)}{\sqrt{2\pi}}
\tag{1.18}
\]
and define
\[
||K||_p = \left(\int \mathrm{h}(K K^*)^{p/2} d\nu_{\Delta}\right)^{1/p}
\tag{1.20}
\]

**Lemma 1** It holds that
\[
\int \mathrm{h} K_{\Delta}^2 d\nu_{\Delta} = \frac{1}{2} |\Delta|,
\tag{1.21}
\]
\[
||K_{\Delta}||_p \leq (p - 1) ||I_{\Delta}||_{2}, \text{ for all } p \geq 2
\tag{1.22}
\]

**Proof.** The first equation is immediate. See [16] for the second inequality. Q.E.D.

Thus we see that \(\kappa K_{\Delta}\) are a.e. bounded with respect to \(d\nu_{\Delta}\), and converges to 0 as \(N \to \infty\). To see to what extent \(K_{\Delta}\) is diagonal, we estimate
\[
\int \mathrm{h} K_{\Delta}^4 d\nu_{\Delta} = \sum_{x_i \in \Delta} \frac{1}{4} \prod_{i=1}^{4} G_{\Delta}(x_1, x_{i+1})
\times \left[2[G^{02}]^{-1}(x_1, x_2)(G^{02}]^{-1}(x_3, x_4) + [G^{02}]^{-1}(x_1, x_3)[G^{02}]^{-1}(x_2, x_4)\right]
\]
where \(x_5 = x_1\). As is proved in [8]
\[
[G_{\Delta}^{02}]^{-1}(x, y) = \frac{1}{2\beta} G_{\Delta}^{-1} - \hat{B}_{\Delta}, \quad \hat{B}_{\Delta}(x, y) = O(\beta^{-2})
\tag{1.23}
\]
The main contribution comes from the term containing \(2[G^{02}]^{-1}(x_1, x_2)\cdots\). To bound this, set \(G_{\Delta}(x, x_{i+1}) = \beta - \delta G(x, x_{i+1})\). Then \(\delta G(x, x) = 0, \delta G(x, x + e_\mu) = 0.25 - O(\beta m^2)\), \((-\Delta)_{xy} = 0\) unless \(|x - y| \leq 1\), and we have
\[
\int \mathrm{h} K_{\Delta}^4 d\nu_{\Delta} \geq \text{const.} \sum_{x_i \in \Delta} \frac{1}{4\beta^2} \left\{\beta^2 \sum_{x_4} \delta_{x_1, x_4} + \sum_{x_4} G^2(x_1, x_4)\right\}
\geq \text{const.} (|\Delta| + |\Delta|^2)
which means that $K_{\Delta}$ is approximately diagonal but off-diagonal parts are still considerably large. However, there is a reason to believe that $W$ functions are of short range and small. In fact we know that

$$
\frac{1}{|G_{\Lambda}|^{-1} + i\kappa \psi_{\Lambda}}(x, y) \leq \frac{1}{|G_{\Lambda}|^{-1} + c(N\beta)^{-1}}(x, y)
$$

for almost all $\psi$. Then $(G_{\Lambda}^{-1} + m^2 + i\kappa \psi_{\Lambda})^{-1}(x, y)$ is negligible if $|x - y| > \sqrt{N\beta}$. Moreover it is shown in two dimension that

$$
\int \frac{1}{|G_{\Lambda}|^{-1} + i\kappa \psi_{\Lambda}}(x, y) d\mu \leq \frac{1}{|G_{\Lambda}|^{-1} + m_{eff}^2}(x, y),
$$

if $d\mu(\psi)$ is Gaussian of mean zero and covariance $[G^{02}]^{-1}$. This logarithmic correction comes from the two-dimensionality. This implies that

$$
\lim_{N\beta \to \infty} \frac{1}{N\beta} \sum_{\Delta} \int \frac{1}{-\Delta + m^2 + i\kappa \psi}(0, x) d\mu = 0
$$

Furthermore $\psi$ in the numerators of $W$ acts as a differential operators since

$$
\psi = \frac{1}{\sqrt{2}} [G^{\Delta}]^{-1/2} \tilde{\psi} \sim \frac{1}{2\sqrt{\beta}} [G_{\Delta}]^{-1/2} \tilde{\psi}
$$

Thus $W(\Delta, \Lambda)$ seems to be small as $N\beta \to \infty$.

We choose $N$ larger than $|\Delta| = L^2$, i.e.,

$$
N^{1/3-\varepsilon} \geq |\Delta| = L^2
$$

This assumption is artificial and its role is to simplify the large field problem to bound the integrals in the region where $|\psi_x|$ are large. So more elaborate idea may remove this condition (it is natural to think that $N \geq 3$ is enough).

To imagine that the non-diagonal terms $W$ are small, we perhaps choose $L$ larger than some power of $\beta$, say $L > (\beta)^{1+\delta}$, $\delta > 0$, but we do not know how to determine it yet though it is now under investigation, see [9].

**Assumption:** We take $N$ larger than $|\Delta| = L^2$ as above, and for sufficiently large $\Delta$, non-local terms $W$ are negligible in this case.

Once $W$ is neglected and $N$ is chosen larger than $|\Delta|$, we can prove the following result uniformly in $\beta$:

**Main Theorem:** Assume $N$ is sufficiently large: $N^{1/3-\varepsilon} \geq |\Delta|$. Neglect non-local terms $W(\Delta, \Lambda)$. Then the two point correlation function

$$
\int \frac{1}{-\Delta + m^2 + i\kappa \psi}(x, y) \prod_{\Delta} d\mu_{\Delta}
$$

decays exponentially fast for all $\beta \geq 0$. 

2 Averaged Green's Function by the measure $d\mu_0$

Let us come back to the present case where $\Delta_i$ are boxes of equal size $L \times L$ ($L \geq 2$) such that $\cup_i \Delta_i = \mathbb{Z}^2$ and $\Delta_i \cap \Delta_j = \emptyset, i \neq j$. Let us estimate

$$G^{(\text{ave})}(x, y) \equiv \int G^{(\psi)}(x, y) d\mu(\psi)$$

(2.1)

where

$$G^{(\psi)}(x, y) \equiv \left( \frac{1}{G^{-1} + i\kappa\psi} \right)(x, y),$$

(2.2a)

$$d\mu(\psi) = \prod \frac{1}{Z_{\Delta}} \det_{3}^{-N/2}(1 + i\kappa G_{\Delta}\psi_{\Delta}) d\nu_{\Delta},$$

(2.2b)

$$d\nu_{\Delta} = \frac{1}{\det^{1/\mathit{2}}(G_{\Delta})} \exp[-(\psi_{\Delta}, G_{\Delta}^{02}\psi_{\Delta})] \prod_{x \in \Delta} \frac{d\psi(x)}{\sqrt{2\pi}}.$$  

(2.2c)

and $\kappa = 2/\sqrt{N}$. Expanding $G^{(\psi)}$ by random walk, we have

$$G^{(\psi)}(x, y) = \sum_{\omega : x \rightarrow y} \prod_{\zeta \in \omega} \frac{1}{(4 + m^2 + i\kappa\psi_{\zeta})^{n_{\zeta}}},$$

(2.3)

where $n_{\zeta} \in \mathbb{N}$ is the visiting number of $\omega$ at $\zeta \in \mathbb{Z}^2$. We set

$$dv = \prod_{\Delta \subset \mathbb{Z}^2} d\nu_{\Delta}$$

(2.4)

We first prove our assertion for the Gaussian case:

**Theorem 2** The following bound holds:

$$\int G^{(\psi)}(x, y) dv \leq \frac{1}{-\Delta + m_{\text{eff}}^2}(x, y)$$

(2.5)

where

$$m_{\text{eff}}^2 = m^2 + \frac{c}{N\beta}$$

(2.6)

with a constant $c > 1$.

**Proof.** Let $\Delta$ be the square of size $L \times L$ centered at the origin, and let $n_x \in \{0, 1, 2, \cdots\}, x \in \Delta$. We estimate

$$D_\Delta(\{n\}) \equiv \int \prod_{x \in \Delta} \frac{1}{(4 + m^2 + i\kappa\psi(x))^{n_x}} d\nu(\psi)$$

(2.7)

For large $\sum_{x \in \Delta} n_x$ such that

$$\sum_{x \in \Delta} n_x \geq \beta(\# \{x \in \Delta; n_x \neq 0\})^2$$

(2.8)
the bound follows by the complex translation estimate by putting $\psi_x \rightarrow \psi_x - i h_x$, where

$$h_x = \frac{c_x}{\beta \sqrt{N}}, \quad c_x = \begin{cases} c > 0 & \text{if } n_x \geq 1 \\ 0 & \text{if } n_x = 0 \end{cases}$$

(2.9)

In fact we have:

$$D_\Delta(\{n\}) \leq \frac{e^{<h, G^2_{\Delta} h>}}{\prod_{x \in \Delta} (4 + m^2 + \kappa h_x)^{n_x}} \leq \frac{e^{\beta^2 (\sum_{x \in \Delta} h_x)^2}}{(4 + m^2 + c(\beta N)^{-1})^{\sum n_x}} \leq \left( \frac{1}{4 + m^2 + c'(\beta N)^{-1}} \right)^{\sum n_x}$$

(2.10)

with a constant $0 < c' < c$.

For small $\{n_x; x \in \Delta\}$, we start with the new expression of $D_\Delta(\{n\})$:

$$\prod_{x \in \Delta} \frac{1}{(n_x - 1)!} \int_0^\infty \prod s_x^{n_x - 1} \exp\left[ -(4 + m^2) \sum s_x - \frac{\kappa^2}{4} <s_\Delta, [G^2_\Delta]^{-1}s_\Delta> \right] \prod ds_x$$

$$= \prod_{x \in \Delta} T^{-n(x)} \int \exp\left[ -\frac{1}{NT^2} <s_\Delta, [G^2_\Delta]^{-1}s_\Delta> \right] \prod_x d\nu_{n_x}(s_x)$$

(2.11)

where $T = 4 + m^2$ and

$$d\nu_{n}(s) = \frac{s^{n-1}e^{-s}}{(n-1)!} ds$$

(2.12)

Since $\int d\nu_{n}(s) = 1$ and $n \log s - s$ takes its maximum at $s = n$, we set $s_x = n_x + \sqrt{n_x} \delta_x$ ($x \in \Delta$) and note that

$$d\nu_{n}(s) = \exp\left[ -\frac{1}{2} \tilde{s}^2 \right] \frac{e^{\delta_n(\tilde{s})}}{<e^{\delta_n(\tilde{s})}>} \frac{d\tilde{s}}{\sqrt{2\pi}}$$

(2.13)

$$\delta_n(\tilde{s}) = -\sqrt{n} \tilde{s} + (n - 1) \log(1 + \frac{\tilde{s}}{\sqrt{n}}) + \frac{1}{2} \tilde{s}^2$$

(2.14)

$$<e^{\delta_n(\tilde{s})}> = \int_{-\sqrt{n}}^{\infty} e^{\delta_n(\tilde{s})} e^{-\tilde{s}^2/2} d\tilde{s} \sqrt{2\pi}$$

$$= 1 + O(1/n) > 1$$

(2.15)

Put

$$\alpha^2 = \frac{1}{N \beta}$$

(2.16)

Then if $\alpha^2 n(x) < 1$ and $N^{-1} < n, [G^2_\Delta]^{-1} n >$ is small, the integral (2.11) is carried out by perturbative calculations.
For large $\alpha^{2}n(x) \geqq 1$ or for non-smooth $n$ such that $N^{-1} < n, [G_{\Delta}^{2}]^{-1}n >> 1$ we use a priori bound. See [8].

In the case where $d\mu$ is Gaussian, we can obtain $G^{(ave)}$ in a closed form. See [8] where $m_{eff}^{2} \sim \log(N\beta)/N\beta$ is obtained.

**Remark 1** We note that this is similar to the pinch singularity encountered in the study of the Anderson localization [5], where

$$\int G(E+i\epsilon, v)(x,y)dP(v)$$

has a convergent random walk expansion, and

$$\int |G(E+i\epsilon, v)(x,y)|^{2}dP(v)$$

does not have.

### 3  Averaged Green's Function by the measure $d\mu(\psi)$

It remains to discuss the effects of the determinants $\det^{-N/2}_{3}(1+\cdots)$. Set

$$S_{\Delta} = \{\psi_{x}; x \in \Delta, \Tr K_{\Delta}^{2} < N^{-1-2\epsilon}\},$$

$$K_{\Delta} = G_{\Delta}^{1/2}\psi_{\Delta}G_{\Delta}^{1/2}$$

(3.1)  

(3.2)

Since

$$\exp[-\Tr K_{\Delta}^{2}] \leq |\det_{2}^{-N/2}(1+i\kappa K_{\Delta})| \leq \left(1 + \frac{4}{N}\Tr K_{\Delta}^{2}\right)^{-N/4}$$

(3.3)

and $\Tr K_{\Delta}^{2} = \sum_{x} \tilde{\psi}_{x}^{2}/2$, we have

$$\int \exp[-\Tr K_{\Delta}^{2}] \prod_{x} \frac{d\tilde{\psi}_{x}}{\sqrt{2\pi}} = \int \exp[-\sum_{x} \frac{1}{2}\tilde{\psi}(x)^{2}] \prod_{x} \frac{d\tilde{\psi}_{x}}{\sqrt{2\pi}} = 1$$

(3.4)

and $\int (1 + \frac{2}{N} \sum \tilde{\psi}(x))^{-N/4} \prod_{x \in \Delta} d\tilde{\psi}_{x}$ is convergent for $2|\Delta| < N$. Even so, it is obvious that $|\det^{-N/2}(1+i\kappa G_{\Delta}\psi)|$ is integrable if and only if $N > 2$ since

$$\det^{-N/2}(1+i\kappa G_{\Delta}\psi) = \det^{-N/2}(G_{\Delta}) \det^{-N/2}(G_{\Delta}^{-1} + i\kappa\psi)$$

$$\sim \det^{-N/2}(G_{\Delta}) \prod_{x \in \Delta} \left(\frac{1}{4 + m^{2} + i\kappa\psi(x)}\right)^{N/2}$$

(3.5)  

(3.6)

holds for $\psi$ such that $|\psi/\sqrt{N}| > O(1)$
3.1 Small Fields, Large Fields and Complex Displacements

Let us estimate

$$D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} \psi_{x}}}{\prod_{x \in \Delta} (4 + m^{2} + i\kappa \psi_{x})^{n_{x}}} \det^{N/2} (1 + i\kappa G_{\Delta}) \prod_{x \in \Delta} d\psi_{x},$$  

(3.7)

$$Z_{\Delta} = \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} \psi_{x}}}{\det^{N/2} (1 + i\kappa G_{\Delta})} \prod_{x \in \Delta} d\psi_{x}$$  

(3.8)

by putting $\psi_{x} \rightarrow \psi_{x} - ih_{x}, \psi_{x} \in \mathbb{R}$. Then

$$D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int \frac{e^{i\sqrt{N}\beta \sum_{x \in \Delta} (\psi_{x} + h_{x})}}{\prod_{x \in \Delta} (4 + m^{2} + \kappa(i\psi_{x} + h_{x}))^{n_{x}}} \det^{N/2} (1 + i\kappa G_{\Delta}^{1/2}(i\psi_{\Delta} + h_{\Delta})) \prod_{x \in \Delta} d\psi_{x}$$

$$= \frac{1}{Z_{\Delta}} \int \frac{\exp[-<\psi - ih, G_{\Delta}^{02}(\psi - ih)>]}{\prod_{x \in \Delta} (4 + m^{2} + \kappa(i\psi_{x} + h_{x}))^{n_{x}}} \det^{N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta})) \prod_{x \in \Delta} d\psi_{x}$$

(3.9)

where

$$K_{\Delta}(\psi_{\Delta}) \equiv G_{\Delta}^{1/2}\psi_{\Delta}G_{\Delta}^{1/2}, \quad K_{\Delta}(h_{\Delta}) \equiv G_{\Delta}^{1/2}h_{\Delta}G_{\Delta}^{1/2}$$

and $K_{\Delta}(h_{\Delta}) \geq 0$ since $h_{x} \geq 0$. We again put $h_{x} = c_{x}/(\sqrt{N}\beta)$ and then

$$\kappa K_{\Delta}(h_{\Delta}) \leq \frac{c|\Delta|}{N}, \quad c = O(1) > 0.$$

(3.10)

We repeat the previous arguments by using $(n - 1)!x^{-n} = \int_{0}^{\infty} s^{n-1}e^{-sx}ds$. Define $I_{n}^{(k)} = \{s; k\sqrt{n} < |s - n| < (k + 1)\sqrt{n}, s \geq 0\}, k = 0, 1, 2, \cdots$, and let $\chi_{x}^{(k)}(s_{x})$ be the characteristic function of the interval $I_{n}^{(k)}$. Then

$$D_{\Delta}(n) = \frac{1}{Z_{\Delta}} \int_{0}^{\infty} \prod_{x \in \Delta} \left(\sum_{k} \chi_{x}^{(k)}(s_{x})\right) s_{x}^{n_{x}-1}ds_{x} \prod_{x \in \Delta} d\psi_{x}$$

$$\times \exp[-\sum_{x} (4 + m^{2} + \kappa(i\psi_{x}(x) + h_{x}))s(x)] \frac{e^{-<\psi_{\Delta} - ih_{\Delta}, G_{\Delta}^{02}(\psi_{\Delta} - ih_{\Delta})>}}{\det^{N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))}\left(3.11\right)$$

$$= \frac{1}{Z_{\Delta}^{(0)}} \frac{1}{\prod_{x} T_{n_{x}}} \int_{0}^{\infty} d\nu_{n_{x}}(s_{x}) \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left(\sum_{k} \chi_{x}^{(k)}(s_{x})\right) d\psi_{x}$$

$$\times \exp\left[-<\psi - ih + i\zeta, G_{\Delta}^{02}(\psi - ih + i\zeta)> - \frac{1}{N} s_{\nu} [G_{\Delta}^{02}]^{-1} \frac{1}{T}s > + \kappa < h_{\Delta}, \frac{s}{T}>\right]$$

$$\times \det^{N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))$$

(3.12)
where \( T_x = 4 + m^2 + \kappa h_x \), \((s/T)_x = s_x/T_x\)

\[
d
\nu_n(s) = \frac{1}{(n-1)!} e^{-s} s^{n-1} ds, \quad \zeta_x = \frac{\kappa}{2} ([G_{\Delta}^{02}]^{-1} - \frac{1}{T} s)(x)
\]

(3.13)

and

\[
Z_{\Delta}^{(0)} = \int \exp[- <(\psi - ih), G_{\Delta}^2(\psi - ih)>] \\
x \det_{3}^{-N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta})) \prod_{x \in \Delta} d\psi_x
\]

(3.14)

We then change the contour of \( \psi_x \) by replacing \( \psi_x + i\zeta_x \) by \( \psi \) (namely we put \( \psi_x \to \psi - i\zeta_x \)). The contours depend on \( \{s_x; x \in \Delta\} \). This yields

\[
D_{\Delta}(n) = \frac{1}{Z_{\Delta}^{(0)}} \prod_{x \in \Delta} \frac{1}{T_x^{n_x}} \int_{0}^{\infty} \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^{\infty} \prod_{x \in \Delta} \left( \sum_{k} \chi_{x}^{(k)}(s_x) \right) d\psi_x \\
\times \exp \left[ - <(\psi - ih), G_{\Delta}^2(\psi - ih)> - \frac{1}{N} <\frac{1}{T} s, [G_{\Delta}^{02}]^{-1} \frac{1}{T} s > + \kappa <h, \frac{s}{T}> \right] \\
x \det_{3}^{-N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}) + \kappa K_{\Delta}(\zeta_{\Delta}))
\]

(3.15)

where

\[
K_{\Delta}(\zeta_{\Delta})(x, y) = \frac{1}{\sqrt{N}} \sum_{\xi} G^{1/2}_{\Delta}(x, \xi) \left( [G_{\Delta}^{02}]^{-1} \frac{s}{T} \right)(\xi) G^{1/2}_{\Delta}(\xi, y),
\]

\[
J_{\Delta}(\zeta_{\Delta}) = \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi_{\Delta})}} K(\zeta_{\Delta}) \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi_{\Delta})}}
\]

and

\[
R_3 = \frac{N}{2} \text{Tr} \left[ \left( \frac{1}{1 + i\kappa K_{\Delta}(\psi_{\Delta})} - 1 + i\kappa K_{\Delta}(\psi_{\Delta}) \right) \kappa K_{\Delta}(\zeta_{\Delta}) + \kappa^2 (J_{\Delta}^2 - K_{\Delta}(\zeta_{\Delta})^2) \right]
\]

(3.16)
3.2 $K_{\Delta}(\psi_{\Delta}), K_{\Delta}(\zeta_{\Delta})$ and $R_{3}$

Let

$$G_{\Delta} = \sum_{i=0}^{|\Delta|-1} e_{i} P_{i}, \quad G_{\Delta}^{o2} = \sum_{i=0}^{|\Delta|-1} \hat{e}_{i} \hat{P}_{i}$$

be the spectral resolutions of the positive matrices $G_{\Delta}$ and $G_{\Delta}^{o2}$ respectively, where $e_{0} \geq e_{1} \geq \cdots \geq e_{|\Delta|-1}, \quad \hat{e}_{0} \geq \hat{e}_{1} \geq \cdots \geq \hat{e}_{|\Delta|-1}, \quad P_{i} P_{j} = \delta_{i,j} P_{i}$ and so on. Then

$$G_{\Delta}^{1/2} = \sum_{i=0}^{|\Delta|-1} \sqrt{e_{i}} P_{i}, \quad [G_{\Delta}^{o2}]^{-1} = \sum_{i=0}^{|\Delta|-1} \frac{1}{\hat{e}_{i}} \hat{P}_{i}$$

(3.17)

It is convenient to introduce the abbreviation for the Green's function with the largest eigenvalue part extracted:

$$G_{\Delta}^{(0)} = \sum_{k \neq 0} e_{k} P_{k} = G_{\Delta} - e_{0}^{-1} P_{0}$$

We let $\{u_{i}\}_{i=0}^{|\Delta|-1}$ and $\{\hat{u}_{i}\}_{i=0}^{|\Delta|-1}$ be the normalized eigenvectors such that

$$G_{\Delta} u_{i} = e_{i} u_{i}, \quad G_{\Delta}^{o2} \hat{u}_{i} = \hat{e}_{i} \hat{u}_{i}$$

(3.19)

Then

$$P_{i} = |u_{i}><u_{i}|, \quad \hat{P}_{i} = |\hat{u}_{i}><\hat{u}_{i}|$$

(3.20)

and for small $\Delta$, we have

$$e_{0} = |\Delta|\beta - O(1), \quad e_{i} = O(1) > 0$$

(3.21)

$$\hat{e}_{0} = |\Delta|\beta^{2} - O(\beta), \quad \hat{e}_{i} = 2\beta e_{i} + O(1)$$

(3.22)

$(i \neq 0)$ and

$$P_{0} \sim \hat{P}_{0} \sim \frac{1}{|\Delta|} |U><U| = \frac{1}{|\Delta|} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

(3.23)

where $U = ^{t}(1,1,\cdots,1) \sim \sqrt{|\Delta|} u_{0}$. Moreover we can symbolically write

$$P_{i} \sim \hat{P}_{i} \sim \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(3.24)
namely $P_i (i \neq 0)$ is a matrix which represents a lattice differentiation since $<u_i, u_0> = 0$. Note that $e_i \leq O(\log |\Delta|)$, $e_0 = \beta |\Delta| - O(|\Delta| \log |\Delta|)$ and

$$
(P_0 \zeta P_0)_{x,y} = \sum_\zeta \frac{1}{|\Delta|^2} \zeta \zeta = (\frac{1}{|\Delta|} \sum \zeta \zeta) P_0, \quad P_0 (\hat{P}_i \zeta) P_0 = O(\beta^{-1}) \tag{3.25}
$$

We insert $\psi = \hat{G}^{-1} \tilde{\psi}/\sqrt{2}$ into $K_\Delta$ and use $\hat{e}_i = 2\beta e_i + O(1) (i \neq 0)$, $P_i = \hat{P}_i + O(\beta^{-1})$ and $\sum_{i \neq 0} P_i = 1 - P_0$ to find that

$$
K_\Delta = \left( \sum\tilde{\psi}(x) \right) P_0 + \frac{\sqrt{|\Delta|}}{2} \left( \sum_{i \neq 0} P_0 (P_i \tilde{\psi}) + \sum_{i \neq 0} (\tilde{\psi} P_i) P_0 \right) + O(\beta^{-1})
$$

$$
K^2_\Delta = \left[ \frac{1}{4} X + (1 - \sqrt{2}) Y^2 \right] P_0 + \frac{\sqrt{2} - 1}{4} \sqrt{|\Delta|} Y \left( P_0 \tilde{\psi} + \tilde{\psi} P_0 \right) + O(\beta^{-1})
$$

where

$$
X = \sum_{x \in \Delta} \tilde{\psi}_x^2, \quad Y = \frac{1}{\sqrt{|\Delta|}} \sum_{x \in \Delta} \tilde{\psi}_x \tag{3.26}
$$

Note that $\text{Tr} K^2_\Delta = \sum \tilde{\psi}_x^2 / 2$ as expected. Just in the same way, we have

$$
K_\Delta(\zeta) = G_\Delta^{1/2} \zeta G_\Delta^{1/2} = G_\Delta^{1/2} \left( \frac{1}{\sqrt{N}} \sum_k \frac{1}{\hat{e}_k} \hat{P}_k \frac{s}{T} \right) G_\Delta^{1/2}
$$

$$
= \left( \sum_x \zeta_x \right) \beta P_0 + \frac{1}{2} \left( \frac{|\Delta|}{\beta N} \right)^{1/2} \left[ \left( [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} \right) P_0 + P_6 ([G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) + [G_\Delta^{(0)}]^{1/2} \zeta [G_\Delta^{(0)}]^{1/2} \right]
$$

and

$$
K^2_\Delta(\zeta) = \beta^2 (\sum \zeta)^2 + \frac{1}{4 \beta N} \left( \sum \zeta \right) \beta P_0 + \frac{1}{2} \left( \frac{|\Delta|}{\beta N} \right)^{1/2} \left( \sum \zeta \right) \left( \sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} \right) P_0
$$

$$
+ \left( \frac{|\Delta|}{N} \right)^{1/2} \left( \sum \zeta \right) + \frac{1}{4 \beta N} (\sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) \left( P_0 ([G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) + ([G_\Delta^{(0)}]^{-1/2} \frac{s}{T})^2 P_0 \right)
$$

$$
+ \frac{\sqrt{|\Delta|}}{2 \sqrt{\beta N}} \left( P_0 \frac{s}{T} \circ \zeta \right) [G_\Delta^{(0)}]^{1/2} + [G_\Delta^{(0)}]^{1/2} \left( \frac{s}{T} \circ \zeta \right) P_0
$$

$$
+ \frac{|\Delta|}{4 N \beta} ([G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) P_0 ([G_\Delta^{(0)}]^{-1/2} \frac{s}{T}) + [G_\Delta^{(0)}]^{1/2} \zeta [G_\Delta^{(0)}]^{1/2}
$$
Here $\zeta_x = N^{-1/2}(G_{\Delta}^{(0)})^{1/2}(s/T)(x) = x_k y_k$ for two vectors $x$ and $y$, $e_0 = \beta|\Delta| - O(|\Delta| \log |\Delta|)$ and we have used $P_0 P_i = 0 (i \neq 0)$ and

$$P_0(G_{\Delta}^{-1/2} \frac{s}{T})[G_{\Delta}^{(0)}]^{1/2} \zeta[G_{\Delta}^{(0)}]^{1/2} = P_0(\frac{s}{T} \circ \zeta)[G_{\Delta}^{(0)}]^{1/2}$$

We can obtain similar expressions for $K(\psi)^n$ etc., and $R_3$ is represented by these functions of $\psi$ and $\zeta$. We decompose our set $\{s_x; s_x \geq 0, x \in \Delta\}$ into 2 regions:

1. **small $s$ region**
2. **large $s$ region**

and each region is also decomposed into large $\psi$ region and small $\psi$ region, where the small $\psi$ field $S_{\Delta}(\psi)$ means the set of $\psi$ such that

$$S_{\Delta}(\psi) = \{\psi_x = \frac{1}{\sqrt{2}}(G_{\Delta}^{-1} \tilde{\psi})(x), \sum_{x \in \Delta} \tilde{\psi}_x^2 \leq N^{1-2\epsilon}\}$$

and small $s$ field $S_{\Delta}(s)$ means the set of $s_x$ such that

$$S_{\Delta}(s) = \left\{s_x = n(x) + \sqrt{n(x)} \tilde{s}(x) \geq 0, \frac{1}{N\beta} \sum_{n.n.} \left(\frac{s_x}{T_x} - \frac{s_y}{T_y}\right)^2 \leq O(1) \right\}$$

3.3 **Small field Region of $s_x$**

For small smooth $\{s_x\}$, we see that $\det_{3}^{-N/2}(1 - \kappa J_{\Delta}(\psi))$ yields a convergent small factor uniformly in $\psi_x$. Put

$$\det_{3}^{-N/2}(1 + \kappa J_{\Delta}(\psi)) = \exp[\mathcal{E}_3]$$

Then

$$|\mathcal{E}_3| = \left| \frac{4}{3 \sqrt{N}} \text{Tr} J_{\Delta}^3 + \cdots \right| = o(1) \text{Tr} (\zeta_{\Delta})^2$$

Contrary to the above, we must be careful about $R_3$ which depend on $\psi$ sensitively.

3.3.1 **small $\psi$ region**

We first assume $\psi$ are small. Let us begin our calculation

$$I = \frac{1}{Z_\Delta^2} \int \exp[\mathcal{E}_3 + R_3] \det_{3}^{-N/2}(1 + i\kappa K_{\Delta}(\psi)) \exp[- <\psi, [G^{(0)}_x][\psi]>] \prod \frac{d\psi_x}{\sqrt{2\pi}}$$

$$= \frac{1}{Z_\Delta^2} \int \exp[\mathcal{E}_3 + R_3] \det_{3}^{-N/2}(1 + i\kappa K_{\Delta})d\nu_{\Delta}$$

$$d\nu_{\Delta} = \exp[-\frac{1}{2} \sum \tilde{\psi}_x^2] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}$$
by decomposing \( \{ \tilde{\psi}_x \in R; x \in \Delta \} \) into small field region

\[
S_\Delta = \{ \sum_x \tilde{\psi}_x^2 < |\Delta| N^\varepsilon \}, \quad \varepsilon \in (0, 1)
\]  

(3.30)

and its compliment \( S^c \), where the normalization constants \( Z^{(0)}_\Delta \) and \( \tilde{Z}^{(0)}_\Delta \) are defined in the obvious way. Thus we evaluate

\[
I = I_S + I_{S^c}
\]

(3.31)

where

\[
I_S = \frac{1}{\tilde{Z}^{(0)}_\Delta} \int_S (1 + \mathcal{E}_3 + R_3 + O(R_3^2)) \det_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta
\]

(3.32)

\[
I_{S^c} = \frac{1}{\tilde{Z}^{(0)}_\Delta} \int_{S^c} \det_1^{-N/2}(1 + i\kappa K_\Delta(\psi - i\hbar + i\zeta)) e^{i\sqrt{N}\beta \sum_x (\psi_x - i\hbar_x + i\zeta_x)} \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}
\]

(3.33)

We first calculate the small field contribution \( I_S \) given by

\[
I_S = \frac{\langle \chi SD_\Delta \rangle}{\langle D_\Delta \rangle} \left\{ 1 + \frac{\langle \chi S \mathcal{E}_3 \rangle}{\langle D_\Delta \rangle} + \frac{\langle \chi S R_3 \rangle}{\langle D_\Delta \rangle} + \frac{\langle \chi S O(R_3^2) \rangle}{\langle D_\Delta \rangle} \right\}
\]

(3.34)

where

\[
D_\Delta \equiv \det_3^{-N/2}(1 + i\kappa K_\Delta), \quad \langle A \rangle = \int A d\nu_\Delta
\]

and

\[
\langle A; B \rangle = \int ABd\nu - (\int A d\nu)(\int B d\nu)
\]

We calculate \( \langle D \rangle \) and \( \langle \chi SD \rangle \) first. We assumed that

\[
\left| \frac{|\Delta| - 2}{2} \right| \leq N^{1/3 - 2\varepsilon}, \quad 0 < \varepsilon << 1
\]

(3.35)

Then

\[
\langle \chi SD \rangle = \int_{\chi S} \det_3^{-N/2}(1 + i\kappa K_\Delta) d\nu_\Delta = 1 - O(N^{-1/3})
\]

(3.36)

To bound \( \langle \chi S^c D \rangle \), we use the bounds (3.3), and set \( r^2 = 2TrK_\Delta^2 = \sum \tilde{\psi}_x^2 \). Then for \( R^2 > \rho_0 = (|\Delta| - 2)/2 \), we have that

\[
\frac{(|\Delta| - 2)!}{(2\pi)^{|\Delta|/2}} \int_{r > R} \left( \frac{1}{1 + \frac{2}{N} r^2} \right)^{N/4} r^{|\Delta| - 1} dr \leq O(\exp[-N^{1/3}])
\]
This means that

\[
\frac{< \chi_S D_\Delta >}{< D >} = \frac{< \chi_S D_\Delta >}{< \chi_S D > + < \chi_S^c D >} = 1 - O(\exp[-cN^{1/3}])
\]  (3.37)

Estimates are straightforward and we see that the most significant contribution is from \( \text{Tr} K_\Delta^2(\psi)K_\Delta(\zeta) \) in \( R_3 \) and we have:

\[
< \chi_S R_3 > = -\frac{|\Delta|}{\sqrt{N}} \left( c_1 \beta \left( \sum_x \zeta_x \right) + \frac{c_2}{\sqrt{\beta |\Delta|}} \left( \sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} \right) \right) - \frac{\sqrt{|\Delta|}}{\sqrt{2\beta N}} \left( \sum [G_\Delta^{(0)}]^{-1/2} \frac{s}{T} \right) - \frac{1}{\sqrt{N}} \text{R}G_\Delta^{(0)} \zeta + \text{(smaller terms)}
\]  (3.38)

where \( c_i = 1 + O(|\Delta|^{-1}) \) (\( i = 1, 2 \)) are positive constants. Moreover we have (see [8]):

\[
\sum_{x \in \Delta} \zeta_x = \frac{1}{\sqrt{N}} \sum_{x \in \Delta} \left( [G_\Delta^{02}]^{-1/2} \frac{s}{T} \right) (x) = \frac{1}{\sqrt{N}} \left( \sum_{x \in \partial \Delta} \frac{1}{\beta} \delta_{\partial \Delta}(x) \frac{s_x}{T_x} + O(\beta^{-3}) \right) \]  (3.39)

\[
\delta_{\partial \Delta}(x) = O\left( \frac{1}{\beta \sqrt{\Delta}} \right) \geq 0 \]  (3.40)

and

\[
\text{Tr} G_\Delta^{(0)} \zeta = \beta \sum \zeta_x - \text{R}e_0 P_0 \zeta
\]

\[
= O\left( \frac{\log |\Delta|}{|\Delta|} \right) \left( \sum \zeta_x \right) - (\beta - \frac{\sigma_0}{|\Delta|}) \sum \frac{s_x}{T_x} + \frac{1}{2\beta \sqrt{N} |\Delta|} \sum \frac{s_x}{T_x}
\]

Then the largest contribution comes from \( < \chi_S R_3 > \) and is negative, and other contributions can be made less than \( \frac{1}{N\beta} \sum \frac{s_x}{T_x} \)

3.3.2 large \( \psi \) region

For \( \{ \tilde{\psi} \} \not\in S_\Delta \), we start with

\[
I_{S^e} = \frac{1}{2\pi} \int_{S^e} \det^{-N/2}(1 + i\kappa K_\Delta(\psi - ih + i\zeta))e^{i\sqrt{N}\beta \sum_x (\psi_x - ih_x + i\zeta)}
\]

\[
\times \exp \left[ -\frac{2}{N} < \frac{s}{T}, [G_\Delta^{02}]^{-1} \frac{s}{T} > + 2\kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}
\]

\[
= \frac{1}{2\pi} \int_{S^e} \det^{-N/2}(1 + i\kappa K_\Delta(\psi))e^{i\sqrt{N}\beta \sum_x \psi_x}
\]

\[
\times \det^{-N/2}(1 + \kappa J_\Delta(h - \zeta))e^{i\sqrt{N}\beta \sum_x (h_x - \zeta_x)}
\]

\[
\times \exp \left[ -\frac{2}{N} < \frac{s}{T}, [G_\Delta^{02}]^{-1} \frac{s}{T} > + 2\kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}
\]  (3.41)

\[
= \frac{1}{2\pi} \int_{S^e} \det^{-N/2}(1 + i\kappa K_\Delta(\psi))e^{i\sqrt{N}\beta \sum_x \psi_x}
\]

\[
\times \exp \left[ -\frac{2}{N} < \frac{s}{T}, [G_\Delta^{02}]^{-1} \frac{s}{T} > + 2\kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > \right] \prod \frac{d\tilde{\psi}_x}{\sqrt{2\pi}}
\]  (3.42)
where
\[ J_{\Delta}(h - \zeta) = \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi)}} K_{\Delta}(h - \zeta) \frac{1}{\sqrt{1 + i\kappa K_{\Delta}(\psi)}} \]  
(3.43)

and then
\[
\begin{align*}
\det^{-N/2}(1 + \kappa J_{\Delta}(h - \zeta)) & e^{\sqrt{N}\beta \sum x(h_x - \zeta_x)} \\
\times \exp \left[ -\frac{2}{N} \left< \frac{s}{T}, [G_{\Delta}^{02}]^{-1} \frac{s}{T} > + 2\kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > \right] \\
= \det^{-N/2}(1 + \kappa J_{\Delta}(h - \zeta)) \\
\times \exp \left[ -\frac{1}{N} \left< \frac{s}{T}, [G_{\Delta}^{02}]^{-1} \frac{s}{T} > + \kappa < h, \frac{s}{T} > + i\kappa < \psi, \frac{s}{T} > + < h, [G_{\Delta}^{02}]h > \right] \\
\times \exp \left[ \text{Tr} \left( \frac{2iK_{\Delta}(\psi)}{1 + i\kappa K_{\Delta}(\psi)} \right) K_{\Delta}(h - \zeta) + \text{Tr}(J_{\Delta}^2(h - \zeta) - K_{\Delta}^2(h - \zeta)) \right]
\end{align*}
\]

and
\[
\begin{align*}
\text{Re} \text{Tr} J_{\Delta}^2(h - \zeta) & \leq \text{Tr} K_{\Delta}^2(h - \zeta) \\
= \frac{1}{N} \left< \frac{s}{T}, [G_{\Delta}^{02}]^{-1} \frac{s}{T} > - \kappa < h, \frac{s}{T} > + < h, [G_{\Delta}^{02}]h > \right>
\end{align*}
\]

Then putting \( S_{\Delta}^{c} = \bigcup_{k=1}^{\infty} S_{k} \) where
\[ S_{k} = \{ \{ \tilde{\psi}_{x} \}; kN^{1-2\epsilon} \leq \sum \tilde{\psi}^{2} \leq (k+1)N^{1-2\epsilon} \} \]
we estimate the integral on each shell of \( S^{c} \):
\[
\int_{S_{k}} \left| \det^{-N/2}(1 + i\kappa K_{\Delta}(\psi)) \right| \prod \frac{d\tilde{\psi}}{\sqrt{2\pi}} \leq \frac{(|\Delta| - 2)!!}{(2\pi)^{|\Delta|/2}} \frac{(|\Delta| - 1)/2}{(1 + 2kN^{-2\epsilon})^{N/4}}
\]

3.3.3 integration over small-smooth \( s_{x} \)

It remains to integrate over \( \{ s_{x} = n_{x} + \sqrt{n_{x}}\tilde{s}_{x} \} \) such that \( 0 < s_{x} < N\beta \) and \( |s_{x} - s_{x+\mu}| < \sqrt{N}\beta \). Since the contribution from \( I_{S_{\Delta}^{c}} \) is negligible, we can apply the previous methods of analysis: we set
\[
\begin{align*}
\nu_{n}(s) &= \frac{1}{(n - 1)!} e^{-n - \sqrt{n}s} (n + \sqrt{n}\tilde{s})^{n - 1} \sqrt{n}d\tilde{s} \\
&= \frac{e^{-n}n^{n}}{(n - 1)!} \sqrt{n} \exp[-\sqrt{n}\tilde{s} + (n - 1) \log(1 + \tilde{s}/\sqrt{n})]d\tilde{s} \\
&= \exp\left[ -\frac{1}{2} s^{2} + O(s/\sqrt{n}) \right] \frac{d\tilde{s}}{\sqrt{\pi}}
\end{align*}
\]

We note that \( K_{\Delta}(x, y) \sim \frac{1}{\sqrt{2|\Delta|}} (\tilde{\psi}(x) + \tilde{\psi}(y)) \) is not of short range, though \( K_{\Delta}(x, y) = O(|\Delta|^{-1/2}) \). This long range nature of the interaction is expected compensated by the Anderson localization like phenomena.
### 3.4 Large field Region of $s_x$

For $\{s_x; s_x > N\beta, \exists x \in \Delta\}$ or for $\{s_x; |s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1\}$, we need an a priori bound to estimate the two-point function. Continuing the argument in the previous section (3.1), we start from

$$D_{\Delta}(n)[C_0]$$

$$= \frac{1}{Z_\Delta} \int \frac{\exp[-<\psi - ih, G^{\Delta 2}_{\Delta}(\psi - ih)>]}{[\prod_{x \in \Delta} (4 + m^2 + \kappa(\psi_x + h_x))^{n_x}] \det_3^{N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))} \prod_{x \in \Delta} d\psi_x$$

$$= \frac{1}{Z_{\Delta}^{(0)}} \frac{1}{\prod_{x \in \Delta} T^{n_x}_x} \int_0^\infty \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^\infty \prod_{x \in \Delta} \left( \sum_k \chi^{(k)}_x(s_x) \right) d\psi_x$$

$$\times \exp \left[ - <(\psi - ih + i\zeta), G^{\Delta 2}_{\Delta}(\psi - ih + i\zeta) > - \frac{1}{N} < \frac{1}{T}s, [G^{\Delta 2}_{\Delta}]^{-1} \frac{1}{T}s > + \kappa < h, \frac{s}{T} > \right]$$

$$\times \det_3^{-N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))$$

(3.44)

We choose $h_x = c_x/\sqrt{N}$. Then

$$\frac{1}{Z_{\Delta}^{(0)}} \frac{1}{\prod_{x \in \Delta} T^{n_x}_x} \int_0^\infty \prod_{x \in \Delta} d\nu_{n_x}(s_x) \int_{-\infty}^\infty \prod_{x \in \Delta} \left( \sum_k \chi^{(k)}_x(s_x) \right) d\psi_x$$

$$\times \exp \left[ - <(\psi - ih + i\zeta), G^{\Delta 2}_{\Delta}(\psi - ih + i\zeta) > - \frac{1}{N} < \frac{1}{T}s, [G^{\Delta 2}_{\Delta}]^{-1} \frac{1}{T}s > + \kappa < h, \frac{s}{T} > \right]$$

$$\times \det_3^{-N/2} (1 + i\kappa K_{\Delta}(\psi_{\Delta}) + \kappa K_{\Delta}(h_{\Delta}))$$

(3.45)

We choose $h_x = c_x/\sqrt{N}$. Then

$$< h, G^{\Delta 2}_{\Delta}h > \leq \frac{(\sum c_x)^2}{N} \leq \frac{|\Delta|^2}{N}$$

(3.46)

and we see

$$\left| \prod_{x \in \Delta} \frac{1}{(4 + m^2 + i\kappa(\psi_x - ih_x))^{n_x}} \right| \leq \left( \frac{1}{4 + m^2 + \frac{c}{\beta N}} \right)^{\Sigma n_x}$$

(3.47)

Then if $\sum n(x)$ is so large that $\sum n(x) h(x)/\sqrt{N} > |\Delta|^2/N$, namely if $\sum n_x > \beta|\Delta|^2$, we easily see that the following a priori bound holds

$$D_{\Delta}(n) \leq \left( \frac{1}{4 + m^2_{\text{eff}}} \right)^{\Sigma n_x}$$

(3.48)

$$m^2_{\text{eff}} = m^2 + \alpha^2, \quad \alpha^2 = \frac{c}{N\beta}$$

(3.49)

Therefore in the following discussion, we assume that $\sum n_x \leq \beta|\Delta|^2$ and $\{s_x = n_x + \sqrt{n_x} \tilde{n}_x, x \in \Delta\}$ satisfy

1. $s_x \geq N\beta, \exists x \in \Delta$, or
2. $|s_x - s_{x'}| > \sqrt{N\beta}, \exists x \in \Delta, \exists x' \in \Delta, |x - x'| = 1$

If (1) occurs, then the factor

$$d_n(s) = \frac{s^n}{(n-1)!} e^{-s} ds$$
restricted on this region yields a small coefficient less than
\[
\exp[-\frac{1}{2}N\beta] \leq \exp[-\sum \frac{n_x}{N\beta}]
\]
If (1) does not take place and (2) happens, then we can implement the complex deformation
\[
\psi_x \rightarrow \psi_x + i\tau\zeta_x,
\]
where \(\zeta = (N)^{-1/2}[G_{\Delta}^{02}]^{-1}(s/T)\) and \(0 < \tau \leq 1\), and we see that the following factor arises from the complex deformation:
\[
\exp[-\frac{1-(1-\tau)^2}{N} < \frac{s}{T}, [G_{\Delta}^{02}]^{-1}\frac{s}{T}>] \leq \exp[-\frac{1-(1-\tau)^2}{2N\beta} < \frac{s}{T}, (-\Delta)\frac{s}{T}>]
\]
(3.50)

On the other hand, since
\[
||\kappa K_{\Delta}(\tau\zeta)||^2 = \frac{4\tau^2}{N^2} < \frac{s}{T}, [G_{\Delta}^{02}]^{-1}\frac{s}{T}>
\]
(3.51)
we have the bound
\[
|\det^{-N/2}_3(1+\kappa K_{\Delta}(\tau\zeta))| \leq \exp\left[O\left(\frac{1}{\sqrt{N}}\right)||K_{\Delta}(\tau\zeta)||^2_2\right]
\]
(3.52)
which is close to 1 and has no effects on the bound (3.50) if \(N\) is large.

### 4 Conclusions and Discussions

We have shown that if the non-local factor
\[
\prod_i \det^{-N/2}_3(1 + W(\Delta_i, \Lambda_i))
\]
are discarded, then the resultant system exhibits exponential clustering for all \(\beta\) if \(N\) is large enough:
\[
<s_0s_x> \sim \int \frac{1}{-\Delta + m^2 + i\kappa\psi}(0, x) \prod d\mu_\Delta(\psi_\Delta),
\]
(4.1)
\[
\leq \exp[-m_{eff}|x|]
\]
(4.2)
where \(m_{eff} = m^2 + c(N\beta)^{-1}\) and
\[
d\mu_\Delta(\psi_\Delta) = \det^{-N/2}_3(1 + i\kappa K_{\Delta})d\nu_\Delta
\]
(4.3)
is the complex measure localized to each block \(\Delta\) of size \(L \times L\) in \(Z^2\). The assumption \(N \gg 1\) is to simplify the large field problem and could be removed by additional efforts. The smallness of \(W(\Delta, \Lambda)\) is due to the Anderson localization type arguments which remains to be justified [8, 9].
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References


