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Triviality of Hierarchical $P(\phi)$ Model

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Abstract

We consider the Kadanoff-Wilson renormalization group (RG) [8] for a class of hierarchical $P(\phi)$ model above four dimensions by using Gawędzki and Kupiainen's analysis. We prove triviality for the class, namely, prove existence of critical trajectory that leads to the Gaussian fixed point.

KEY WORDS: Hierarchical model; triviality; renormalization group; $P(\phi)$ model.

1 Introduction

Hierarchical spin model is an equilibrium statistical mechanical system introduced by Dyson, Bleher and Sinai [3] [1] [2]. This model is known as a model suitable for tracing block spin renormalization group (RG) trajectories, i.e., the RG transformation is reduced to the following nonlinear transformation $\mathcal{R}$ of a function (single spin potential) $v = v(\phi)$:

$$\exp[-\mathcal{R}v(\phi)] \\ = \frac{\int \exp[-\frac{1}{2}L^d[v(L^{-(d-2)/2}\phi + z) + v(L^{-(d-2)/2}\phi - z)]]d\nu(z)}{\int \exp[-L^4 v(z)]d\nu(z)} \tag{1}$$

where $d\nu(z) = \frac{1}{(2\pi)^{1/2}}\exp(-\frac{1}{2}z^2)dz$, and L is an even integer valued constant. It is easy to see that the trivial function $v(\phi) \equiv 0$ is a fixed point of $\mathcal{R}$, which we call the Gaussian fixed point. If, for a class of single spin potentials, RG trajectories with initial potentials in the class, converge to the Gaussian fixed point, then we say that the class of functions is trivial. Gawedzki and Kupiainen studied this recursion in detail, and proved (among other

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things) the triviality for ϕ^4 models with some small ϕ^4 coupling constant in 4 dimensions [4] [5] [6]. See [6] for a review of their results together with the relation of (1) and the hierarchical spin model. The purpose of our work is to extend the results of Gawedzki and Kupiainen and prove triviality for a wider class of potentials. To be specific, We consider the following class of single spin potentials:

$$v_0(\phi) = \mu\phi^2 + \lambda P(\phi), \tag{2}$$
$$P(\phi) = \sum_{k=2}^{N} a_{2k} : \phi^{2k} :, \tag{3}$$

where $:\phi^{2k}:$ is given by

$$\int_{-\infty}^{\infty} L^d \sum_{\pm} : (L^{-(d-2)/2}\phi \pm z)^{2k} : d\nu(z) = L^{2k-(k-1)d} : \phi^{2k} : . \tag{4}$$

(For example : $\phi^6 := \phi^6 - \frac{15}{1-L^{-2}}\phi^4 - \frac{45}{1-L^{-4}}\phi^2 + \frac{90}{(1-L^{-2})(1-L^{-4})}\phi^2 +$ "const".) Let us define a class of initial single spin potentials $\mathcal{V}_0(N, L, D, C_1, n_0)$ satisfying the following conditions for constants L, D, C_1, and n_0,

(Pa) for $|\mathbf{Im}\phi| < C_1 n_0^{1/2N}$, $\exp[-v_0(\phi)]$ is analytic, positive for real ϕ, even, and satisfies

$$|e^{-(v_0)_{\geq 4}(\phi)}| \leq \exp[D - \sum_{k=2}^{N} a_{2k,0}^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,0}(\mathbf{Im}\phi)^{2k}], \tag{5}$$

where $\{A_{2k}\}$ are universal constants, and $a_{2k,0} = \lambda \cdot a_{2k}$

(Pb) for $|\phi| < C_1 n_0^{1/2N}$, $(v_0)_{\geq 4}(\phi)$ is analytic,

$$(v_0)_{\geq 4}(\phi) = \lambda_0 \sum_{k=2}^{N} : \phi^{2k} : + (v_0)_{\geq 2N+2}(\phi) \tag{6}$$

with

$$\frac{C_{--}L^{-4}}{n_0} \leq a_{4,0} \leq \frac{C_{++}L^{-4}}{n_0}, \quad C_{--}(N) > \frac{1}{48}, C_{++}(N) < \frac{1}{24}, \tag{7}$$
$$C_0 L^{-4} n_0^{-1} < a_{2k,0} < C_0' L^{-4} n_0^{-1}, C_0 > 0 \tag{8}$$
$$|(v_0)_{\geq 2N+2}(\phi)| \leq n_0^{-3/2N}. \tag{9}$$

We will prove the following for our class.

Theorem 1.1 In $d \geq 4$, there exist positive constants:

$$D(N),\ \bar{C}_1(N, L, D) \geq L,\ \bar{n}_0(N, L, D, C_1) \geq L^{48},$$

such that the following holds. Let $C_1 \geq \bar{C}_1(N, L, D)$, $n_0 \geq \bar{n}_0(N, L, D, C_1)$. Define the RG as (1). Then there exists $\mu_{crit} \in \mathbf{R}$ such that the iterates v_n of the recursion converge to zero uniformly on compacts in $\mathbf{C}^1$, if we start from $v_0 \in \mathcal{V}_0(N, L, D, C_1, n_0)$ with $\mu_0 = \mu_{crit}$.

To prove of the triviality for (1) with potentials of the form (Pa)-(Pb), we will show that the parameters will enter the region where the Theorem of Gawędzki and Kupiainen [6] can be applied (i.e. G-K region), after some iterations (finite time iterations) of the RG. The point of our proof is to change the induction hypothesis after some iterations to reflect the dominant terms in the potential. The proof goes along the following line. In the beginning, we are in the region where $(v_n)_{\geq 2N}(\phi)$ is dominant. For properly chosen initial data, $(v_n)_{\geq 2N}(\phi)$ decreases rapidly, and we then go into the region where ϕ^{2N-2} term of $v_n(\phi)$ is comparable to $(v_n)_{\geq 2N}(\phi)$. As the recursion proceeds, the ϕ^{2N-2} term becomes positive and dominant, and then ϕ^{2N-4} becomes positive and dominant etc. After all, $v_n(\phi)$ enters the G–K region. To trace the trajectory, we will divide up the induction into $N + 1$ parts along the trajectory and impose different induction hypothesis for the $a_{2k,n}$ dominant regime for $k = N, N - 1, \cdots, 2, 1$. (Compare the induction hypotheses L1.2a and L1.2b with L1.3a and L1.3b, respectively.) We will prove this by means of two lemmas. First, for $N > m > 2, n \geq 0$, let $\mathcal{V}_n^m(N, L, D, C_1, n_0)$ be the class of potentials v_n satisfying:

L1.2a for $|\mathbf{Im}\phi| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, $\exp[-v_n(\phi)]$ is analytic, positive for real ϕ, even, and

$$|e^{-(v_n)_{\geq 4}(\phi)}| \leq \exp[D - \sum_{k=2}^{N} a_{2k,n}^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n}(\mathbf{Im}\phi)^{2k}], \tag{10}$$

L1.2b for $|\phi| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, $(v_n)_{\geq 4}(\phi)$ is analytic, and

$$(v_n)_{\geq 4}(\phi) \ = \ \sum_{k=2}^{N} a_{2k,n}\phi^{2k} + (v_n)_{\geq 2N+2}(\phi), \tag{11}$$

with

$$|a_{4,n} - L^{(d-2k)n}a_{2k,0}| \leq nL^{(d-2k)n}n_0^{-1-2/N},\ \text{for } k = 1, \cdots, N \tag{12}$$

$$|(v_n)_{\geq 2N+2}| \leq (n_0^{-3/2N})L^{-n/N}. \tag{13}$$

Lemma 1.2 Let $3 \le m \le N$ There exist constants

$$D(N), \bar{C_1}(N,L,D) \ge L, \bar{n_0}(N,L,D,C_1) \ge L^{48} \tag{14}$$

such that the following holds. Let $1/2N > \delta > 0$, $C_1 \le \bar{C_1}(N,L,D)$, $n_0 \ge \bar{n_0}(N,L,D,C_1)$ and $n \ge 0$ satisfy the inequality

$$(L^{(d-2m)n} n_0^{-1})^{1/2m} \ge \begin{cases} (L^{(d-2m+2)n} n_0^{-1})^{1/(2m-2)} & \text{if } m > 3, \\ (n_0+n)^{-1/4} & \text{if } m = 3. \end{cases} \tag{15}$$

Suppose also that $v_0 \in \mathcal{V}_0(N,L,D,C_1,n_0)$, and $v_n \in \mathcal{V}_n^m(N,L,D,C_1,n_0)$. Then, there exists a closed interval $J_n \subset I_n = [-(n_0+n)^{-1-\delta}, (n_0+n)^{-1-\delta}]$ such that for μ_n running through J_n, $v_{n+1} \in \mathcal{V}_{n+1}^m(N,L,D,C_1,n_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

Since $\mathcal{V}_0(N,L,D,C_1,n_0) = \mathcal{V}_0^N(N,L,D,C_1,n_0)$, we can iterate Lemma 1.2 for $m = N$, and for $n \ge 0$ as long as (15) is satisfied. For $3 \le m \le N-1$, put

$$n_m = \min\{n \in \mathbf{N} | (L^{(d-2m)n} n_0^{-1})^{1/2m} \le (L^{(d-2m+2)n} n_0^{-1})^{1/(2m-2)}\}. \tag{16}$$

Obviously, $\frac{1}{d}\log_L n_0 \le n_m < \log_L n_0$. By Lemma 1.2 for $m = N$,

$$v_{n_{N-1}} \in \mathcal{V}_{n_{N-1}}^N(N,L,D,C_1,n_0) = \mathcal{V}_{n_{N-1}}^{N-1}(N,L,D,C_1,n_0). \tag{17}$$

Therefore we can restart applying Lemma 1.2 for $m = N-1$. Since

$$\mathcal{V}_{n_{m-1}}^m(N,L,D,C_1,n_0) = \mathcal{V}_{n_{m-1}}^{m-1}(N,L,D,C_1,n_0) \tag{18}$$

for each m, this can be continued until $n = n_3$. Let

$$n_2 = \min\{n : (n_0+n)^{1/4} \le (L^{2n} n_0)^{1/6}\}, \tag{19}$$

and let us define a class of single spin potentials $\mathcal{V}_{n_2+n}^2(N,L,D,C_1,n_0)$ satisfying:

L1.3a for $|\mathbf{Im}\phi| < C_1(n_0+n_2+n)^{1/4}$, $\exp[-v_{n_2+n}]$ is analytic and positive for real ϕ, even, and

$$|e^{-(v_{n_2+n})_{\ge 4}(\phi)}| \\ \le \exp[D - \sum_{k=2}^{N} a_{k,n_2}^{1/2k} |\phi|^2 + \sum_{k=2}^{N} A_{2k} a_{2k,n_2+n} (\mathbf{Im}\phi)^{2k}], \tag{20}$$

L1.3b for $|\phi| < C_1(n_0 + n_2 + n)^{1/4}$, $(v_{n_2+n})_{\geq 4}(\phi)$ is analytic,

$$(v_{n_2+n})_{\geq 4}(\phi) = \sum_{k=2}^{N} a_{2k,n} : \phi^{2k} : +(v_{n_2+n})_{\geq 2N+2}(\phi), \tag{21}$$

with

$$|a_{4,n_2+n} - a_{4,0}| \leq (n_2 + n) n_0^{-1-2/N}, \tag{22}$$

$$|a_{2k,n_2+n} - L^{(d-2k)(n_2+n)} n_0| \leq (n_2 + n) L^{(d-2k)(n_2+n-1)} n_0^{-1-2/N}, \tag{23}$$

$$|(v_{n_1+n})_{\geq 2N+2}(\phi)| \leq L^{-3n-n_2/N} n_0^{-3/2N}. \tag{24}$$

Lemma 1.3 There exist constants

$$N, D(N), \bar{C_1}(N, L, D) \geq L, \bar{n_0}(N, L, D, C_1) \geq L^{48}$$

such that the following holds. Let $N^{-1} > \delta > 0$, $C_1 \geq \bar{C_1}(N, L, D)$, $n_0 \geq \bar{n_0}(N, L, D, C_1)$, $\log_L n_0 \geq n \geq 0$. $v_0(\phi) \in \mathcal{V}_0(N, L, D, C_1, n_0)$, and $v_{n_2+n} \in \mathcal{V}^2_{n_2+n}(N, L, D, C_1, n_0)$. Then, there exists a closed interval $J_{n_2+n} \subset I_{n_2+n} = [-(n_0 + n_2 + n)^{-1-\delta}, (n_0 + n_2 + n)^{-1-\delta}]$ such that for μ_{n_2+n} running through J_{n_2+n}, $v_{n_2+n+1} \in \mathcal{V}^2_{n_2+n+1}$. Further, the map $\mu_{n_2+n} \mapsto \mu_{n_2+n+1}$ sweeps I_{n_2+n+1} continuously.

The proof of Lemma 1.3 is close to the proof of Lemma 1.2. A different point from Lemma 1.2 is the difference in the condition of the region where $v_{n_2+n}(\phi)$ satisfies analyticity. In fact we require that $\exp[-v_{n_2+n}(\phi)]$ is analytic for $|\mathbf{Im}\phi| < C_1(n_0+n_2+n)^{1/4}$ in Lemma 1.3. Because ϕ^4 term becomes dominant compared with $(v_{n_1+n})_{\geq 6}(\phi)$ this time. With Lemma 1.3 we can continue iterations, and we can make sure that after a finite number of iterations, this potential is in the region where Gawędzki and Kupiainen studied [6]:

G-Ka $e^{-(v_n)_{\geq 4}(\phi)}$ is analytic in $|\mathbf{Im}\phi| < C_1(n_0 + n)^{1/4}$, positive for real ϕ, even and

$$|\exp[-(v_n)_{\geq 4}(\phi)]| \leq \exp[D - \lambda_n^{1/2}|\phi|^2 + A_1\lambda_n(\mathbf{Im}\phi)^4], \tag{25}$$

G-Kb for $|\phi| < C_1(n_0 + n)^{1/4}$, $(v_n)_{\geq 4}(\phi)$ is analytic,

$$(v_n)_{\geq 4}(\phi) = \lambda_n\phi^4 + (v_n)_{\geq 6}(\phi) \tag{26}$$

with

$$\frac{C_- L^{-4}}{n_0 + n} \leq \lambda_n \leq \frac{C_+ L^{-4}}{n_0 + n}, C_- = \frac{1}{48}, C_+ = \frac{1}{24}, \tag{27}$$

$$|(v_n)_{\geq 6}(\phi)| \leq (n_0 + n)^{-3/4} \tag{28}$$

In this class $\mathcal{V}_n^{G-K}(L, D, C_1, n_0)$, Gawędzki and Kupiainen proved the following,

Theorem 1.4 (Gawędzki and Kupiainen) There exist constants D, $\bar{C}_1(L, D), \bar{n}_0(L, D, C_1)$ such that the following holds. Let $C_1 \geq \bar{C}_1(L, D)$, $n_0 \geq \bar{n}_0(L, D, C_1)$ and $n \geq 0$.

Put

$$v_n(\phi) = \mu_n - \frac{6\lambda_n}{1 - L^{-2}}\phi^2 + (v_n)_{\geq 4}(\phi) \tag{29}$$

where $(v_n)_{\geq 4}(\phi) \in \mathcal{V}_n^{G-K}(L, D, C_1, n_0)$. Then, there exists a closed interval $J_n \subset I_n$ such that for μ_n running through J_n, $(v_{n+1})_{\geq 4}(\phi) = v_{n+1}(\phi) - \mu_{n+1}\phi^2 + \frac{6\lambda_{n+1}}{1-L^{-2}}\phi^2 \in \mathcal{V}_{n+1}^{G-K}(L, D, C_1, n_0)$. Further, the map $\mu_n \mapsto \mu_{n+1}$ sweeps I_{n+1} continuously.

2 Proof of Lemma 1.2

Now we start to prove Lemma 1.2. Let $2 < m < N$, we will only prove that $v_n'(\phi) = v_{n+1}(\phi)$ is in $\mathcal{V}_{n+1}^m(N, L, D, C_1, n_0)$, if μ_n is in I_n. As before, we separate the cases into two ; small field case or large field case corresponding to the cases either $|\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$, or $|\mathbf{Im}\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$ respectively. In the small field case, we prove that $v_n'(\phi)$ satisfies **L1.2b$'$**, the condition **L1.2b** with n being replaced by $n + 1$, by using the Taylor expansion, and some estimation of the Gaussian integrals as in [6]. As for the large field region, we only investigate global behavior of $v_n'(\phi)$, i.e., we confirm that $v_n'(\phi)$ satisfies (13) of **L1.2a$'$**, the condition **L1.2a** with n being replaced by $n+1$. We use K for calculable absolute constants, whose values will vary in each occurrence.

2.1 Small field region analysis

Let $v_n \in \mathcal{V}_n^m$. We must also prepare some notations. Write $\chi_1(z) = \chi(|z| < (L^{(2m-4)n}n_0)^{1/2m})$ and throughout this subsection, we assume that ϕ is in the region $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. Note that we have to put C_1 to satisfy the inequality $|L^{-1}\phi \pm z| < C_1(L^{(2m-4)n}n_0)^{1/2m}$ for $|z| < (L^{(2m-4)n}n_0)^{1/2m}$ and $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. Next, decompose $v_{n+1}(\phi)$ as follows,

$$v_{n+1}(\phi) = v_n'(\phi) = \tilde{v}_n'(\phi) + \approx{v}_n'(\phi), \tag{30}$$

$$e^{-\tilde{v}_n'(\phi)} = \int \exp[-\frac{L^4}{2}\sum_{\pm} v_n(L^{-1}\phi \pm z)]d\nu_1(z)/(\phi = 0)_{small}, \tag{31}$$

where

$$(\phi = 0)_{small} \;=\; \int \exp[-L^4 v_n(z)] d\nu_1(z), \tag{32}$$

$$d\nu_1(z) \;\equiv\; \chi_1(z) e^{-z^2/2} \frac{dz}{\sqrt{2\pi}}. \tag{33}$$

2.1.1 Estimation of $\tilde{v_n'}(\phi)$

Let us take a logarithm of (31).

$$\begin{aligned}\tilde{v_n'}(\phi) &= \sum_{k=1}^{N} L^{4-2k}(a_{2k,n} - c_{2k,n})\phi^{2k} \\ &\quad - \log \int e^{-w_\phi(z)} d\nu_1(z) + \log(\phi = 0)_{small},\end{aligned} \tag{34}$$

where $c_{2k,n}, w_\phi(z)$ are given by

$$\sum_{k=1}^{N} a_{2k,n} : \phi^{2k} := \sum_{k=1}^{N} (a_{2k,n} - c_{2k,n})\phi^{2k}, \tag{35}$$

$$w_\phi(z) = w_0(z) + w_2(z)\phi^2 + w_4(z)\phi^4 + w_6(z)\phi^6 + w_{\geq 8}(\phi, z), \tag{36}$$

$$\begin{aligned}&w_0(z) = L^4 v_n(z) \\ &w_{2p}(z) \\ &= \sum_{k=1}^{N} L^{4-2p}\{\binom{2k}{2p}(a_{2k,n} - c_{2k,n})z^{2p} + \frac{d^{2(N-p)}}{dz^{2(N-p)}}(v_n)_{\geq 2N+2}(z)\}\phi^{2N-2p},\end{aligned} \tag{37}$$

for $p = 0, \cdots, N-1$ and

$$\begin{aligned}w_{\geq 2N+2}(\phi, z) = \frac{L^{-4}\phi^{2N+2}}{(2N+1)!}\{&\int_0^1 dt(1-t)^{2N+1} \frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(L^{-1}t\phi + z) \\ &+ \int_0^1 dt(1-t)^{2N+1} \frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(L^{-1}t\phi - z)\}.\end{aligned} \tag{38}$$

From the conditions **L1.2a** - **L1.2b**, $v_n(\phi)$ is even and analytic. We can estimate $\frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(\phi)$ on the support of $d\nu_1(z)$ as follows by using the Cauchy formula and (13),

$$\begin{aligned}&|(v_n)_{\geq 2N+2}(z)| \\ &\leq \frac{1}{(2N+2)!}\int_0^1 dt(1-t)^{2N+1}|z^{2N+2}\frac{d^{2N+2}}{dz^{2N+2}}(v_n)_{\geq 2N+2}(tz)| \\ &\leq \frac{C_1}{(2N+2)!(C_1-1)^{2N+3}} n_0^{-\frac{3}{2N}} n_0^{-\frac{(N+1)}{m}} L^{-(\frac{1}{N}+\frac{(N+1)(m-2)}{m})n}|z|^{2N+2}.\end{aligned} \tag{39}$$

$\frac{d^2}{dz^2}(v_n)_{\geq 2N+2}(z)$ to $\frac{d^{2N}}{dz^{2N}}(v_n)_{\geq 2N+2}(z)$ can be estimated as (39). From the perturbation expansion:

$$-\log \int e^{-w_\phi(z)} d\nu_1(z)$$
$$= -\log \int d\nu_1(z) + \langle w_\phi(z) \rangle_0 - \int_0^1 dt(1-t) \langle w_\phi(z); w_\phi(z) \rangle_t, \qquad (40)$$

where

$$\langle \cdots \rangle_t \equiv \int \cdots e^{-tw_\phi(z)} d\nu_1(z) / \int e^{-tw_\phi(z)} d\nu_1(z). \qquad (41)$$

Now, we shall estimate each part of (40). Using the estimation of the Gaussian integrations, we get

$$\begin{aligned} \langle w_\phi(z) \rangle_0 &= L^4 \langle v_n(z) \rangle_0 \\ &+ \sum_{p=0}^{N-1} \sum_{k=1}^{N} L^{4-2k} \binom{2k}{2p} (a_{2k,n} - c_{2k,n}) \phi^{2N-2p} (2p-1)!! \\ &+ \sum_{k=2}^{N} \overset{\sim 0,0}{R_{2k}} (L, n_0, n) \phi^{2k} + \langle w_{\geq 2N+2}(\phi, z) \rangle_0, \end{aligned} \qquad (42)$$

where, the terms $\overset{\sim 0,0}{R_{2i}} (L, n_0, n), i = 1, \cdots, N$ satisfy

$$| \overset{\sim 0,0}{R_{2i}} (L, n_0, n)| \leq (n_0^{-3/2N}) n_0^{-(N+1)/m} L^{-(1/N+(N+1)(m-2)/m)n}. \qquad (43)$$

From (39) and the similar estimates for $\frac{d^2}{dz^2}(v_n)_{\geq 2N+2}, \cdots, \frac{d^{2N}}{dz^{2N}}(v_n)_{\geq 2N+2}$, we obtain,

$$|\langle w_{\geq 2N+2}(\phi, z) \rangle_0| \leq L^{4-n/N} (1 + (n_0)^{-1/m} L^{(4-2m)n/m}) (n_0^{-3/2N}). \qquad (44)$$

Next we estimate

$$\begin{aligned} \int_0^1 dt(1-t) \langle w_\phi(z); w_\phi(z) \rangle_t &= \int_0^1 dt(1-t) \sum_{i,j} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t \\ &= \int_0^1 dt(1-t) \langle w_0(z); w_0(z) \rangle_t + \int_0^1 dt(1-t) \sum_{i,j \neq 0} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t, \end{aligned} \qquad (45)$$

where

$$\tilde{w}_{2i} = \begin{cases} w_{2i}(z) \phi^{2i} & i = 0, \cdots, 2N \\ w_{\geq 2N+2}(\phi, z) & i = N+1. \end{cases}$$

The cumulants are

$$\begin{aligned} \langle \tilde{w}_{2i}; \tilde{w}_{2j} \rangle_t &= \langle e^{-tw_\phi(z)} \rangle_0^{-1} \langle \tilde{w}_{2i} \tilde{w}_{2j} e^{-tw_\phi(z)} \rangle_0 \\ &- \langle e^{-tw_\phi(z)} \rangle_0^{-2} \langle \tilde{w}_{2i} e^{-tw_\phi(z)} \rangle_0 \langle \tilde{w}_{2j} e^{-tw_\phi(z)} \rangle_0. \end{aligned} \qquad (46)$$

Note that the support of $d\nu_1(z)$ is $|z| < (L^{(2m-4)n}n_0)^{1/2m}$. From (15), we get the uniform estimate $|w_\phi(z)| \le K \cdot L^{2N}C_1^{2N}$ for $|z| < (L^{(2m-4)n}n_0)^{1/2m}$ and $|\phi| < \frac{10LC_1}{11}(L^{(2m-4)n}n_0)^{1/2m}$. Hence,

$$
\begin{aligned}
&|\sum_{(i,j)\neq(0,0)} \langle \tilde{w}_{2i}; \tilde{w}_{2j}\rangle_t| \\
&\qquad \le e^{K\cdot L^{2N}C_1^{2N}} \sum_{(i,j)\neq(0,0)} (\langle|\tilde{w}_{2i}||\tilde{w}_{2j}|\rangle_0 + \langle|\tilde{w}_{2i}|\rangle_0\langle|\tilde{w}_{2j}|\rangle_0). \qquad (47)
\end{aligned}
$$

From (37)-(38), we can estimate $|\int_0^1 dt(1-t)\sum_{(i,j)\neq(0,0)}\langle\tilde{w}_{2i};\tilde{w}_{2j}\rangle_t|$ similarly as in (39), and we obtain

$$
\begin{aligned}
&|\text{2nd term of RHS of (45)}| \\
&\qquad \le Ke^{K\cdot C_1^{2N}}L^{-2}n_0^{-2}(|\phi|^2 + \sum_{k=2}^{N} L^{-(4-2k)n-2}|\phi|^{2k}) \\
&\qquad\qquad + |\text{higher order terms}|. \qquad (48)
\end{aligned}
$$

The higher order terms are estimated as follows,

$$
|\text{higher order terms}| \le Ke^{K\cdot L^N C_1^{2N}} L^{4(N-1)-n/N}C_1^{4(N-1)}(n_0^{-4/2N}). \qquad (49)
$$

Next, we estimate $\int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t$. Since $\langle w_0(z); w_0(z)\rangle_t$ is analytic function in $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)}n_0)^{1/2m}$, by Cauchy formula we get

$$
\begin{aligned}
&|\int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t - \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}| \\
&\qquad \le K\exp(K\cdot L^{2N}C_1^{2N})\cdot L^{-2}n_0^{-2}|\phi|^2. \qquad (50)
\end{aligned}
$$

So we have,

$$
\begin{aligned}
&|\int_0^1 dt(1-t)\langle w_\phi(z); w_\phi(z)\rangle_t - \int_0^1 dt(1-t)\langle w_0(z); w_0(z)\rangle_t|_{\phi=0}| \\
&\qquad \le K\exp(K\cdot L^{2N}C_1^{2N})L^{-2}n_0^{-2}(|\phi|^2 + \cdots + L^{-(4-2N)n}|\phi|^{2N}) \\
&\qquad\qquad + |\text{higher order terms}|, \qquad (51) \\
&|\text{higher order terms}| \le Ke^{K\cdot L^{2N}C_1^{2N}} L^{4(N-1)-n/N}C_1^{4(N-1)}(n_0^{-4/2N}). \qquad (52)
\end{aligned}
$$

These coefficients are large, but not terrible, because we can take n_0 sufficiently large. In the following, we put $n_0^{1/2N} \ge K\cdot C_1^{4(N-1)}L^{4(N-1)}e^{K\cdot L^{2N}C_1^{2N}}$. From (34) and (40), we infer that

$$
\overset{\sim}{v_n'}(\phi) = \sum_{k=1}^{N} L^{4-2k}(a_{2k,n} - c_{2k,n})\phi^{2k}
$$

$$+\sum_{p=1}^{N}\sum_{k=1}^{N} L_{2k}^{4-2k} C_{2p}(a_{2k,n}-c_{2k,n})(2N-2p-1)!!\phi^{2p}$$
$$+\sum_{k=1}^{2N} \tilde{R}_{2k}\ (N,L,n_0,n)\phi^{2k}+\ (\tilde{v_n})'_{\geq 2N+2}\ (\phi), \tag{53}$$

where, the terms $\tilde{R}_{2i}\ (N,L,n_0,n), i=1,\cdots,N$ satisfy

$$|\ \tilde{R}_{2i}\ (N,L,n_0,n)| \leq L^{-10-(4-2i)n} n_0^{-2+1/2N} + |\ \tilde{R}_{2i}^{0,0}\ (N,L,n_0,n)|,$$
$$i=1,\cdots,N, \tag{54}$$

and from (44) and (52), $(\tilde{v_n})'_{\geq 2N+2}\ (\phi)$ satisfy

$$|\ (\tilde{v_n})'_{\geq 8}\ (\phi)| \leq L^{4-n/N}(1+L^{-(4-2m)n/m}(n_0)^{-1/m}+L^{-4})(n_0^{-3/2N}), \tag{55}$$

for $|\phi| < \frac{10}{11}LC_1(L^{(4-2m)n}n_0)^{1/2m}$. Notice that

$$(\phi=0)_{small} = \log\int d\nu_1(z) - \langle w_0(z)\rangle_0 + \int_0^1 dt(1-t)\langle w_0(z);w_0(z)\rangle_t|_{\phi=0}.$$

So we can check that the constant term $(\phi=0)_{small}$ vanishes. The estimate (55) is a little weaker than what we want (see (13)). So, we need a stronger estimate. Since $\tilde{v}'_n\ (\phi)$ is analytic in $|\phi| < \frac{10}{11}LC_1(L^{-(4-2m)n}n_0)^{1/2m}$, $\phi^{-2N-2}\ (\tilde{v_n})'_{\geq 8}\ (\phi)$ is also analytic in $|\phi| < \frac{10}{11}LC_1(L^{-(4-2m)n}n_0)^{1/2m}$. We obtain from the maximum principle

$$|\ (\tilde{v_n})'_{\geq 2N+2}\ (\phi)| \leq \left(\frac{|\phi|}{(10L/11)C_1(L^{-(4-2m)n}n_0)^{1/2m}}\right)^{2N+2}(n_0^{-3/2N})$$
$$\times(L^{4-n/N}(1+L^{-(4-2m)n/m}(n_0)^{-1/m}+L^{-4})), \tag{56}$$

so that for $|\phi| < C_1(L^{-(4-2m)(n+1)}n_0)^{1/2m}$,

$$|\ (\tilde{v_n})'_{\geq 2N+2}\ (\phi)| \leq \left(\frac{11}{10}\right)^{2N+2} L^{-(2N+2)(1-(4-2m)/2m)}$$
$$\times(L^{4-n/N}(1+L^{-(4-2m)n/m}(n_0)^{-1/m}+L^{-4})(n_0^{-3/2N})). \tag{57}$$

2.1.2 Estimation of $\approx\!v'_n\ (\phi)$ for $|\phi| < \frac{10}{11}LC_1(L^{-(4-2m)n}n_0)^{1/2m}$

Represent (30) as

$$\overset{\approx}{v}'_n\ (\phi) \;=\; \log\left(1+\frac{\int\exp[-\frac{1}{2}L^4\sum_{\pm}v_n(L^{-1}\phi\pm z)](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v}'_n(\phi)}(\phi=0)_{small}}\right)$$
$$+\log(\phi=0)_{small} - \log(\phi=0). \tag{58}$$

We want to prove that $\overset{\approx}{v'_n}(\phi)$ is analytic in $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$ and sufficiently smaller than $\tilde{v'_n}(\phi)$. To prove these properties, we have only to prove that

$$\frac{\int \exp[-\frac{1}{2}L^4\sum_{\pm} v_n(L^{-1}\phi \pm z)](1-\chi_1(z))d\nu(z)}{e^{-\tilde{v'_n}(\phi)}(\phi=0)_{small}} \tag{59}$$

is analytic and sufficiently small in $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. First of all, we estimate the denominator of (59). We can show that the denominator is bounded from below by a constant which depends on C_1, but not on n_0. From **L1.2b**, and (54) together with uniform estimate of $w_0(z)$ under the condition of (15), we estimate denominator as follows,

$$|\text{denominator of (59)}| \;\geq\; \exp[-K\cdot L^{2N}C_1^{2N}]. \tag{60}$$

Next, we estimate the numerator part of (59),

$$|\text{numerator of (59)}| \leq \int(1-\chi_1(z))\prod_{\pm}|\exp[-v_n(L^{-1}\phi \pm z)]|^{L^4/2}d\nu(z). \tag{61}$$

Using (10) of **L1.2a** for $|L^{-1}\phi \pm z| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, we have

$$|\text{numerator of (59)}| \leq \exp[K + L^4D + \sum_{k=2}^{N} A_{2k}C'_0C_1^{2k} - \frac{1}{4}(L^{(2m-4)n}n_0)^{1/m}]. \tag{62}$$

So,

$$|(59)| < \exp[K\cdot L^{2N}C_1^{2N} + L^4D + \sum_{k=2}^{2N} A_{2k}C'_0C_1^{2k} - \frac{1}{4}(L^{(2m-4)n}n_0)^{\frac{1}{m}}]. \tag{63}$$

For given L, D and C_1, we can take n_0 large enough to obtain

$$\text{RHS of (63)} \leq \exp[-\frac{1}{8}(L^{(2m-4)n}n_0)^{1/m}]. \tag{64}$$

This estimate is also valid for $\log(\phi=0)-\log(\phi=0)_{small}$. According to (64), we can show that $\overset{\approx}{v'_n}(\phi)$ is analytic and

$$|\overset{\approx}{v'_n}(\phi)| \leq 2e^{-1/8(L^{(2m-4)n}n_0)^{1/m}}. \tag{65}$$

2.1.3 Estimation of coefficients

Now, we assume that $|\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$ i.e. ϕ is in the small field region of $v'_n(\phi)$. Notice that the small field region is in the region $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$, so we can use the argument above. Thus, $\overset{\approx}{v'_n}(\phi)$ is analytic in the small field region of v'_n, and we can obtain power series expansion of $\overset{\approx}{v'_n}(\phi)$. With the use of Cauchy's estimate, we see that coefficients of ϕ^2 to ϕ^{2N} satisfy,

$$|\frac{1}{k!}\frac{d^k}{d\phi^k}\overset{\approx}{v'_n}(0)| \;\leq\; e^{-1/8(L^{(2m-4)n}n_0)^{1/m}}, k=2,4,\cdots,2N. \tag{66}$$

Using the bounded convergence theorem, we see that $\frac{1}{2}\frac{d^2}{d\phi^2}\overset{\approx}{v'_n}(0), \frac{1}{4!}\frac{d^4}{d\phi^4}\overset{\approx}{v'_n}(0), \cdots \frac{1}{2N!}\frac{d^{2N}}{d\phi^{2N}}\overset{\approx}{v'_n}(0)$ are continuous functions of μ_n on I_n. From (57) and (65), if n_0 is sufficiently large, then we have

$$|(v_n)'_{\geq 2N+2}(\phi)| \;\leq\; L^{-(n+1)/N}(n_0^{-3/2N}), \tag{67}$$

for $|\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$. From (4), (53), (54), and (66), we know that

$$\begin{aligned} |a_{2k,n+1} - L^{4-2k}a_{2k,n}| &= |R_{2k}(N,L,n_0,n) + \frac{1}{2k!}\frac{d^{2k}}{d\phi^{2k}}\overset{\approx}{v'_n}(0)| \\ &\leq L^{(4-2k)n}n_0^{-1-2/N}, k=3,\cdots 2N. \end{aligned} \tag{68}$$

Thus, if n_0 is sufficiently large, we have

$$|a_{2k,n+1} - L^{(4-2k)(n+1)}a_{2k,0}| \;<\; (n+1)L^{(4-2k)n}n_0^{-1-2/N} \tag{69}$$

which proves (13) of **L1.2b$'$**. From (53), (54), we know

$$|a_{4,n+1} - a_{4,n}| \;\leq\; n_0^{-1-2/N}. \tag{70}$$

Thus, we have

$$|a_{4,n+1} - a_{4,0}| < (n+1)n_0^{-1-2/N}, \tag{71}$$

which completes the proof of **L1.2b$'$**. Similarly, we get estimation of coefficient μ'_n as follows,

$$|\mu'_n - L^2\mu_n| \leq K \times n_0^{-1-2/N}. \tag{72}$$

We know that map $R : \mu \mapsto \mu'$ is continuous, and image $R(I_n)$ include I_{n+1}. So that we can take for J_{n+1} a connected component of this inverse image $R^{-1}(I_{n+1}) \subset I_n$.

This ends the analysis of the small field properties.

2.2 Large field region analysis

Next, we prove that $e^{-(v_n)'(\phi)}$ satisfy the condition **L1.2a′**. First, we prove it in the case where $|\mathbf{Re}\phi| > C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$. Next, we prove it in $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$ i.e. this region includes the small field region of $v_n'(\phi)$.

2.2.1 The case where $|\mathbf{Re}\phi| > C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$

Note that the definition of the RG (1) has the following expression

$$e^{-v_n'(\phi)} = \int \prod_{\pm} \exp[-v_n(L^{-1}\phi \pm z)]^{L^4/2} d\nu(z)/(\phi = 0). \tag{73}$$

$|\mathbf{Im}(L^{-1}\phi \pm z)| < C_1(L^{(2m-4)n}n_0)^{1/2m}$, if $|\mathbf{Im}\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$. From the condition **L1.2a**,

$$\begin{aligned}|e^{-(v_n)'_{\geq 4}(\phi)}| &\leq \exp[L^4 D - L^2 \sum_{k=2}^{N} a_{2k,n}^{1/2k}|\phi|^2 + \sum_{k=2}^{N} L^{4-2k} A_{2k} a_{2k,n} (\mathbf{Im}\phi)^{2k}] \\ &\times \int_{-\infty}^{\infty} e^{-L^4 \mu_n z^2 - L^4 \sum_{k=2}^{2N} a_{2k,n}^{1/2k} z^2} d\nu(z)/(\phi = 0). \end{aligned} \tag{74}$$

Note that, $\{a_{2k,n}\}$ are positive and sufficiently small, hence, this integral part and $(\phi = 0)$ estimated as absolute constants, so we get

$$\begin{aligned}&\text{RHS of (74)} \\ &\quad \leq \exp[L^4 D - L^2 \sum_{k=2}^{N} a_{2k,n}^{1/2k}|\phi|^2 + \sum_{k=2}^{N} L^{4-2k} A_{2k} a_{2k,n} (\mathbf{Im}\phi)^{2k} + K]. \end{aligned} \tag{75}$$

If D and L are given, we take C_1 sufficiently large and then we take n_0 sufficiently large. Thus, we obtain

$$\begin{aligned}&|\exp(-(v_n')_{\geq 4}(\phi))| \\ &\quad < \exp[D - \sum_{k=2}^{2N} a_{2k,n+1}^{1/2k}|\phi|^2 + \sum_{k=2}^{2N} A_{2k} a_{2k,n+1} (\mathbf{Im}\phi)^{2k}], \end{aligned} \tag{76}$$

for $|\mathbf{Im}\phi| < C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$, $|\mathbf{Re}\phi| > C_1(L^{(2m-4)(n+1)}n_0)^{1/2m}$.

2.2.2 The case where $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$

Now we prove remainder part of large field region. Let $\mu_n \in I_n$, and $|\phi| < \frac{10}{11}LC_1(L^{(2m-4)n}n_0)^{1/2m}$. From (55), (69), (71), (72), and $K(n_0+n)^{1/4} >$

$(L^{(2m-4)n}n_0)^{1/2m}$ for $m \geq 3$, we have

$$\begin{aligned}|e^{-((v_n)')_{\geq 4}(\phi)}| &\leq \exp[K\sum_{k=2}^{2N} L^{-2}C_1^{2k}n_0^{-1/k}] \\ &\quad \times \exp[-\sum_{k=2}^{N} a_{2k,n+1}(\mathbf{Re}\phi^{2k}) + L^4 n_0^{-1/2}]. \qquad (77)\end{aligned}$$

And, we estimate $a_{2k,n+1}(\mathbf{Re}\phi^{2k})$ as follows,

$$\begin{aligned}a_{2k,n+1}(\mathbf{Re}\phi^{2k}) &\geq a_{2k,n+1}(\frac{1}{4}(\mathbf{Re}\phi)^{2k} - K(\mathbf{Im}\phi)^{2k}) \\ &\geq -\frac{1}{2}D_{2k} + 2(a_{2k,n+1})^{1/k}|\phi|^2 - A_{2k}a_{2k,n+1}(\mathbf{Im}\phi)^{2k}. \qquad (78)\end{aligned}$$

Notice that D_{2k} does not depend on C_1, n_0 or n. Put $D = \sum_{k=2}^{N} D_{2k}$. From (77) to (78),

$$\begin{aligned}|e^{-((v_n)')_{\geq 4}(\phi)}| &\leq \exp[D - \sum_{k=2}^{N}(a_{2k,n+1})^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n+1}(\mathbf{Im}\phi)^{2k}] \\ &\quad \times \exp[-\frac{1}{2}D + K\cdot L^{-2}C_1^2 n_0^{-1/2}] \\ &\quad \times \exp[K\cdot L^{4/3}C_1^{2N}(L^{(4-2m)(n+1)}n_0)^{1/m} + L^4 n_0^{-1/2}], \qquad (79)\end{aligned}$$

which is smaller than

$$\exp[D - \sum_{k=2}^{N} a_{2k,n+1}^{1/k}|\phi|^2 + \sum_{k=2}^{N} A_{2k}a_{2k,n+1}(\mathbf{Im}\phi)^{2k}], \qquad (80)$$

if n_0 is sufficiently large. Proof of Lemma 1.2 is completed.

3 Proof of Theorem 1.1

Finally, we prove Theorem 1.1, using Lemma 1.2, Lemma 1.3 and Theorem 1.4. First of all, we notice that it is possible to take constants L, $D(N)$, $C_1(N, L, D)$, $n_0(N, L, D, C_1)$ to satisfy Lemma 1.2, Lemma 1.3, and Theorem 1.4. We can check that potential $v(\phi)$ can be iterated n_2 times if initial parameters satisfy the conditions **(Pa)** and **(Pb)** because of Lemma 1.2. Notice that $v_{n_2}(\phi)$, the potential after n_2 iterations, satisfies the conditions **L1.3a** and **L1.3b** with $n = 0$, and so Lemma 1.3 can be applied to this potential. We have to iterate $\mathcal{R}$ using Lemma 1.3, sufficiently many times so that the iterated potentials satisfy the G-K conditions. Put

$$\begin{aligned}n_1 = \min\{n \in \mathbf{N} : |(v_{n_2+n})_{\geq 6}(\phi)| &< (n_0+n_2+n)^{-3/4} \\ &\text{for } |\phi| < C_1(n_0+n_2+n)^{1/4}\}. \qquad (81)\end{aligned}$$

Then,

$$a_{6,n_1+n_2-1} < (n_0 + n_1 + n_2 - 1)^{-9/4}. \tag{82}$$

By calculation, n_1 can be estimated as $n_1 < K \log_L n_0$. Since, $a_{2k,n_1+n_2} \geq 0$, and by (22)

$$\begin{aligned} a_{4,n_1+n_2} - c_{4,n_1+n_2} &< a_{4,0} + (n_1+n_2)n_0^{-1-2/N} \\ &< \frac{C_{++}}{L^4} n_0^{-1} + 2(\log_L n_0) n_0^{-1-2/N} < \frac{C_+}{L^4}(n_0+n_1+n_2)^{-1}. \end{aligned} \tag{83}$$

Similarly, by (82) we have

$$a_{4,n_1+n_2} - c_{4,n_1+n_2} > \frac{C_-}{L^4}(n_0+n_1+n_2)^{-1}. \tag{84}$$

So, we checked the condition **G-Kb** completely. Next, let us check the condition **G-Ka**. Notice that analyticity, positivity for real ϕ, and even function of $v_{n_1+n_2}(\phi)$ are checked easily. Now, We check the bound of $v_{n_1+n_2}(\phi)$

$$\begin{aligned} |\exp[-v_{n_1+n_2}(\phi)]| \leq \exp[D - \sum_{k=2}^{2N} a_{2k,n_1+n_2}^{1/k} |\phi|^2] \\ \times \exp[+\sum_{k=2}^{2N} A_{2k} a_{2k,n_1+n_2} (\mathbf{Im}\phi)^{2k}]. \end{aligned} \tag{85}$$

Notice that $-\sum_{k=3}^{2N} a_{2k,n_1+n_2}^{1/2k} |\phi|^2 + \sum_{k=3}^{2N} A_{2k} a_{2k,n_1+n_2} (\mathbf{Im}\phi)^{2k}$ is nonpositive for $(\mathbf{Im}\phi) < C_1(n_0+n_1+n_2)^{1/4}$ from the definitions of n_1 and n_2. So we have the following inequality

$$|\exp[-v_{n_1+n_2}(\phi)]| \leq \exp[D - a_{4,n_1+n_2}^{1/2} |\phi|^2 + A_4 a_{4,n_1+n_2} (\mathbf{Im}\phi)^4]. \tag{86}$$

We have checked all of the G-K conditions. Since $a_{2k,n_1+n_2-1}, k \geq 3$ is sufficiently small by (82), we know

$$|\mu_{n_1+n_2} - L^2(\mu_{n_1+n_2-1} - c_{2,n_1+n_2-1} + \frac{6\lambda_{n_1+n_2-1}}{1-L^{-2}})| \leq K \cdot n_0^{-1-2/N}. \tag{87}$$

As in the proof lemma 1.2 and Lemma 1.3, we can take for $J_{n_1+n_2}$ a suitable connected component. So, we can adapt Theorem Gawędzki and Kupiainen [6]. Now, Theorem 1.1 is finished.

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