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Kyoto University
Random Point Fields for Para-Particles of order 3

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Abstract
Random point fields which describe gases consist of para-particles of order three are given by means of the canonical ensemble approach. The analysis for the case of the para-fermion gases is discussed in full detail.

1 Introduction
The purpose of this note is to apply the method which we have developed in [TIa] to statistical mechanics of gases which consist of para-particles of order 3. We begin with quantum mechanical thermal systems of finite fixed numbers of para-bosons and/or para-fermions in the bounded boxes in \( \mathbb{R}^d \). Taking the thermodynamic limits, random point fields on \( \mathbb{R}^d \) are obtained. We will see that the point fields obtained in this way are those of \( \alpha = \pm 1/3 \) given in [ShTa03].

We use the representation theory of the symmetric group. ( cf. e.g. [JK81, S91, Si96]) Its basic facts are reviewed briefly, in section 2, along the line on which the quantum theory of para-particles are formulated. We state the results in section 3. Section 4 devoted to the full detail of the discussion on the thermodynamic limits for para-fermion's case.
2 Brief review on Representation of the symmetric group

We say that $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n$ is a Young frame of length $n$ for the symmetric group $S_N$ if

\[ \sum_{j=1}^{n} \lambda_j = N, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0. \]

We associate the Young frame $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ with the diagram of $\lambda_1$-boxes in the first row, $\lambda_2$-boxes in the second row, ..., and $\lambda_n$-boxes in the $n$-th row. A Young tableau on a Young frame is a bijection from the numbers $1, 2, \cdots, N$ to the $N$ boxes of the frame.

Let $M_p^n$ be the set of all the Young frames for $S_N$ which have lengths less than or equal to $p$. For each frame in $M_p^n$, let us choose one tableau from those on the frame. The choices are arbitrary but fixed. $\mathcal{T}_p^n$ denotes the set of all the tableaux chosen in this way. The row stabilizer of tableau $T$ is denoted by $\mathcal{R}(T)$, i.e., the subgroup of $S_N$ consists of those elements that keep all rows of $T$ invariant, and $C(T)$ the column stabilizer whose elements preserve all columns of $T$.

Let us introduce the three elements

\[ a(T) = \frac{1}{\# R(T)} \sum_{\sigma \in R(T)} \sigma, \quad b(T) = \frac{1}{\# C(T)} \sum_{\sigma \in C(T)} \text{sgn}(\sigma) \sigma \]

and

\[ e(T) = \frac{d_T}{N!} \sum_{\sigma \in R(T)} \sum_{\tau \in C(T)} \text{sgn}(\tau) \sigma \tau = c_T a(T) b(T) \]

of the group algebra $\mathbb{C}[S_N]$ for each $T \in \mathcal{T}_p^n$, where $d_T$ is the dimension of the irreducible representation of $S_N$ corresponding to $T$ and $c_T = d_T \# R(T) \# C(T)/N!$. As is known,

\[ a(T_1) a(T_2) = b(T_3) \sigma a(T_1) = 0 \quad \text{(2.1)} \]

hold for any $\sigma \in S_N$ if $T_2 \rightarrow T_1$. The relations

\[ a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1) e(T_2) = 0 \quad (T_1 \neq T_2) \quad \text{(2.2)} \]

also hold for $T, T_1, T_2 \in \mathcal{T}_p^n$. For later use, let us introduce

\[ d(T) = e(T) a(T) = c_T a(T) b(T) a(T) \quad \text{(2.3)} \]

for $T \in \mathcal{T}_p^n$. They satisfy

\[ d(T)^2 = d(T), \quad d(T_1) d(T_2) = 0 \quad (T_1 \neq T_2) \quad \text{(2.4)} \]
which are shown readily from (2.2) and (2.1). The inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{C}[S_N]$ is
definite by
\[ \langle \sigma, \tau \rangle := \delta_{\sigma\tau} \quad \text{for } \sigma, \tau \in S_N \]
and the sesqui-linearity.

The left representation $L$ and the right representation $R$ of $S_N$ on $\mathbb{C}[S_N]$ are defined
by
\[ L(\sigma)g = L(\sigma) \sum_{\tau \in S_N} g(\tau)\tau = \sum_{\tau \in S_N} g(\tau^{-1})\tau \]
and
\[ R(\sigma)g = R(\sigma) \sum_{\tau \in S_N} g(\tau)\tau = \sum_{\tau \in S_N} g(\tau)\tau^{-1} = \sum_{\tau \in S_N} g(\tau\sigma)\tau, \]
respectively. Here and hereafter we identify $g : S_N \rightarrow \mathbb{C}$ and $\sum_{\tau \in S_N} g(\tau)\tau \in \mathbb{C}[S_N].$
They are extended to the representation of $\mathbb{C}[S_N]$ on $\mathbb{C}[S_N]$ as
\[ L(f)g = fg = \sum_{\sigma, \tau} f(\sigma)g(\tau)\sigma\tau = \sum_{\sigma} (\sum_{\tau} f(\sigma\tau^{-1})g(\tau))\sigma \]
and
\[ R(f)g = \hat{f} = \sum_{\sigma, \tau} g(\sigma)f(\tau)\sigma\tau^{-1} = \sum_{\sigma} (\sum_{\tau} g(\sigma\tau)f(\tau))\sigma, \]
where $\hat{f} = \sum_{\tau} \hat{f}(\tau)\tau = \sum_{\tau} f(\tau^{-1})\tau = \sum_{\tau} f(\tau)^{-1}.$

The character of the irreducible representation of $S_N$ corresponding to tableau $T \in \mathcal{T}_p^N$ is obtained by
\[ \chi_T(\sigma) = \sum_{\tau \in S_N} (\tau, L(\sigma)R(e(T))\tau) = \sum_{\tau \in S_N} (\tau, \sigma\tau e(T)). \]
We introduce a tentative notation
\[ \chi_g(\sigma) \equiv \sum_{\tau \in S_N} (\tau, L(\sigma)R(g)\tau) = \sum_{\tau, \gamma \in S_N} (\tau, \sigma\tau\gamma^{-1})g(\gamma) = \sum_{\tau \in S_N} g(\tau^{-1}\sigma\tau) \quad (2.5) \]
for $g = \sum_{\tau} g(\tau)\tau \in \mathbb{C}[S_N].$ Then $\chi_T = \chi_{e(T)}$ holds.

Now let us consider representations of $S_N$ on Hilbert spaces. Let $\mathcal{H}_L$ be a certain $L^2$
space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its $N$-fold Hilbert space
tensor product. Let $U$ be the representation of $S_N$ on $\otimes^N \mathcal{H}_L$ defined by
\[ U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \quad \text{for } \varphi_1, \ldots, \varphi_N \in \mathcal{H}_L, \]
or equivalently by
\[ (U(\sigma)f)(x_{1}, \ldots, x_N) = f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) \quad \text{for } f \in \otimes^N \mathcal{H}_L. \]
Obviously, $U$ is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}.$ We extend $U$ for $\mathbb{C}[S_N]$ by
linearity. Then $U(a(T))$ is an orthogonal projection because of $U(a(T))^* = U(a(T)) = U(a(T))$ and (2.2). So are $U(b(T))', U(d(T))'$s and $P_0 = \sum_{T \in \mathcal{T}^N_p} U(d(T)).$ Note that
$\text{Ran } U(d(T)) = \text{Ran } U(e(T))$ because of $d(T)e(T) = e(T)$ and $e(T)d(T) = d(T).$
Para-statistics and Random point fields

3.1 Para-bosons of order 3

Let us consider a quantum system of $N$ para-bosons of order $p$ in the box $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$. We refer the literatures [MeG64, HaT69, StT70] for quantum mechanics of para-particles. (See also [OK69].) The arguments of these literatures indicate that the state space of our system is given by $\mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L$, where $\mathcal{H}_L = L^2(\Lambda_L)$ with Lebesgue measure is the state space of one particle system in $\Lambda_L$. We need the heat operator $G_L = e^{\beta \Delta_L}$ in $\Lambda_L$, where $\Delta_L$ is the Laplacian in $\Lambda_L$ with periodic boundary conditions.

It is obvious that there is a CONS of $\mathcal{H}_{L,N}^{pB}$, which consists of the vectors of the form $U(d(T))\varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)}$, which are the eigenfunctions of $\otimes^N G_L$. Then, we define the point field $\mu_{L,N}^{pB}$ of $N$ free para-bosons of order $p$ as in section 2 of [TIa] and its generating functional is given by

$$\int e^{-\langle f, \xi \rangle} d\mu_{L,N}^{pB}(\xi) = \frac{\text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)P_{pB}]}{\text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{pB}]}\tag{3.1}$$

where $f$ is a nonnegative continuous function on $\Lambda_L$ and $\tilde{G}_L = G_L^{1/2} e^{-f} G_L^{1/2}$.

**Lemma 3.1**

$$\int e^{-\langle f, \xi \rangle} d\mu_{L,N}^{pB}(\xi) = \frac{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in S_N} \chi_T(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in S_N} \chi_T(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]}\tag{3.1}$$

$$= \frac{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_T \{\tilde{G}_L(x_i, x_j)\} \prod_{1 \leq i, j \leq N} dx_i dx_j}{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_T \{G_L(x_i, x_j)\} \prod_{1 \leq i, j \leq N} dx_i dx_j}\tag{3.2}$$

**Remark 1:** $\mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L$ is determined by the choice of the tableaux $T$'s. The spaces corresponding to different choices of tableaux are different subspaces of $\otimes^N \mathcal{H}_L$. However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, $\chi_T(\sigma)$ depends only on the frame on which the tableau $T$ is defined.

**Remark 2:** $\det T A = \sum_{\sigma \in S_N} \chi_T(\sigma) \prod_{i=1}^N A_{\sigma(i)}$ in (3.2) is called immanant.

**Proof:** Since $\otimes^N G$ commutes with $U(\sigma)$ and $a(T)e(T) = e(T)$, we have

$$\text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(d(T))] = \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))U(a(T))] = \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))].$$

(3.3)
On the other hand, we get from (2.5) that

\[
\sum_{\sigma \in S_N} \chi_\sigma(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)] = \sum_{\sigma \in S_N} g(\tau^{-1} \sigma \tau) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\tau^{-1} \sigma \tau \tau^{-1})]
\]

\[
= N! \sum_{\sigma} g(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)] = N! \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(g)],
\]

where we have used the cyclicity of the trace and the commutativity of \( U(\tau) \) with \( \otimes^N G \).

Putting \( g = e(T) \) and using (3.3) and \( P_{PB} = \sum_{T \in \mathcal{T}_p^N} U(d(T)) \), we obtain the first equation. The second one is obvious. \( \square \)

Now, let us consider the thermodynamic limit

\[ L, N \to \infty, \quad N/L^d \to \rho > 0. \quad (3.5) \]

We need the heat operator \( G = e^{\beta \Delta} \) on \( L^2(\mathbb{R}^d) \). In the following, \( f \) is a nonnegative continuous function having a compact support. It is supposed to be fixed in the thermodynamic limit. Its support will be contained in \( \Lambda_L \) for large enough \( L \).

We get the limiting random point field \( \mu_{pB}^{3B} \) on \( \mathbb{R}^d \) for the low density region.

**Theorem 3.2** The finite random point field for para-bosons of order 3 defined above converge weakly to the random point field whose Laplace transform is given by

\[
\int e^{-\langle f, \xi \rangle} d\mu_{pB}^{3B}(\xi) = \mathrm{Det}[1 + \sqrt{1 - e^{-\beta |p|^2}} r_* G(1 - r_* G)^{-1} \sqrt{1 - e^{-\beta |p|^2}}]^{-3}
\]

in the thermodynamic limit, where \( r_* \in (0, 1) \) is determined by

\[
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta |p|^2}}{1 - r_* e^{-\beta |p|^2}} = (r_* G(1 - r_* G)^{-1})(x, x),
\]

if

\[
\frac{\rho}{3} < \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta |p|^2}}{1 - e^{-\beta |p|^2}}.
\]

Remark : The high density region \( \rho \geq 3\rho_c \) is related to the Bose-Einstein condensation. We need a different analysis for the region. See [TIb] for the case of \( p = 1 \) and 2.

### 3.2 Para-fermions of order 3

For Young tableau \( T \), \( T' \) denotes the tableau obtained by exchanging the rows and the columns of \( T \), i.e., \( T' \) is the transpose of \( T \). The transpose \( \lambda' \) of the frame \( \lambda \) can be defined similarly. Then, \( T' \) lives on \( \lambda' \) if \( T \) lives on \( \lambda \). It is obvious that

\[ \mathcal{R}(T') = \mathcal{C}(T), \quad \mathcal{C}(T') = \mathcal{R}(T). \]
The generating functional of the point field $\mu_{L,N}^{pF}$ for $N$ para-fermions of order $p$ in the box $\Lambda_L$ is given by

$$
\int e^{-<j,\xi> }d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in \mathcal{T}_p^N} \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}) U(d(T'))]}{\sum_{T\in \mathcal{T}_p^N} \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G) U(d(T'))]}
$$

as in the case of para-bosons of order $p$. And the following expressions also hold.

**Lemma 3.3**

$$
\int e^{-<j,\xi> }d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_T'(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)]}{\sum_{T\in \mathcal{T}_p^N} \sum_{\sigma \in \mathcal{S}_N} \chi_T'(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)]}
$$

(3.6)

$$
\int e^{-<j,\xi> }d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_{T'} \{\tilde{G}_L(x_i, x_j)\} dx_1 \cdots dx_N}{\sum_{T\in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_{T'} \{G_L(x_i, x_j)\} dx_1 \cdots dx_N}
$$

(3.7)

**Theorem 3.4** The finite random point fields for para-fermions of order 3 defined above converge weakly to the point field $\mu_3^{3F}$ whose Laplace transform is given by

$$
\int e^{-<j,\xi> }d\mu_3^{3F}(\xi) = \text{Det}[1 - \sqrt{1 - e^{-r_*G(1 + r_*G)^{-1}}\sqrt{1 - e^{-f}}}]^3
$$

in the thermodynamic limit (3.5), where $r_* \in (0, \infty)$ is determined by

$$
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta|p|^2}}{1 + r_* e^{-\beta|p|^2}} = (r_* G(1 + r_* G)^{-1})(x, x).
$$

(3.8)

### 4 Proof of Theorem 3.4

In the rest of this paper, we use results in [T1a] frequently. We refer them e.g., Lemma I.3.2 for Lemma 3.2 of [T1a]. Let $\psi_T$ be the character of the induced representation $\text{Ind}_{\mathcal{R}(T)}^{\mathcal{S}_N}[1]$, where 1 is the one dimensional representation $\mathcal{R}(T) \ni \sigma \rightarrow 1$, i.e.,

$$
\psi_T(\sigma) = \sum_{\tau \in \mathcal{S}_N} \langle \tau, L(\sigma) R(a(T)) \tau \rangle = \chi_{a(T)}(\sigma).
$$

Since the characters $\chi_T$ and $\psi_T$ depend only on the frame on which the tableau $T$ lives, not on $T$ itself, we use the notation $\chi_\lambda$ and $\psi_\lambda$ ($\lambda \in M_p^N$) instead of $\chi_T$ and $\psi_T$, respectively.
Let \( \delta \) be the frame \((p-1, \cdots, 2, 1, 0) \in M_p^N\). Generalize \( \psi_\mu \) to those \( \mu = (\mu_1, \cdots, \mu_p) \in \mathbb{Z}^p \) which satisfies \( \sum_{j=1}^p \mu_j = N \) by

\[
\psi_\mu = 0 \quad \text{for} \quad \mu \in \mathbb{Z}^p - \mathbb{Z}_+^p
\]

and

\[
\psi_\mu = \psi_{\pi \mu} \quad \text{for} \quad \mu \in \mathbb{Z}_+^p \quad \text{and} \quad \pi \in S_p \quad \text{such that} \quad \pi \mu \in M_p^N,
\]

where \( \mathbb{Z}_+ = \{0\} \cup \mathbb{N} \). Then the determinantal form \([JK81]\) can be written as

\[
\chi_\lambda = \sum_{\pi \in S_p} \text{sgn} \pi \psi_{\lambda + \delta - \pi \delta}. \tag{4.1}
\]

Let us recall the relations

\[
\chi_{T'}(\sigma) = \text{sgn} \sigma \chi_T(\sigma), \quad \psi_{T'}(\sigma) = \text{sgn} \sigma \psi_T(\sigma),
\]

where

\[
\psi_{T'}(\sigma) = \sum_{\tau} \langle \tau, L(\sigma)R(b(T'))\tau \rangle = \chi_{b(T')}(\sigma)
\]
denotes the character of the induced representation \( \text{Ind}_{C(T')}^{S_N}[\text{sgn}] \), where \( \text{sgn} \) is the representation \( C(T') = \mathcal{R}(T) \ni \sigma \to \text{sgn} \sigma \). Then we have a variant of (4.1)

\[
\chi_{\lambda'} = \sum_{\pi \in S_p} \text{sgn} \pi \psi_{\lambda' + \delta' - (\pi \delta')} \tag{4.2}
\]

Now we consider the denominator of (3.6). Let \( T \in \mathcal{T}_p^N \) live on \( \mu = (\mu_1, \cdots, \mu_p) \in M_p^N \). Thanks to (3.4) for \( g = b(T') \), we have

\[
\sum_{\sigma \in S_N} \varphi_{T'}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)U(\sigma)) = N! \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G)U(b(T')))
\]

\[
= N! \prod_{j=1}^p \text{Tr}_{\otimes^j \mathcal{H}_L}((\otimes^j G)A_{\mu_j}),
\]

where \( A_n = \sum_{\tau \in S_n} \text{sgn}(\tau)U(\tau)/n! \) is the anti-symmetrization operator on \( \otimes^n \mathcal{H}_L \). In the last step, we have used

\[
b(T') = \prod_{j=1}^p \sum_{\sigma \in \mathcal{R}_j} \frac{\text{sgn} \sigma}{\# \mathcal{R}_j},
\]

where \( \mathcal{R}_j \) is the symmetric group of \( \mu_j \) numbers which lie on the \( j \)-th row of the tableau \( T \). Then (4.2) yields

\[
\sum_{\sigma \in S_N} \chi_\lambda(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G_L)U(\sigma)) = \sum_{\pi \in S_p} \text{sgn} \pi \sum_{\sigma \in S_N} \varphi_{\lambda' + \delta' - (\pi \delta')} \psi_{\lambda' + \delta' - (\pi \delta')} \text{Tr}_{\otimes^N \mathcal{H}_L}((\otimes^N G_L)U(\sigma))
\]
\[= N! \sum_{\pi \in S_p} \prod_{j=1}^{p} \Tr_{\otimes^{(\lambda_j-j+p(j))}} (\otimes^{(\lambda_j-j+p(j))} G_L) A_{\lambda_j-j+p(j)}\]

Here we understand that \( \Tr_{\otimes^n} ((\otimes^n G) A_n) = 1 \) if \( n = 0 \) and \( = 0 \) if \( n < 0 \) in the last expression. Let us recall the defining formula of Fredholm determinant

\[\text{Det}(1 + J) = \sum_{n=0}^{\infty} \Tr_{\otimes^n} ((\otimes^n J) A_n)\]

for a trace class operator \( J \). We use it in the form

\[\Tr_{\otimes^n} ((\otimes^n G) A_n) = \oint_{S_r(0)} \frac{dz}{2\pi iz^{n+1}} \text{Det}(1 + zG_L), \quad (4.3)\]

where \( r > 0 \) can be set arbitrary. Note that the right hand side equals to 1 for \( n = 1 \) and to 0 for \( n < 0 \). Then we have the following expression of the denominator of (3.6)

\[\sum_{\lambda \in \mathcal{M}^n_p} \sum_{\pi \in S_p} \chi_{\lambda'}(\pi) \Tr_{\otimes^n} ((\otimes^n G_L) U(\pi))\]

\[= N! \sum_{\lambda \in \mathcal{M}^n_p} \prod_{j=1}^{p} \frac{\Det(1 + z_j G_L)}{2\pi iz_j^{\lambda_j-j+p(j)+1}}. \quad (4.4)\]

The similar formula for the numerator also holds.

Now we concentrate on the case of \( p = 3 \). To make the thermodynamic limit procedure explicit, let us take a sequence \( \{L_N\}_{N \in \mathbb{N}} \) which satisfies \( N/L_N^d \rightarrow \rho \) as \( N \rightarrow \infty \). In the followings, \( r = r_k \in [0, \infty) \) denotes the unique solution of

\[\Tr r G_{L_N} (1 + r G_{L_N})^{-1} = k \quad (4.5)\]

for \( 0 \leq k \leq N \). We suppress the \( N \) dependence of \( r_k \). The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.5) is a continuous and monotone function of \( r \). See Lemma I.3.2, for details. We put

\[v_k = \Tr [r_k G_{L_N} (1 + r_k G_{L_N})^{-2}] \quad (4.6)\]

and

\[D_{k,l,m} = \oint_{S_r(0)^3} \frac{\left[\prod_{j=1}^{3} \Det(1 + z_j G_{L_N})\right] (z_1 - z_2)(z_2 - z_3)}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} dz_1 dz_2 dz_3, \]
for $k, l, m \in \mathbb{Z}$. Note that $D_{k,l,m} = 0$ if at least one of $k, l, m$ is negative. Summing over $\lambda_1$ and $\lambda_3$ in (4.4) for $p = 3$, we get

$$
\sum_{\lambda \in M_1^N} \sum_{\sigma \in S_N} \chi_N(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_{L_N}}[(\otimes^N G_{L_N}) U(\sigma)] = N! \left( \sum_{l=1}^{[N/3]+1} D_{N+3-2l,l,l-1} + \sum_{l=[N/3]+2}^{[N/2]+1} D_{l,l,N+2-2l} \right).
$$

Since $r > 0$ of the contour $S_r(0)$ is arbitrary, we may change the complex integral variables $z_j = r_j \eta_j$ with $\eta_j \in S_1(0)$ for $j = 1, 2, 3$. Thanks to the property of Fredholm determinant, we have

$$
\text{Det}[1 + z_j G_{L_N}] = \text{Det}[1 + r_j G_{L_N}] \text{Det}[1 + (\eta_j - 1)r_j G_{L_N}(1 + r_j G_{L_N})^{-1}]
$$

Now, we can put

$$
\mathcal{F}_{k,l,m} = \frac{r_0^{3k_0} v_0^{5/2}}{\text{Det}[1 + r_0 G_{L_N}]^3} D_{k,l,m} = R_{k,l,m} v_0^{5/2} I_{k,l,m},
$$

where

$$
R_{k_1,k_2,k_3} = \prod_{j=1}^{3} \frac{r_0^{k_j} \text{Det}[1 + r_j G_{L_N}]}{r_j^{k_j} \text{Det}[1 + r_0 G_{L_N}]}
$$

and

$$
I_{k_1,k_2,k_3} = \oint \oint \oint_{S_1(0)^3} \left( \prod_{j=1}^{3} \text{Det}[1 + (\eta_j - 1)r_j G_{L_N}(1 + r_j G_{L_N})^{-1}] \right)
$$

$$
\times (r_1 \eta_1 - r_2 \eta_2)(r_2 \eta_2 - r_3 \eta_3) \frac{d\eta_1 d\eta_2 d\eta_3}{(2\pi i)^3 \eta_1^{k_1+1} \eta_2^{k_2+1} \eta_3^{k_3+1}}.
$$

Here $k_0 = (N + 2)/3$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$ satisfy $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 3k_0$. We use the abbreviation $r_\nu$ and $v_\nu$ for $r_{k_\nu}$ and $v_{k_\nu}$ ($\nu = 0, 1, 2, 3$), respectively. Here, let us recall that $r_0 \rightarrow r_*$ in the thermodynamic limit because of $k_0/L^d \rightarrow \rho/3$, (3.8) and Lemma I.3.5.

Define a sequence $\{f_N\}_{N \in \mathbb{N}}$ of nonnegative functions on $\mathbb{R}$ by

$$
f_N(x) = \begin{cases} 
\mathcal{F}_{l,l,N+2-2l} & \text{for } \sqrt{N+2} x \in [l-1-(N+2)/3, l-(N+2)/3) \\
\mathcal{F}_{l,l,N+3-2l,l-1} & \text{for } \sqrt{N+2} x \in [l-1-(N+2)/3, l-(N+2)/3) \\
0 & \text{otherwise.}
\end{cases}
$$

Then the denominator of (3.6) becomes

$$
N! \sqrt{N+2} \frac{\text{Det}[1 + r_0 G_{L_N}]^3}{r_0^{3k_0} v_0^{5/2}} \int_{-\infty}^{\infty} f_N(x) \, dx.
$$
We introduce $\tilde{D}_{k,l,m}$, $\tilde{\mathcal{F}}_{k,l,m}$ and $\tilde{f}_{N}$ using $\tilde{G}_{L_{N}}$ instead of $G_{L_{N}}$ in $D_{k,l,m}$, $\mathcal{F}_{k,l,m}$ and $f_{N}$ and so on, to get the expression

$$E_{L,N}^{3F}[e^{-<f,f>}] = \frac{\text{Det}[1 + \tilde{r}_{0}\tilde{G}_{L_{N}}]^{3} r_{0}^{3k_{0}} v_{0}^{5/2} \int_{-\infty}^{\infty} \tilde{f}_{N}(x) \, dx}{\text{Det}[1 + r_{0}G_{L_{N}}]^{3} r_{0}^{3k_{0}} v_{0}^{5/2} \int_{-\infty}^{\infty} f_{N}(x) \, dx}.$$  

From Lemma I.3.6, we have

$$\frac{\tilde{v}_{0}}{v_{0}} \rightarrow 1 \quad (4.7)$$

in the thermodynamic limit. Similarly, we obtain

$$\frac{r_{0}^{k_{0}} \text{Det}[1 + \tilde{r}_{0}\tilde{G}_{L_{N}}]}{\tilde{r}_{0}^{k_{0}} \text{Det}[1 + r_{0}G_{L_{N}}]} \rightarrow \text{Det}[1 - \sqrt{1 - e^{-\beta r_{*}G(1 + r_{*}G)^{-1}\sqrt{1 - e^{-\beta}}}}]$$

from the proof of Theorem I.3.1 (see Eq. (a-c), where we should read $N$ as $k_{0}$, $z_{N}$ as $r_{0}$ and $\alpha = -1$). Thus Theorem 3.4 is proved, if we get the following lemma:

**Lemma 4.1** Under the thermodynamic limit,

$$\int_{-\infty}^{\infty} \tilde{f}_{N}(x) \, dx, \int_{-\infty}^{\infty} f_{N}(x) \, dx \rightarrow \int_{-\infty}^{\infty} e^{-2\rho x^{2}/\kappa_{sucb}} \frac{dx}{(2\pi)^{3/2}}$$

hold, where

$$\kappa = \int \frac{dp}{(2\pi)^{d}} \frac{r_{*}e^{-\beta|p|^{2}}}{1 + r_{*}e^{-\beta|p|^{2}}}.$$  

**Proof:** Let $k, r, v \in [0, \infty)$ satisfy the relations

$$k = \text{Tr} [rG_{L_{N}}(1 + rG_{L_{N}})^{-1}], \quad v = \text{Tr} [rG_{L_{N}}(1 + rG_{L_{N}})^{-2}]. \quad (4.8)$$

1° There exist positive constants $c_{1}$ and $c_{2}$ which depend only on the density $\rho$ such that

$$r_{j} \leq c_{1}, \quad r_{j} - r_{l} \leq c_{1} \frac{k_{j} - k_{l}}{k_{l}}, \quad c_{2}k_{j} \leq v_{j} \leq k_{j},$$

hold for $k_{j}, k_{l} > 0$ satisfying $k_{j} > k_{l}$.

We have $v \leq k$ and $r \leq r_{N}$ for $k \leq N$. Recall $r_{N}$ converges to the constant $r^{*}$ which determined by

$$\int \frac{dp}{(2\pi)^{d}} \frac{r^{*}e^{-\beta|p|^{2}}} {1 + r^{*}e^{-\beta|p|^{2}}} = \rho.$$  

Then $\{r_{N}\}$ is bounded from above. Hence we have $r \leq r_{N} \leq c_{1}$ and $v \geq k/(1 + r_{N}) \geq k/(1 + c_{1})$ since $0 \leq G_{L_{N}} \leq 1$. Thanks to $dk/dr = v/r \geq k/c_{1}$, we get $c_{1} \int_{r_{N}}^{k} dk / k \geq \int_{r_{N}}^{k} dr$, which yields the second inequality. ◇
There exist positive constants $c_0', c_1'$ and $c_2'$ which depend only on $\rho$ such that

$$A_{k,n} = \oint_{S_1(0)} \text{Det}[1+(\eta-1)rG_{L_N}(1+rG_{L_N})^{-1}] \frac{(\eta-1)^n}{2\pi i \eta^{k+1}} \frac{d\eta}{2\pi i} \quad (n = 0, 1, 2, k = 0, 1, \cdots, N)$$

satisfy

$$A_{k,0} = (1 + o(1)) / \sqrt{2\pi v}, \quad A_{k,2} = (-1 + o(1)) / \sqrt{2\pi v^3} \quad \text{for large } k \leq N$$

and

$$|A_{k,0}| \leq c_0'/\sqrt{1+k}, \quad |A_{k,1}| \leq c_1'/\sqrt{1+k^3}, \quad |A_{k,2}| \leq c_2'/\sqrt{1+k^3} \quad \text{for all } k = 0, 1, \cdots, N.$$

Put

$$h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) e^{-ikx/\sqrt{v}} \text{Det}[1+(e^{ix/\sqrt{v}}-1)rG_{L_N}(1+rG_{L_N})^{-1}],$$

as in the proof of Proposition I.A.2. Then, we have

$$|h_k(x)| \leq e^{-2x^2/(\pi^2)} \in L^1(\mathbb{R}) \quad (4.9)$$

and

$$h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\gamma v/3]}(x) x e^{-x^2/2} e^{\delta} \rightarrow e^{-x^2/2} \quad \text{as } N \geq k \rightarrow \infty \quad (4.10)$$

where $|\delta| \leq 4|x^3|/9\sqrt{3v}$. Setting $\eta = \exp(ix/\sqrt{v})$, we have

$$A_{k,n} = \int_{\infty}^{\infty} \frac{(e^{ix/\sqrt{v}}-1)^n h_k(x)}{2\pi \sqrt{v}} dx.$$

Then, $|A_{k,0}| \leq c'/\sqrt{v} < c''/\sqrt{k}$ for $k = 1, 2, \cdots, N$. On the other hand, Cauchy's integral formula yields $A_{0,0} = 1$, readily. So we get the bound $|A_{k,0}| \leq c_0'/\sqrt{1+k}$. Now the asymptotic behavior of $A_{k,0}$ can be derived by the use of dominated convergence theorem and (4.10).

For $n = 1$, we have

$$A_{k,1} = \frac{i}{2\pi v} \int_{-\infty}^{\infty} x h_k(x) dx + R,$$

where

$$|R| \leq \int \frac{x^2}{4\pi \sqrt{v^3}} h_k(x) dx = O(1/\sqrt{v^3}).$$

The integrand of first term can be written as

$$x h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) x e^{-x^2/2} + \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) x (e^{\delta} - 1) e^{-x^2/2}.$$
The integral of the first term of the right hand side is 0. While the second term is bounded by $|x|\delta h(x)$, since $|e^\delta - 1| \leq |\delta| e^{\delta v_0}$. For the third term, we use (4.9). Then we get the bound $|\int xh_k(x) dx| \leq c''/\sqrt{v}$ for $k \geq 1$. Together with $A_{0,1} = 0$, the bounds for $A_{k,1}$ are derived. Similarly, we get the formulae for $A_{k,2}$.

3° Let $(k_1, k_2, k_3) \in \mathbb{Z}_+$ satisfies

\[ k_1 \geq k_2 \geq k_3, \quad k_1 + k_2 + k_3 = 3k_0 = N + 2 \]

and

\[ k_1 = k_2 \quad \text{or} \quad k_2 = k_3 + 1. \]

Then the estimates

\[ |v_0^{5/2}I_{k_1, k_2, k_3}| \leq c \left( \frac{k_0}{1+k_3} \right)^{5/2} \leq c' e^{(k_0-k_3)^2/4k_0} \]

hold for all such $(k_1, k_2, k_3)$ and

\[ v_0^{5/2}I_{k_1, k_2, k_3} = \frac{v_0^{5/2}(1+o(1))}{(2\pi)^{3/2}v_1^{1/2}v_2^{3/2}v_3^{1/2}} \]

holds for large $N$ and $(k_1, k_2, k_3)$, where $c, c'$ are positive constants depending only on $\rho$.

In fact, expanding

\[(r_1\eta_1-r_2\eta_2)(r_2\eta_2-r_3\eta_3) = (r_1(\eta_1-1)-r_2(\eta_2-1)+r_1-r_2)(r_2(\eta_2-1)-r_3(\eta_3-1)+r_2-r_3)\]

in the integrand of $I_{k_1, k_2, k_3}$, we get the first inequality from 1° and 2°. The second inequality is obvious. Similarly, the asymptotic behavior follows.

4°

\[ R_{k_1, k_2, k_3} = e^{-\sum_{i=1}^3 (k_0-k_i)^2/2v_j^i} \]

holds where $v_j^i = \text{Tr} [r_j^i G_{L_N} (1 + r_j^i G_{L_N}^{-2})]$ for a certain middle point $r_j^i$ between $r_0$ and $r_j$. Especially, we have the bound

\[ R_{k_1, k_2, k_3} \leq e^{-(k_0-k_3)^2/2k_0}. \]

Recall that $G_{L_N}$ is a non-negative trace class self-adjoint operator. If we put

\[ \psi(t) = \log \det [1 + e^t G_{L_N}] = \text{Tr} [\log (1 + e^t G_{L_N})], \]

we have

\[ \psi'(t) = \text{Tr} [e^t G_{L_N} (1 + e^t G_{L_N})^{-1}], \quad \psi''(t) = \text{Tr} [e^t G_{L_N} (1 + e^t G_{L_N})^{-2}]. \]
In the equality
\[ \psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0) = \int_t^{t_0} (s - t_0)\psi''(s)\,ds + t_0(\psi'(t_0) - \psi'(t)), \]
apply
\[ \int_t^{t_0} (s - t_0)\psi''(s)\,ds = \int_t^{t_0} ds \int_{t_0}^{s} du \psi''(s), \frac{\psi''(u)}{\psi'(u)} = -\frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)} \]
where $u_c$ is a middle point of $t$ and $t_0$. Then we obtain
\[ \frac{e^{t_0\psi'(t_0)}}{e^{t\psi(t)}}, \frac{\text{Det}[1 + e^{t}G_{L_N}]}{\text{Det}[1 + e^{t_0}G_{L_N}]} = e^{\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0)} = e^{t_0(\psi'(t_0) - \psi'(t)) - \frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)}}. \]
Set $e^t = r_j$ and $e^{t_0} = r_0$. Then $\psi'(t) = k_j$, $\psi'(t_0) = k_0$, $\psi''(t) = v_j$, and $\psi''(t_0) = v_0$ hold. Taking the product of those equalities for $j = 1, 2$ and $3$, we get the desired expression, since $3k_0 = k_1 + k_2 + k_3$. \hfill \Box

5° Recall that the functions $\varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L)$ $(k \in \mathbb{Z}^d)$ constitute a C.O.N.S. of $L^2(\Lambda_L)$, where $G_L\varphi_k^{(L)} = e^{-\beta|2\pi k/L|^2}\varphi_k^{(L)}$ holds for all $k \in \mathbb{Z}^d$. Then, we obtain
\[ v_0 \frac{L^d}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \left\{ \frac{\beta|2\pi k/L|^2}{1 + \beta|2\pi k/L|^2} \right\} \to \kappa, \]
in the thermodynamic limit, since $k_0/L^d \to \rho/3$ and $r_0 \to r_*$ hold.

From 3° and 4°, we have a bound
\[ |F_{k_1, k_2, k_3}| \leq c'e^{-(k_0 - k_3)^2/4k_0} \] (4.11)
and
\[ F_{k_1, k_2, k_3} = \frac{v_0^{5/2}(1 + o(1))}{(2\pi)^3/2v_1^{3/2}v_2^{3/2}v_3^{1/2}} e^{-\Sigma_j (k_0 - k_j)/2v_j'}, \] (4.12)
for large $N, k_1, k_2, k_3$, where $v_j'$ is a mean value which we have written $\psi''(u_c)$ in 4°. For $l = 1, 2, \cdots, [N/3] + 1$, $\sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3]$ implies $|l - 1 - (N + 2)/3| \geq \sqrt{N + 2}|x|$, hence we get the bound
\[ f_N(x) = F_{N+3-2l,l,l-1} \leq c'e^{-(N+2)x^2/4k_0} \leq c'e^{-3x^2/4}. \]
We also get $f_N(x) \leq c'\exp(-3x^2/4)$ for the other cases, similarly.

For fixed $x \in \mathbb{R}$, we choose $l \in \mathbb{Z}$ such that $\sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3]$. Then we have $v_j' / v_0 \to 1$ ($j = 1, 2, 3$) and
\[ \sum_{j=1}^{3} \frac{(k_0 - k_j)^2}{v_j'} = \frac{4N}{v_0} x^2 + o(1).\]
Hence, we obtain \( f_N(x) \rightarrow (2\pi)^{-3/2} \exp(-2\rho x^2/\kappa) \) in the thermodynamic limit. Thus the dominated convergence theorem yields the desired result for \( f_N \). Because of (4.7), the one for \( \tilde{f}_N \) can be proved similarly.

\[ \square \]

参考文献


