数理解析研究所講究録
1482巻 2006年 153-162

An asymptotic formula for marginal running coupling constants and universality of loglog corrections‡

埼玉医科大学物理学教室 向田寿光 (Hisamitsu Mukaida)

Department of Physics, Saitama Medical College, 981 Kawakado, Iruma-gun, Saitama, 350-0496, Japan
`mukaida@saitama-med.ac.jp`

Abstract

Given a two-loop beta function for multiple marginal coupling constants, we derive an asymptotic formula for the running coupling constants driven to an infrared fixed point. It can play an important role in universal loglog corrections to physical quantities.

1. Introduction

Log and loglog corrections to physical quantities in critical phenomena generally appear in a statistical system at the critical dimension. In the language of renormalization group (RG) [1], those corrections arise from marginally irrelevant coupling constants in the system. As the length scale we are looking at becomes larger, the coupling constants effectively change, obeying a renormalization-group equation (RGE), and approach an infrared fixed point if initial values of the trajectories are on the critical surface.

Universality (i.e., property independent of initial data) of the logarithmic corrections is closely related to the long-distance behavior of the running coupling constants. For example, consider the $S = 1/2$ Heisenberg antiferromagnetic chain [2]. Low energy properties of this model are described by $k = 1$ SU(2) Wess-Zumino-Witten model with a perturbation whose running coupling constant g is obeyed the following RGE:

$$\frac{dg}{dt} = -g^2 - \frac{1}{2}g^3, \tag{1}$$

where t is related to a length scale L of renormalization-group transformation (RGT) by $t = \log L$. Vanishing linear term and the negative sign in front of g^2 indicate that g is marginally irrelevant.

Finite-size effect in the ground state energy E_0 with L sites is given by the formula [3, 4]

$$\frac{-6L}{\pi v}\left(E_0 - Le_\infty\right) = 1 + \frac{3}{8}g^3 + \mathrm{O}\left(g^4\right), \tag{2}$$

‡ Talk given at seminar on "Applications of Renormalization Group Methods in Mathematical Sciences", RIMS, Kyoto University, Sept. 2005

where e_∞ is the ground-state energy per site in the infinite-volume limit. We can easily integrate the RGE and find the asymptotic form as

$$\begin{aligned} g(t) &= \frac{1}{t} - \frac{\log t}{2t^2} + \mathrm{O}\left(\frac{1}{t^2}\right) \\ &= \frac{1}{\log L} - \frac{\log\log L}{2\log L^2} + \mathrm{O}\left(\frac{1}{\log L^2}\right) \end{aligned} \tag{3}$$

Note that the first and the second terms are universal in the sense that they are independent of an initial condition, while higher order terms depend on it. Inserting this asymptotic form into (2), one finds

$$-\frac{6L}{\pi v}[E_0 - Le_\infty] = 1 + \frac{3}{8\log^3 L}\left\{1 - \frac{3}{2}\frac{\log\log L}{\log L}\right\} + \mathrm{O}\left(\frac{1}{\log L^4}\right), \tag{4}$$

where the first term in the curly brace comes from the leading term in (3), while the second term from the subleading term. Thus universality of the finite-size effect (4) is closely related to universality of the asymptotic form of g, which is general feature in the case when coupling constants are all marginal.

Although we can integrate RGE explicitly in the case of a single coupling constant, as in the case of (1), we cannot generally perform the same procedure in the case of multiple coupling constants. Difficulty stems from vanishing linear terms, which makes linearization impossible. Therefore, it is worthwhile to determine an asymptotic form analogous to (3) in the case of multiple marginal coupling constants, which is the main purpose of this report.

An algebraic method for finding asymptotic form was proposed in Refs [7, 8], where the beta function is restricted to the lowest order. Since the lowest-order beta function for marginal coupling constants is homogeneous, the RGE is invariant under scaling transformation [7]. One can define another RG transformation to the RGE, thanks to the scale invariance §. The new RGE generally has a linear term, which allows us to obtain the asymptotic form without explicit integration.

However, we cannot apply the above method to the present case, because there are no such scale invariances in the beta function obtained up to the next-to-leading order. Hence we need to find an alternative method to remedy the problem for linear terms to vanish.

In the next section, we present a change of variables in the RGE that allows us to apply the linearization. In section 3, we switch from the resultant RGE to an equivalent integral equation. In section 4, we show a sufficient condition for loglog corrections be universal. A universal asymptotic formula for the solution in the long-distance limit is also derived under the sufficient condition. In section 5, applying our result, we rederive the universal asymptotic formula for the running coupling constants in the classical XY

§ A general idea of RG, applied as a tool for asymptotic analysis of non-linear differential equations, is developed in Refs. [10, 11].

model, as an example. The result is consistent with the original article by Amit *et al.* [6]. The final section is devoted to summary of our procedure.

2. Changing variables of RGE

We consider an RGE for marginal coupling constants denoted by $\boldsymbol{g}(t) = (g_1(t), ..., g_n(t))$. We regard the space of the coupling constants as the n dimensional Euclidean space $\boldsymbol{R}^n$. Suppose that we have obtained the RGE up to the next-to-leading order, which is to say we start with the following RGE

$$\frac{\mathrm{d}\boldsymbol{g}(t)}{\mathrm{d}t} = \boldsymbol{V}(\boldsymbol{g}(t)) + \boldsymbol{F}(\boldsymbol{g}(t)). \tag{5}$$

The leading and the subleading terms of the beta function are described by $\boldsymbol{V}$ and $\boldsymbol{F}$ respectively. It is assumed that they possess the following scaling property:

$$\boldsymbol{V}(k\boldsymbol{g}) = k^2\boldsymbol{V}(\boldsymbol{g}), \qquad \boldsymbol{F}(k\boldsymbol{g}) = k^3\boldsymbol{F}(\boldsymbol{g}). \tag{6}$$

Suppose that they are defined in an open subset E of $\boldsymbol{R}^n$ and belong to $C^2(E)$ i.e., their second derivative exist and continuous on E. We also assume that the origin O belongs to the closure $\bar{E}$ of E.

It is a general feature of an RGE of marginal coupling constants that there are no linear terms, which causes difficulty in deriving an asymptotic formula. We introduce new variables to bypass this problem. First we replace t by

$$u \equiv \frac{1}{\epsilon}\log(\epsilon t + 1), \tag{7}$$

where ϵ is a parameter with

$$0 < \epsilon < 1. \tag{8}$$

As we will see later, ϵ is introduced in order to control an effect of the subleading term $\boldsymbol{F}$. Next we change $\boldsymbol{g}$ by $\boldsymbol{c}$, where

$$\boldsymbol{g}(t) = \mathrm{e}^{-\epsilon u}\boldsymbol{c}(u). \tag{9}$$

Using the scaling property (6), the RGE (5) is written as

$$\frac{\mathrm{d}\boldsymbol{c}(u)}{\mathrm{d}u} = \epsilon\boldsymbol{c}(u) + \boldsymbol{V}(\boldsymbol{c}(u)) + \mathrm{e}^{-\epsilon u}\boldsymbol{F}(\boldsymbol{c}(u)). \tag{10}$$

Now we can extract the linear part from the first two terms. We assume that a non-trivial solution $\boldsymbol{c}^* \in E$ for

$$\epsilon\boldsymbol{c}^* + \boldsymbol{V}(\boldsymbol{c}^*) = \boldsymbol{0} \tag{11}$$

exists. This means that $\boldsymbol{V}(\boldsymbol{c}^*)$ points to the origin. Therefore, $\boldsymbol{V}$ has an incoming straight line through $\boldsymbol{c}^*$ as an integral curve, according to the scaling property (6). See Figure 1. Note that $\boldsymbol{c}^*$ is linear in ϵ. In fact, $\boldsymbol{a}^*$ defined by

$$\boldsymbol{c}^* \equiv \epsilon\boldsymbol{a}^* \tag{12}$$

solves

$$a^* + V(a^*) = 0. \tag{13}$$

Equation (12) indicates that c^* approaches the origin as ϵ becomes smaller. Therefore, an effect of the subleading term F in a neighborhood of c^* is suppressed if we take ϵ sufficiently small.

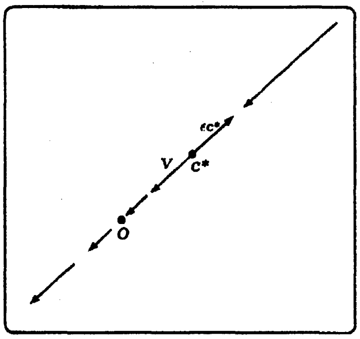

Figure 1. When c^* satisfies (11), $V(c^*)$ points to the origin. Since V satisfies (6), there is incoming straight line through c^* as an integral curve of V.

We analyze (10) in a neighborhood of c^*. Define

$$b(u) \equiv c(u) - c^* \tag{14}$$

and write

$$V(c(u)) = V(c^*) + DV(c^*)b(u) + v(b(u)), \tag{15}$$

where $DV(c^*)$ is the derivative of V at c^*, which is represented by the $n \times n$ matrix as

$$DV(c^*)_{ij} = \frac{\partial V_i}{\partial c_j}(c^*). \tag{16}$$

The RGE (10) is written as

$$\frac{\mathrm{d}b(u)}{\mathrm{d}u} = Mb(u) + H(u, b(u)), \tag{17}$$

where

$$M \equiv \epsilon I_n + DV(c^*) = \epsilon\left(I_n + DV(a^*)\right) \tag{18}$$

$$H(u, b(u)) \equiv v(b(u)) + \mathrm{e}^{-\epsilon u} F(c^* + b(u)) \tag{19}$$

with I_n being the $n \times n$ unit matrix. Namely, M is linear in ϵ.

To sum up, if we know a solution $b(u)$ for (17), we get a solution for (5) by

$$g(t) = \mathrm{e}^{-\epsilon u}\left(c^* + b(u)\right), \quad u = \frac{1}{\epsilon}\log(\epsilon t + 1). \tag{20}$$

3. Integral equation

In this section, following reference [5] (Section 2.7), we derive an integral equation satisfied by a solution $\boldsymbol{b}(u)$ for (17) driven to $\boldsymbol{0}$.

We assume that there are no eigenvalues with zero real part in M. Suppose that M has k eigenvalues having a negative real part, and $n-k$ eigenvalues with positive real part. If the positive eigenmodes are fine-tuned to vanish, $|\boldsymbol{b}(u)|$ becomes smaller as $u \to \infty$. We find from (20) that the corresponding $\boldsymbol{g}(t)$ approaches the origin from the $\boldsymbol{c}^*$-direction. In order to show the existence of such solutions, we decompose M into a block diagonal form. Namely,

$$R^{-1}MR = \begin{pmatrix} \epsilon P & 0 \\ 0 & \epsilon Q \end{pmatrix} \equiv \epsilon\Lambda, \tag{21}$$

where ϵP is $k \times k$ matrix whose eigenvalues have a negative real part. Similarly, ϵQ is an $(n-k)\times(n-k)$ matrix, where its eigenvalues have a positive real part. Note that P, Q and Λ are independent of ϵ because M is linear in ϵ. Define the tilde operation

$$\tilde{\boldsymbol{x}} = R^{-1}\boldsymbol{x}, \qquad \tilde{\boldsymbol{X}} = R^{-1}\circ \boldsymbol{X} \circ R \tag{22}$$

for a point $\boldsymbol{x} \in E$ and a map $\boldsymbol{X} : E \to \boldsymbol{R}^n$, e.g., $\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{c}}) = R^{-1}\boldsymbol{F}(R\tilde{\boldsymbol{c}}) = R^{-1}\boldsymbol{F}(\boldsymbol{c})$. The RGE (17) can be written as

$$\frac{\mathrm{d}\tilde{\boldsymbol{b}}(u)}{\mathrm{d}u} = \epsilon\Lambda\tilde{\boldsymbol{b}}(u) + \tilde{\boldsymbol{H}}(u, \tilde{\boldsymbol{b}}(u)). \tag{23}$$

Let

$$U(u) \equiv \begin{pmatrix} \mathrm{e}^{P\epsilon u} & 0 \\ 0 & 0 \end{pmatrix}, \qquad T(u) \equiv \begin{pmatrix} 0 & 0 \\ 0 & \mathrm{e}^{Q\epsilon u} \end{pmatrix}. \tag{24}$$

Then

$$\frac{\mathrm{d}U}{\mathrm{d}u} = \epsilon\Lambda U(u), \qquad \frac{\mathrm{d}T}{\mathrm{d}u} = \epsilon\Lambda T(u) \tag{25}$$

and

$$\mathrm{e}^{\epsilon\Lambda u} = U(u) + T(u). \tag{26}$$

We focus on a solution that behaves as $\tilde{\boldsymbol{b}}(u) \to \boldsymbol{0}$ as $u \to \infty$. The integral equation corresponding to it is

$$\tilde{\boldsymbol{b}}(u) = U(u)\boldsymbol{p} + \int_0^u \mathrm{d}u' U(u-u')\tilde{\boldsymbol{H}}(u', \tilde{\boldsymbol{b}}(u')) - \int_u^\infty \mathrm{d}u' T(u-u')\tilde{\boldsymbol{H}}(u', \tilde{\boldsymbol{b}}(u')), \tag{27}$$

where $\boldsymbol{p} = (p_1, ..., p_k, 0, ..., 0)$ specifies an initial condition in the following way:

$$\begin{aligned} &\tilde{b}(0)_i = p_i \quad \text{for} \quad i = 1, ..., k. \\ &\tilde{b}(0)_i = -\left(\int_0^\infty \mathrm{d}u' T(-u')\tilde{\boldsymbol{H}}(u', \tilde{\boldsymbol{b}}(u'))\right)_i \quad \text{for} \quad i = k+1, ..., n. \end{aligned} \tag{28}$$

We can show that (27) has a unique solution if ϵ and $\boldsymbol{p}$ are sufficiently small. Moreover, we find that the solution satisfies

$$\left|\tilde{\boldsymbol{b}}(u)\right| \leq J\mathrm{e}^{-\alpha\epsilon u} \tag{29}$$

for some $J > 0$ [9]. Here α is a positive number, such that $-\alpha$ is strictly greater than the real part of every eigenvalue of P.

4. Universal asymptotic form of $\tilde{\boldsymbol{b}}(u)$

In this section, we derive a universal asymptotic form of $\tilde{\boldsymbol{b}}(u)$ by applying (29) to the right-hand side in (27). (Here, "universal" means that the asymptotic form is independent of $\boldsymbol{p}$.)

For this purpose, we give a more concrete form of P. Let λ_l $(l = 1, ..., n_-)$ be the distinct eigenvalues of P with the multiplicity d_l. We denote by W_l the generalized eigenspace associated with λ_l. Clearly, $\dim W_l = d_l$. Taking an appropriate basis for $\boldsymbol{R}^n$, P is represented as a block diagonal form. Here the l th block P_l is a $d_l \times d_l$ upper triangle matrix whose diagonal components take a common value λ_l. Furthermore, the basis allows us to assume that $N_l \equiv P_l - \lambda_l I_{d_l}$ is a nilpotent matrix, namely $N^{\nu_l - 1} \neq 0$, $N^{\nu_l} = 0$ for some $1 \leq \nu_l \leq d_l$. An arbitrary element $\boldsymbol{x} \in \boldsymbol{R}^n$ can be decomposed as

$$\boldsymbol{x} = \sum_{l=1}^{n_-} \boldsymbol{x}^{(l)} + \boldsymbol{x}^{(+)}, \quad \boldsymbol{x}^{(l)} \in W_l, \tag{30}$$

where $\boldsymbol{x}^{(+)}$ is an element of the subspace spanned by the positive eigenmodes of M. Applying $U(u)$ to the both sides, we have

$$U(u)\boldsymbol{x} = \sum_{l=1}^{n_-} \mathrm{e}^{\epsilon\lambda_l u} \sum_{k=0}^{\nu_l - 1} \frac{(\epsilon u)^k N_l^k}{k!} \boldsymbol{x}^{(l)}, \tag{31}$$

with the convention $N_l^0 = I_{d_l}$ even if N_l is the zero matrix.

We can show that [8]

$$M\boldsymbol{c}^* = -\epsilon \boldsymbol{c}^*, \tag{32}$$

which is equivalent to

$$P\boldsymbol{c}^* = -\boldsymbol{c}^*. \tag{33}$$

Then we set

$$\lambda_1 = -1. \tag{34}$$

Using (31) and (34), we can estimate the right-hand side of (27). The first term is written as

$$U(u)\boldsymbol{p} = \sum_{l=1}^{n_-} \boldsymbol{G}_1^{(l)}(u)\mathrm{e}^{\lambda_l \epsilon u}, \tag{35}$$

where $G_1^{(l)}(u)$ is a polynomial of degree at most $\nu_l - 1$. Since (35) explicitly depends on $\boldsymbol{p}$, it is non-universal.

In order to obtain a universal asymptotic form, universal terms should dominate over (35) when $u \to \infty$. Let us find a condition that such terms appear from the remaining part. The integral containing U in (27) is divided as

$$\int_0^u U(u-u')\mathrm{e}^{-\epsilon u'}\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{c}}^*)\,\mathrm{d}u' + \int_0^u U(u-u')\left(\tilde{\boldsymbol{H}}(u',\tilde{\boldsymbol{b}}(u')) - \mathrm{e}^{-\epsilon u'}\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{c}}^*)\right)\mathrm{d}u'. \tag{36}$$

Using (31) to $\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{c}}^*)$, the first integral is easily calculated. It is important to notice that the case of $l = 1$ is treated separately, because the factor $\exp(-\epsilon\lambda_1 u')$ in $U(u-u')$ cancels $\exp(-\epsilon u')$ in the integrand. When $l = 1$ and $k = \nu_1 - 1$ in (31), the cancellation brings a term proportional to $u^{\nu_1}\exp(-\epsilon u)$, which is not contained in $\boldsymbol{G}_1^{(l)}(u)$. Writing this explicitly, the integral is expressed as

$$\int_0^u U(u-u')\mathrm{e}^{-\epsilon u}\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{c}}^*)\,\mathrm{d}u' = \frac{(\epsilon u)^{\nu_1}\,\mathrm{e}^{-\epsilon u}}{\epsilon\nu_1!}N_1^{\nu_1-1}\tilde{\boldsymbol{F}}^{(1)}(\tilde{\boldsymbol{c}}^*) + \sum_{l=1}^{n_-}\mathrm{e}^{\lambda_l\epsilon u}\boldsymbol{G}_2^{(l)}(u). \tag{37}$$

Here, it is straightforward to check that $\boldsymbol{G}_2^{(l)}(u)$ is a polynomial whose degree is at most $\nu_l - 1$. The second term in the right-hand side of (37) contributes to the same order as (35), so that its universal behavior is obscured by non-universal feature of $U(u)\boldsymbol{p}$. On the other hand, the first term can dominate over (35) in the case when

$$\Re\lambda_l < -1 \quad (l = 2, \ldots, n_-)\,. \tag{38}$$

We can similarly estimate the second term of (36) by employing (29). Since the cancellation of the exponential factor does not occur in this term, we get

$$\left|\int_0^u U(u-u')\left(\boldsymbol{H}(u',\tilde{\boldsymbol{b}}(u')) - \mathrm{e}^{-\epsilon u}\tilde{\boldsymbol{F}}(\tilde{\boldsymbol{c}}^*)\right)\mathrm{d}u'\right| < \sum_{l=0}^{n_-} G_3^{(l)}(u)\mathrm{e}^{\lambda_l\epsilon u} \tag{39}$$

for all $u \geq 0$. Again, $G_3^{(l)}(u)$ is a polynomial of degree at most $\nu_l - 1$.

As for the last integral in (27), there is a number $B > 0$ such that

$$\left|\int_u^\infty T(u-u')\boldsymbol{H}(u',\tilde{\boldsymbol{b}}(u'))\mathrm{d}u'\right| < B\mathrm{e}^{-\epsilon u}, \tag{40}$$

for all $u \geq 0$. Collecting (35), (37), (39) and (40) we conclude that: for all $u \geq 0$, there is some polynomial $G^{(l)}(u)$ of degree at most $\nu_l - 1$ $(l = 1, \ldots, \nu_-)$, such that

$$\left|\tilde{\boldsymbol{b}}(u) - \frac{(\epsilon u)^{\nu_1}\,\mathrm{e}^{-\epsilon u}}{\epsilon\nu_1!}N_1^{\nu_1-1}\tilde{\boldsymbol{F}}^{(1)}(\tilde{\boldsymbol{c}}^*)\right| < \sum_{l=0}^{n_-} G^{(l)}(u)\mathrm{e}^{\lambda_l\epsilon u}. \tag{41}$$

If the condition (38) holds, the most dominant term in the right-hand side of (41) when $u \to \infty$ is $u^{\nu_1-1}\mathrm{e}^{-\epsilon u}$. Thus, we obtain

$$\tilde{\boldsymbol{b}}(u) = \frac{(\epsilon u)^{\nu_1}\,\mathrm{e}^{-\epsilon u}}{\epsilon\nu_1!}N_1^{\nu_1-1}\tilde{\boldsymbol{F}}^{(1)}(\tilde{\boldsymbol{c}}^*) + \mathrm{O}\left(u^{\nu_1-1}\mathrm{e}^{-\epsilon u}\right). \tag{42}$$

Let us turn back to the original variables by (20). We get, under the condition (38),

$$\tilde{g}(t) = \frac{1}{\epsilon t+1}\left(\epsilon\tilde{a}^* + \frac{\epsilon^3\left(\log\left(\epsilon t+1\right)\right)^{\nu_1}}{\epsilon\left(\epsilon t+1\right)\nu_1!}N_1^{\nu_1-1}\tilde{F}^{(1)}\left(\tilde{a}^*\right)\right) + \mathrm{O}\left(\frac{\left(\log\left(\epsilon t+1\right)\right)^{\nu_1-1}}{\left(\epsilon t+1\right)^2}\right) \tag{43}$$

Removing the tildes, we get

$$g(t) = \frac{a^*}{t} + \frac{(\log t)^{\nu_1}}{t^2\nu_1!}R\bar{N}_1^{\nu_1-1}R^{-1}F\left(a^*\right) + \mathrm{O}\left(\frac{(\log t)^{\nu_1-1}}{t^2}\right), \tag{44}$$

as $t \to \infty$. Here $\bar{N}_1$ is $n \times n$ matrix defined as

$$\bar{N}_1 = \left(\begin{array}{c|c} N_1 & 0\cdots 0 \\ \hline 0 & \\ \vdots & \mathbf{0} \\ 0 & \end{array}\right). \tag{45}$$

It is worthwhile to note that the $(\log t)^{\nu_1}$ term appears, which can bring about a $(\log\log L)^{\nu_1}$ correction in general. This is a generalization of (3) to the case of multiple coupling constants.

5. Application to the two-dimensional XY model

In this section, we illustrate our method using the two-dimensional classical XY model [12]. The beta function up to subleading order of this model and the two-point correlation function containing loglog correction are originally derived by Amit *et al* . [6]. They obtained the asymptotic form of the coupling constants by explicitly integrating the RGE. Here we rederive the asymptotic form within our formulation.

The 2D classical XY model has the following RGE [6, 13]:

$$\begin{aligned} \frac{\mathrm{d}g_1}{\mathrm{d}t} &= -g_2^2 - B_1 g_2^2 g_1 \\ \frac{\mathrm{d}g_2}{\mathrm{d}t} &= -g_1 g_2 - A_1 g_2^3, \end{aligned} \tag{46}$$

where $g_2 > 0$ and $2A_1 + B_1 = 3/2$. It indicates that

$$V(g) = \begin{pmatrix} -g_2^2 \\ -g_1 g_2 \end{pmatrix}, \quad F(g) = \begin{pmatrix} -B_1 g_2^2 g_1 \\ -A_1 g_2^3 \end{pmatrix}. \tag{47}$$

Solving (13), we get a non-trivial solution

$$a^* = (1,1). \tag{48}$$

Inserting this into (18), one finds

$$M = \epsilon\begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}. \tag{49}$$

The eigenvalues and corresponding eigenvectors of M are

$$-1 \leftrightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad 2 \leftrightarrow \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \tag{50}$$

Namely, the space of negative eigenmodes of M is one dimension, i.e., $\nu_1 = 1$. It indicates that the critical surface along $\boldsymbol{a}^*$ is in fact a line. The transformation matrix R and the diagonalized matrix Λ are obtained from the eigenvectors and the eigenvalues respectively. The result is

$$R = \begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}, \qquad \Lambda = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}. \tag{51}$$

Furthermore, we have

$$\bar{N}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{52}$$

Therefore, from (44), we conclude that

$$\begin{aligned} \boldsymbol{g}(t) &= \frac{\boldsymbol{a}^*}{t} - \frac{1}{3}(2A_1 + B_1)\frac{\log t}{t^2}\boldsymbol{a}^* + \mathrm{O}\left(\frac{1}{t^2}\right) \\ &= \frac{\boldsymbol{a}^*}{t} - \frac{1}{2}\frac{\log t}{t^2}\boldsymbol{a}^* + \mathrm{O}\left(\frac{1}{t^2}\right) \end{aligned} \tag{53}$$

for the critical line. This is consistent with the original result.

6. Summary

Here we summarize the procedure reaching final result (44). Suppose that we obtain an RGE

$$\frac{d\boldsymbol{g}(t)}{dt} = \boldsymbol{V}(\boldsymbol{g}(t)) + \boldsymbol{F}(\boldsymbol{g}(t)).$$

i) Find a solution $\boldsymbol{a}^*$ for

$$\boldsymbol{a}^* + \boldsymbol{V}(\boldsymbol{a}^*) = \boldsymbol{0}.$$

ii) Compute the matrix M

$$M_{ij} = \delta_{ij} + \frac{\partial V_i}{\partial a_j}(\boldsymbol{a}^*)$$

iii) Compute the regular matrix R such that

$$R^{-1}MR = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix},$$

where P (resp. Q) is define by negative (positive) eigenvalues, with

$$P = \left(\begin{array}{c|c|c} P_1 & \cdots & \cdots \\ \hline \cdots & P_2 & \cdots \\ \hline \cdots & \cdots & \cdots \end{array}\right), \quad P_l = \lambda_l I_{d_l} + N_l.$$

iv) Let P_1 be the matrix for the eigenvalue -1. If the real part of the other negative eigenvalues are less than -1, the asymptotic form is

$$g(t) = \frac{a^*}{t} + \frac{(\log t)^{\nu_1}}{t^2 \nu_1!} R \bar{N}_1^{\nu_1 - 1} R^{-1} F(a^*) + \mathrm{O}\left(\frac{(\log t)^{\nu_1 - 1}}{t^2}\right),$$

where $\bar{N}_1$ is

$$\bar{N}_1 = \left(\begin{array}{c|c} N_1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & \mathbf{0} \\ 0 & \end{array}\right).$$

Acknowledgments

It is my pleasure to thank C. Itoi for informing me of this attractive problem. I am also grateful to K. R. Ito for giving me the opportunity of this presentation.

References

[1] Wilson K G and Kogut J 1974 *Phys. Rep.* **12C** 75.
[2] M -H. Kato 1998 Ph. D thesis, Nihon University.
[3] I. Affleck 1986 *Phys. Rev. Lett.* **56** 746.
[4] J. L. Cardy 1986 *J. Phys. A: Math. Gen.* **19** L1093; 1987 *J. Phys. A: Math. Gen.* **20** 5039
[5] Perko L 2002 *Differential Equations and Dynamical Systems* 3rd ed (New York: Springer-Verlag).
[6] Amit D J, Goldschmidt Y Y, and Grinstein G G 1980 *J. Phys. A: Math. Gen.* **13** 585.
[7] Itoi C and Mukaida H 1999 *Phys. Rev.* E **60** 3688, *Preprint* cond-mat/9711308
[8] Mukadia H 2004 *Phys. Rev.* E, **70** 017101, *Preprint* cond-mat/0308461.
[9] Mukadia H 2004 *J. Phys. A: Math. Gen.* **37** 10815, *Preprint* cond-mat/0405466.
[10] Chen L -Y, Goldenfeld N and Oono Y 1996 *Phys. Rev.* E **54** 376.
[11] Bricmont J and Kupiainen A 1992 *Commun. Math. Phys.* **150** 193;
Bricmont J, Kupiainen A and Lin G 1994 *Commun. Pure. Appl. Math.* **47** 893.
[12] Berezinskiĭ V L 1970 *Zh. Éksp. Teor. Fiz.* **59** 907 [1971 *Sov. Phys. JETP* **32** 493];
Kosterlitz J M and Thouless D J 1973 *J. Phys. C: Solid State Phys.* **6** 1181.
[13] J. Balog, and Á. Hegedűs 2000 *J. Phys. A: Math. Gen.* **33** 6543.