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Kyoto University
Random Point Fields for Para-Particles of order 3

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概要
Random point fields which describe gases consist of para-particles of order three are given by means of the canonical ensemble approach. The analysis for the case of the para-fermion gases is discussed in full detail.

1 Introduction
The purpose of this note is to apply the method which we have developed in [T1a] to statistical mechanics of gases which consist of para-particles of order 3. We begin with quantum mechanical thermal systems of finite fixed numbers of para-bosons and/or para-fermions in the bounded boxes in $\mathbb{R}^d$. Taking the thermodynamic limits, random point fields on $\mathbb{R}^d$ are obtained. We will see that the point fields obtained in this way are those of $\alpha = \pm 1/3$ given in [ShTa03].

We use the representation theory of the symmetric group. (cf. e.g. [JK81, S91, Si96]) Its basic facts are reviewed briefly, in section 2, along the line on which the quantum theory of para-particles are formulated. We state the results in section 3. Section 4 devoted to the full detail of the discussion on the thermodynamic limits for para-fermion's case.

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2 Brief review on Representation of the symmetric group

We say that \((\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n\) is a Young frame of length \(n\) for the symmetric group \(S_N\) if
\[
\sum_{j=1}^{n} \lambda_j = N, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0.
\]
We associate the Young frame \((\lambda_1, \lambda_2, \cdots, \lambda_n)\) with the diagram of \(\lambda_1\)-boxes in the first row, \(\lambda_2\)-boxes in the second row, ..., and \(\lambda_n\)-boxes in the \(n\)-th row. A Young tableau on a Young frame is a bijection from the numbers 1, 2, ..., \(N\) to the \(N\) boxes of the frame.

Let \(M_p^N\) be the set of all the Young frames for \(S_N\) which have lengths less than or equal to \(p\). For each frame in \(M_p^N\), let us choose one tableau from those on the frame. The choices are arbitrary but fixed. \(\mathcal{T}_p^N\) denotes the set of all the tableaux chosen in this way. The row stabilizer of tableau \(T\) is denoted by \(\mathcal{R}(T)\), i.e., the subgroup of \(S_N\) consists of those elements that keep all rows of \(T\) invariant, and \(C(T)\) the column stabilizer whose elements preserve all columns of \(T\).

Let us introduce the three elements
\[
a(T) = \frac{1}{\# \mathcal{R}(T)} \sum_{\sigma \in \mathcal{R}(T)} \sigma, \quad b(T) = \frac{1}{\# \mathcal{C}(T)} \sum_{\sigma \in \mathcal{C}(T)} \text{sgn}(\sigma)\sigma
\]
and
\[
e(T) = \frac{d_T}{N!} \sum_{\sigma \in \mathcal{R}(T)} \sum_{\tau \in \mathcal{C}(T)} \text{sgn}(\tau)\sigma\tau = c_T a(T)b(T)
\]
of the group algebra \(\mathbb{C}[S_N]\) for each \(T \in \mathcal{T}_p^N\), where \(d_T\) is the dimension of the irreducible representation of \(S_N\) corresponding to \(T\) and \(c_T = d_T \# \mathcal{R}(T) \# \mathcal{C}(T) / N!\). As is known,
\[
a(T_1) a(T_2) = b(T_1) a(T_1) = 0 \quad (2.1)
\]
hold for any \(\sigma \in S_N\) if \(T_2 \rightarrow T_1\). The relations
\[
a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1)e(T_2) = 0 \quad (T_1 \neq T_2) \quad (2.2)
\]
also hold for \(T, T_1, T_2 \in \mathcal{T}_p^N\). For later use, let us introduce
\[
d(T) = e(T)a(T) = c_T a(T)b(T)a(T) \quad (2.3)
\]
for \(T \in \mathcal{T}_p^N\). They satisfy
\[
d(T)^2 = d(T), \quad d(T_1)d(T_2) = 0 \quad (T_1 \neq T_2) \quad (2.4)
\]
which are shown readily from (2.2) and (2.1). The inner product $\langle \cdot, \cdot \rangle$ of $C[S_N]$ is defined by

$$\langle \sigma, \tau \rangle = \delta_{\sigma \tau} \quad \text{for } \sigma, \tau \in S_N$$

and the sesqui-linearity.

The left representation $L$ and the right representation $R$ of $S_N$ on $C[S_N]$ are defined by

$$L(\sigma)g = L(\sigma) \sum_{\tau \in S_N} g(\tau) \tau = \sum_{\tau \in S_N} g(\tau) \sigma \tau = \sum_{\tau \in S_N} g(\sigma^{-1} \tau) \tau$$

and

$$R(\sigma)g = R(\sigma) \sum_{\tau \in S_N} g(\tau) \tau = \sum_{\tau \in S_N} g(\tau) \tau \sigma^{-1} = \sum_{\tau \in S_N} g(\tau \sigma) \tau,$$

respectively. Here and hereafter we identify $g : S_N \to C$ and $\sum_{\tau \in S_N} g(\tau) \tau \in C[S_N]$. They are extended to the representation of $C[S_N]$ on $C[S_N]$ as

$$L(f)g = fg = \sum_{\sigma, \tau} f(\sigma) g(\tau) \sigma \tau = \sum_{\sigma} \left( \sum_{\tau} f(\sigma \tau^{-1}) g(\tau) \right) \sigma$$

and

$$R(f)g = g \hat{f} = \sum_{\sigma, \tau} g(\sigma) f(\tau) \sigma \tau^{-1} = \sum_{\sigma} \left( \sum_{\tau} g(\sigma \tau) f(\tau) \right) \sigma,$$

where $\hat{f} = \sum_{\tau} \hat{f}(\tau) \tau = \sum_{\tau} f(\tau^{-1}) \tau = \sum_{\tau} f(\tau) \tau^{-1}$.

The character of the irreducible representation of $S_N$ corresponding to tableau $T \in \mathcal{T}_N^N$ is obtained by

$$\chi_T(\tau) = \sum_{\tau \in S_N} \langle \tau, L(\sigma) R(e(T)) \tau \rangle = \sum_{\tau \in S_N} \langle \tau, \sigma \tau e(\overline{T}) \rangle.$$

We introduce a tentative notation

$$\chi_g(\sigma) \equiv \sum_{\tau \in S_N} \langle \tau, L(\sigma) R(g) \tau \rangle = \sum_{\tau, \gamma \in S_N} \langle \tau, \sigma \tau \gamma^{-1} \rangle g(\gamma) = \sum_{\tau \in S_N} g(\tau^{-1} \sigma \tau) \quad (2.5)$$

for $g = \sum_{\tau} g(\tau) \tau \in C[S_N]$. Then $\chi_T = \chi_{e(T)}$ holds.

Now let us consider representations of $S_N$ on Hilbert spaces. Let $\mathcal{H}_L$ be a certain $L^2$ space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its $N$-fold Hilbert space tensor product. Let $U$ be the representation of $S_N$ on $\otimes^N \mathcal{H}_L$ defined by

$$U(\sigma) \varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)} \quad \text{for } \varphi_1, \ldots, \varphi_N \in \mathcal{H}_L,$$

or equivalently by

$$(U(\sigma)f)(x_1, \ldots, x_N) = f(x_{\sigma(1)}, \ldots, x_{\sigma(N)}) \quad \text{for } f \in \otimes^N \mathcal{H}_L.$$

Obviously, $U$ is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. We extend $U$ for $C[S_N]$ by linearity. Then $U(a(T))$ is an orthogonal projection because of $U(a(T))^* = U(a(T)) = U(a(T))$ and (2.2). So are $U(b(T))$, $U(d(T))$, and $P_{PB} = \sum_{T \in \mathcal{T}_N^N} U(d(T))$. Note that $\text{Ran } U(d(T)) = \text{Ran } U(e(T))$ because of $d(T)e(T) = e(T)$ and $e(T)d(T) = d(T)$. 
3 Para-statistics and Random point fields

3.1 Para-bosons of order 3

Let us consider a quantum system of $N$ para-bosons of order $p$ in the box $\Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d$. We refer the literatures \[\text{MeG64, HaT69, StT70}\] for quantum mechanics of para-particles. (See also \[\text{OK69}\].) The arguments of these literatures indicate that the state space of our system is given by $\mathcal{H}^{pB}_{L,N} = P_{pB} \otimes^N \mathcal{H}_L$, where $\mathcal{H}_L = L^2(\Lambda_L)$ with Lebesgue measure is the state space of one particle system in $\Lambda_L$. We need the heat operator $G_L = e^{\beta \Delta_L}$ in $\Lambda_L$, where $\Delta_L$ is the Laplacian in $\Lambda_L$ with periodic boundary conditions.

It is obvious that there is a CONS of $\mathcal{H}^{pB}_{L,N}$ which consists of the vectors of the form $U(d(T))\varphi_{k_1}^{(L)} \otimes \cdots \otimes \varphi_{k_N}^{(L)}$, which are the eigenfunctions of $\otimes^N G_L$. Then, we define the point field $\mu_{L,N}^{pB}$ of $N$ free para-bosons of order $p$ as in section 2 of \[\text{TIa}\] and its generating functional is given by

$$
\int e^{-<f,\xi>} d\mu_{L,N}^{pB}(\xi) = \frac{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{pB}]}{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{pB}]} [f]
$$

where $f$ is a nonnegative continuous function on $\Lambda_L$ and $\tilde{G}_L = G_L^{1/2}e^{-f}G_L^{1/2}$.

Lemma 3.1

$$
\int e^{-<f,\xi>} d\mu_{L,N}^{pB}(\xi) = \frac{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in S_N} \chi_T(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_p^N} \sum_{\sigma \in S_N} \chi_T(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]}
$$

(3.1)

$$
= \frac{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_T \{G_L(x_i, x_j)\}_{1 \leq i,j \leq N} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_p^N} \int_{\Lambda_L^N} \det_T \{G_L(x_i, x_j)\}_{1 \leq i,j \leq N} dx_1 \cdots dx_N}
$$

(3.2)

Remark 1: $\mathcal{H}^{pB}_{L,N} = P_{pB} \otimes^N \mathcal{H}_L$ is determined by the choice of the tableaux $T$'s. The spaces corresponding to different choices of tableaux are different subspaces of $\otimes^N \mathcal{H}_L$. However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, $\chi_T(\sigma)$ depends only on the frame on which the tableau $T$ is defined.

Remark 2: $\det A = \sum_{\sigma \in S_N} \chi_T(\sigma) \prod_{i=1}^N A_{\sigma(i)}$ in (3.2) is called immanant.

Proof: Since $\otimes^N G$ commutes with $U(\sigma)$ and $a(T)e(T) = e(T)$, we have

$$
\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(d(T))] = \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))U(a(T))]
$$

$$
= \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[U(a(T))(\otimes^N G_L)U(e(T))] = \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))].
$$

(3.3)
On the other hand, we get from (2.5) that
\[
\sum_{\sigma \in \mathcal{S}_N} g(\sigma) \operatorname{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)] = \sum_{\tau, \sigma} g(\tau^{-1} \sigma \tau) \operatorname{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\tau)U(\sigma)U(\tau^{-1})]
\]
\[
= N! \sum_{\sigma} g(\sigma) \operatorname{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(g)]
\]
where we have used the cyclicity of the trace and the commutativity of $U(\tau)$ with $\otimes^N G$. Putting $g = e(T)$ and using (3.3) and $P_{pB} = \sum_{T \in \mathcal{T}_{p}^{N}} U(d(T))$, we obtain the first equation. The second one is obvious.

Now, let us consider the thermodynamic limit
\[
L, N \to \infty, \quad N/L^d \to \rho > 0.
\]
We need the heat operator $G = e^{\beta A}$ on $L^2(\mathbb{R}^d)$. In the following, $f$ is a nonnegative continuous function having a compact support. It is supposed to be fixed in the thermodynamic limit. Its support will be contained in $\Lambda_L$ for large enough $L$.

We get the limiting random point field $\mu^{3B}_\rho$ on $\mathbb{R}^d$ for the low density region.

**Theorem 3.2** The finite random point field for para-bosons of order 3 defined above converge weakly to the random point field whose Laplace transform is given by
\[
\int e^{-\langle f, \xi \rangle} d\mu^{3B}_\rho(\xi) = \operatorname{Det}[1 + \sqrt{1 - r_* \beta} G(1 - r_* \beta)^{-1} \sqrt{1 - e^{-\beta|p|^2}}]^{-3}
\]
in the thermodynamic limit, where $r_* \in (0, 1)$ is determined by
\[
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta|p|^2}}{1 - r_* e^{-\beta|p|^2}} = (r_* G(1 - r_* G)^{-1})(x, x),
\]
if
\[
\frac{\rho}{3} < \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta|p|^2}}{1 - e^{-\beta|p|^2}}.
\]
**Remark**: The high density region $\rho \geq 3 \rho_c$ is related to the Bose-Einstein condensation. We need a different analysis for the region. See [T1b] for the case of $p = 1$ and 2.

### 3.2 Para-fermions of order 3

For Young tableau $T$, $T'$ denotes the tableau obtained by exchanging the rows and the columns of $T$, i.e., $T'$ is the transpose of $T$. The transpose $\lambda'$ of the frame $\lambda$ can be defined similarly. Then, $T'$ lives on $\lambda'$ if $T$ lives on $\lambda$. It is obvious that
\[
\mathcal{R}(T') = \mathcal{C}(T), \quad \mathcal{C}(T') = \mathcal{R}(T).
\]
The generating functional of the point field $\mu_{L,N}^{pF}$ for $N$ para-fermions of order $p$ in the box $\Lambda_{L}$ is given by

$$
\int e^{-<j, \xi>} d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in T_{p}^{N}} \text{Tr}_{\otimes^{N}H_{L}}[(\otimes^{N}\bar{G})U(d(T'))]}{\sum_{T\in T_{p}^{N}} \text{Tr}_{\otimes^{N}H_{L}}[(\otimes^{N}G)U(d(T'))]}
$$

as in the case of para-bosons of order $p$. And the following expressions also hold.

**Lemma 3.3**

$$
\int e^{-<f, \xi>} d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T\in T_{p}^{N}} \sum_{\sigma \in S_{N}} \chi_{T}'(\sigma) \text{Tr}_{\otimes^{N}H_{L}}[(\otimes^{N}G_{L})U(\sigma)]}{\sum_{T\in T_{p}^{N}} \text{det}_{T'}\{G_{L}(x_{j}, x_{j})\} dx_{1} dx_{N}} \quad (3.6)
$$

$$
= \frac{\sum_{T\in T_{p}^{N}} \int_{\Lambda_{L}^{N}} \text{det}_{T'}\{\bar{G}_{L}(x_{i}, x_{j})\} dx_{1} \cdots dx_{N}}{\sum_{T\in T_{p}^{N}} \int_{\Lambda_{L}^{N}} \text{det}_{T'}\{G_{L}(x_{j}, x_{j})\} dx_{1} \cdots dx_{N}} \quad (3.7)
$$

**Theorem 3.4** The finite random point fields for para-fermions of order 3 defined above converge weakly to the point field $\mu_{p}^{3F}$ whose Laplace transform is given by

$$
\int e^{-<j, \xi>} d\mu_{p}^{3F}(\xi) = \text{Det}[1 - \sqrt{1 - e^{-f}} r_{*}G(1 + r_{*}G)^{-1}\sqrt{1 - e^{-f}}]^{3}
$$

in the thermodynamic limit (3.5), where $r_{*} \in (0, \infty)$ is determined by

$$
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^{d}} \frac{r_{e^{-\beta|p|^{2}}}}{1 + r_{e^{-\beta|p|^{2}}}} = (r_{*}G(1 + r_{*}G)^{-1}(x, x)). \quad (3.8)
$$

4 Proof of Theorem 3.4

In the rest of this paper, we use results in [T1a] frequently. We refer them e.g., Lemma I.3.2 for Lemma 3.2 of [T1a]. Let $\psi_{T}$ be the character of the induced representation $\text{Ind}_{R(T)}^{S_{N}}[1]$, where 1 is the one dimensional representation $R(T) \ni \sigma \to 1$, i.e.,

$$
\psi_{T}(\sigma) = \sum_{T' \in S_{N}} <\tau, L(\sigma)R(a(T))\tau> = \chi_{a(T)}(\sigma).
$$

Since the characters $\chi_{T}$ and $\psi_{T}$ depend only on the frame on which the tableau $T$ lives, not on $T$ itself, we use the notation $\chi_{\lambda}$ and $\psi_{\lambda}$ ( $\lambda \in M^{N}_{p}$ ) instead of $\chi_{T}$ and $\psi_{T}$, respectively.
Let $\delta$ be the frame $(p-1, \cdots, 2, 1, 0) \in M_{p}^{N}$. Generalize $\psi_{\mu}$ to those $\mu = (\mu_{1}, \cdots, \mu_{p}) \in \mathbb{Z}_{p}^{*}$ which satisfies $\sum_{j=1}^{p} \mu_{j} = N$ by

$$\psi_{\mu} = 0 \quad \text{for} \quad \mu \in \mathbb{Z}_{p}^{*} - \mathbb{Z}_{p}^{+}$$

and

$$\psi_{\mu} = \psi_{\pi \mu} \quad \text{for} \quad \mu \in \mathbb{Z}_{p}^{+} \quad \text{and} \quad \pi \in S_{p} \quad \text{such that} \quad \pi \mu \in M_{p}^{N},$$

where $\mathbb{Z}_{p}^{+} = \{0\} \cup \mathbb{N}$. Then the determinantal form [JK81] can be written as

$$\chi_{\lambda} = \sum_{\pi \in S_{p}} \text{sgn} \pi \psi_{\lambda+\delta-\pi\delta}. \quad (4.1)$$

Let us recall the relations

$$\chi_{T'}(\sigma) = \text{sgn} \sigma \chi_{T}(\sigma), \quad \varphi_{T'}(\sigma) = \text{sgn} \sigma \psi_{T}(\sigma),$$

where

$$\varphi_{T'}(\sigma) = \sum_{\tau} <\tau, L(\sigma)R(b(T'))\tau > = \chi_{b(T')}(\sigma)$$

denotes the character of the induced representation $\text{Ind}_{C(T')}^{S_{N}}[\text{sgn}]$, where $\text{sgn}$ is the representation $C(T') = \mathcal{R}(T') \ni \sigma \mapsto \text{sgn} \sigma$. Then we have a variant of (4.1)

$$\chi_{\lambda'} = \sum_{\pi \in S_{p}} \text{sgn} \pi \varphi_{\lambda'+\delta'-\pi\delta'}. \quad (4.2)$$

Now we consider the denominator of (3.6). Let $T \in \mathcal{T}_{p}^{N}$ live on $\mu = (\mu_{1}, \cdots, \mu_{p}) \in M_{p}^{N}$. Thanks to (3.4) for $g = b(T')$, we have

$$\sum_{\sigma \in S_{N}} \varphi_{T'}(\sigma) \text{Tr}_{\otimes^{N}H_{L}}((\otimes^{N}G)U(\sigma)) = N! \text{Tr}_{\otimes^{N}H_{L}}((\otimes^{N}G)U(b(T')))$$

$$= N! \prod_{j=1}^{p} \text{Tr}_{\otimes^{\mu_{j}}H_{L}}((\otimes^{\mu_{j}}G)A_{\mu_{j}}),$$

where $A_{n} = \sum_{\tau \in S_{n}} \text{sgn}(\tau)U(\tau)/n!$ is the anti-symmetrization operator on $\otimes^{n}H_{L}$. In the last step, we have used

$$b(T') = \prod_{j=1}^{p} \sum_{\sigma \in \mathcal{R}_{j}} \frac{\text{sgn} \sigma}{\# \mathcal{R}_{j}},$$

where $\mathcal{R}_{j}$ is the symmetric group of $\mu_{j}$ numbers which lie on the $j$-th row of the tableau $T$. Then (4.2) yields

$$\sum_{\sigma \in S_{N}} \chi_{\lambda'}(\sigma) \text{Tr}_{\otimes^{N}H_{L}}((\otimes^{N}G_{L})U(\sigma)) = \sum_{\pi \in S_{p}} \text{sgn} \pi \sum_{\sigma \in S_{N}} \varphi_{\lambda'+\delta'-\pi\delta'}(\sigma) \text{Tr}_{\otimes^{N}H_{L}}((\otimes^{N}G_{L})U(\sigma))$$
Here we understand that $\text{Tr}_{\otimes^n s_{\mathcal{H}_L}}((\otimes^n G)A_n) = 1$ if $n = 0$ and $= 0$ if $n < 0$ in the last expression. Let us recall the defining formula of Fredholm determinant

$$\text{Det}(1 + J) = \sum_{n=0}^{\infty} \text{Tr}_{\otimes^n s_{\mathcal{H}_L}}((\otimes^n J)A_n)$$

for a trace class operator $J$. We use it in the form

$$\text{Tr}_{\otimes^n s_{\mathcal{H}_L}}((\otimes^n G)A_n) = \oint_{S_r(0)} \frac{dz}{2\pi iz^{n+1}} \text{Det}(1 + zG_L),$$

where $r > 0$ can be set arbitrary. Note that the right hand side equals to 1 for $n = 1$ and to 0 for $n < 0$. Then we have the following expression of the denominator of (3.6)

$$\sum_{\lambda \in \mathcal{M}_p^N} \sum_{\pi \in s_p} \chi_{\lambda'}(\pi) \text{Tr}_{\otimes^N s_{\mathcal{H}_L}}((\otimes^N G_L)U(\sigma))$$

for the numerator also holds.

Now we concentrate on the case of $p = 3$. To make the thermodynamic limit procedure explicit, let us take a sequence $\{L_N\}_{N \in \mathbb{N}}$ which satisfies $N/L_N^d \rightarrow \rho$ as $N \rightarrow \infty$. In the followings, $r = r_k \in [0, \infty)$ denotes the unique solution of

$$\text{Tr} rG_{L_N} (1 + rG_{L_N})^{-1} = k$$

for $0 \leq k \leq N$. We suppress the $N$ dependence of $r_k$. The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.5) is a continuous and monotone function of $r$. See Lemma I.3.2, for details. We put

$$v_k = \text{Tr} [r_k G_{L_N} (1 + r_k G_{L_N})^{-2}]$$

and

$$D_{k,l,m} = \oint \oint \oint_{S_r(0)^3} \frac{[\prod_{j=1}^3 \text{Det}(1 + z_j G_{L_N})](z_1 - z_2)(z_2 - z_3)}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} dz_1 dz_2 dz_3,$$
for $k, l, m \in \mathbb{Z}$. Note that $D_{k,l,m} = 0$ if at least one of $k, l, m$ is negative. Summing over $\lambda_1$ and $\lambda_3$ in (4.4) for $p = 3$, we get

$$
\sum_{\lambda \in M^N} \sum_{\sigma \in S_N} \chi(\sigma) \text{Tr}_{\Theta_N^{lN} H_{L_N}}[(\otimes^N G_{L_N}) U(\sigma)] = N! \left( \sum_{i=1}^{[N/3]+1} D_{N+3-2i,i,i-1} + \sum_{i=[N/3]+2}^{[N/2]+1} D_{i,i,N+2-2i} \right).
$$

Since $r > 0$ of the contour $S_r(0)$ is arbitrary, we may change the complex integral variables $z_j = r_j \eta_j$ with $\eta_j \in S_1(0)$ for $j = 1, 2, 3$. Thanks to the property of Fredholm determinant, we have

$$\text{Det}[1 + z_j G_{L_N}] = \text{Det}[1 + r_j G_{L_N}] \text{Det}[1 + (\eta_j - 1)r_j G_{L_N} (1 + r_j G_{L_N})^{-1}].$$

Now, we can put

$$\mathcal{F}_{k,l,m} = \frac{r_0^{3k_0} v_0^{5/2}}{\text{Det}[1 + r_0 G_{L_N}]^3} D_{k,l,m} = R_{k,l,m} v_0^{5/2} I_{k,l,m},$$

where

$$R_{k_1,k_2,k_3} = \prod_{j=1}^{3} \frac{r_0^{k_j} \text{Det}[1 + r_j G_{L_N}]}{r_j^{k_j} \text{Det}[1 + r_0 G_{L_N}]}$$

and

$$I_{k_1,k_2,k_3} = \oint \oint \oint_{S_1(0)^3} \left( \prod_{j=1}^{3} \text{Det}[1 + (\eta_j - 1)r_j G_{L_N} (1 + r_j G_{L_N})^{-1}] \right)$$

$$\times (r_1 \eta_1 - r_2 \eta_2)(r_2 \eta_2 - r_3 \eta_3) \frac{d\eta_1 d\eta_2 d\eta_3}{(2\pi i)^3 \eta_1^{k_1+1} \eta_2^{k_2+1} \eta_3^{k_3+1}}.$$

Here $k_0 = (N+2)/3$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$ satisfy $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 3k_0$. We use the abbreviation $r_{k_\nu}$ and $v_{k_\nu}$ for $r_{k_\nu}$ and $v_{k_\nu}$, respectively. Here, let us recall that $r_0 \to r_*$ in the thermodynamic limit because of $k_0/L^d \to \rho/3$, (3.8) and Lemma I.3.5.

Define a sequence $\{f_N\}_{N \in \mathbb{N}}$ of nonnegative functions on $\mathbb{R}$ by

$$f_N(x) = \begin{cases} 
\mathcal{F}_{l,i,N+2-2l} & \text{for } \sqrt{N+2} x \in [l-1-(N+2)/3, l-(N+2)/3) \\
\mathcal{F}_{N+3-2l,i,l-1} & \text{for } \sqrt{N+2} x \in [l-1-(N+2)/3, l-(N+2)/3) \\
0 & \text{otherwise.}
\end{cases}$$

Then the denominator of (3.6) becomes

$$N! \sqrt{N+2} \left( \frac{r_0^{3k_0} v_0^{5/2}}{\text{Det}[1 + r_0 G_{L_N}]^3} \right) \int_{-\infty}^{\infty} f_N(x) \, dx.$$
We introduce \( \tilde{D}_{k,l,m}, \tilde{\mathcal{F}}_{k,l,m} \) and \( \tilde{f}_N \) using \( \tilde{G}_{L_{N}} \) instead of \( G_{L_{N}} \) in \( D_{k,l,m}, \mathcal{F}_{k,l,m} \) and \( f_N \) and so on, to get the expression

\[
E_{L,N}^{3F}[e^{-<f,\xi>}] = \frac{\text{Det}[1 + \tilde{r}_0 \tilde{G}_{L_{N}}]^{3/2}}{\text{Det}[1 + r_0 G_{L_{N}}]^{3/2}} \int_{-\infty}^{\infty} \tilde{f}_N(x) dx \cdot \frac{v_0^{5/2}}{\tilde{v}_0^{5/2}} = \frac{\int_{\infty}^{\infty} \tilde{f}_N(x) dx}{\int_{\infty}^{\infty} f_N(x) dx}.
\]

From Lemma I.3.6, we have

\[
\tilde{v}_0 \rightarrow v_0 (4.7)
\]

in the thermodynamic limit. Similarly, we obtain

\[
\frac{r_0^{k_0} \text{Det}[1 + \tilde{r}_0 \tilde{G}_{L_{N}}]}{\tilde{r}_0^{k_0} \text{Det}[1 + r_0 G_{L_{N}}]} \rightarrow \text{Det}[1 - \sqrt{1-e^{-f}} G(1+r G)^{-1} \sqrt{1-e^{-f}}]
\]

from the proof of Theorem I.3.1 (see Eq. (a-c), where we should read \( N \) as \( k_0, z_N \) as \( r_0 \) and \( \alpha = -1 \)). Thus Theorem 3.4 is proved, if we get the following lemma:

**Lemma 4.1** Under the thermodynamic limit,

\[
\int_{-\infty}^{\infty} \tilde{f}_N(x) dx, \int_{-\infty}^{\infty} f_N(x) dx \rightarrow \int_{-\infty}^{\infty} e^{-2\rho x^2/\kappa} \frac{dx}{(2\pi)^{3/2}}
\]

hold, where

\[
\kappa = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-|p|^2}}{(1 + r_* e^{-|p|^2})^2}.
\]

**Proof:** Let \( k, r, v \in [0, \infty) \) satisfy the relations

\[
k = \text{Tr} [r G_{L_N} (1 + r G_{L_N})^{-1}], \quad v = \text{Tr} [r G_{L_N} (1 + r G_{L_N})^{-2}].
\]

1° There exist positive constants \( c_1 \) and \( c_2 \) which depend only on the density \( \rho \) such that

\[
r_j \leq c_1, \quad r_j - r_l \leq c_2 k_j - k_l, \quad c_2k_j \leq v_j \leq k_j,
\]

hold for \( k_j, k_l > 0 \) satisfying \( k_j > k_l \).

We have \( v \leq k \) and \( r \leq r_N \) for \( k \leq N \). Recall \( r_N \) converges to the constant \( r^* \) which determined by

\[
\int \frac{dp}{(2\pi)^d} \frac{r^* e^{-|p|^2}}{1 + r^* e^{-|p|^2}} = \rho.
\]

Then \( \{r_N\} \) is bounded from above. Hence we have \( r \leq r_N \leq c_1 \) and \( v \geq k/(1 + r_N) \geq k/(1 + c_1) \) since \( 0 \leq G_{L_N} \leq 1 \). Thanks to \( dk/dr = v/r \geq k/c_1 \), we get \( c_1 \int_{r_l}^{r_k} dk/k \geq \int_{r_l}^{r_k} dr \), which yields the second inequality. \( \diamond \)
2. There exist positive constants $c', c'_1$ and $c'_2$ which depend only on $\rho$ such that

$$A_{k,n} = \oint_{S_{1}(0)} \mathrm{D} \mathrm{e} \mathrm{t} \left[ 1 + (\eta - 1) r G_{L_{N}} \left( 1 + r G_{L_{N}} \right)^{-1} \right] \frac{(\eta - 1)^n d\eta}{2\pi i n^{k+1}} \quad (n = 0, 1, 2, \ldots, N)$$

satisfy

$$A_{k,0} = (1 + o(1))/\sqrt{2\pi v}, \quad A_{k,2} = (-1 + o(1))/\sqrt{2\pi v^3} \quad \text{for large } k \leq N$$

and

$$|A_{k,0}| \leq c'_0/\sqrt{1 + k}, \quad |A_{k,1}| \leq c'_1/\sqrt{1 + k^3},$$

$$|A_{k,2}| \leq c'_2/\sqrt{1 + k^3} \quad \text{for all } k = 0, 1, \ldots, N.$$

Put

$$h_k(x) = \chi_{[-\pi\sqrt{v}, \pi\sqrt{v}]}(x) e^{-ix/\sqrt{v}} \mathrm{D} \mathrm{e} \mathrm{t} \left[ 1 + (e^{ix/\sqrt{v}} - 1) r G_{L_{N}} \left( 1 + r G_{L_{N}} \right)^{-1} \right],$$

as in the proof of Proposition I.A.2. Then, we have

$$|h_k(x)| \leq e^{-2x^2/\pi^2} \in L^1(\mathbb{R}) \quad (4.9)$$

and

$$h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) xe^{-x^2/2} + \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x)(e^{\delta} - 1)e^{-x^2/2} \quad \text{as } N \geq k \rightarrow \infty \quad (4.10)$$

where $|\delta| \leq 4|x^3|/9\sqrt{3v}$.

Setting $\eta = \exp(ix/\sqrt{v})$, we have

$$A_{k,n} = \int_{\infty}^{\infty} \frac{(e^{ix/\sqrt{v}} - 1)^n h_k(x)}{2\pi \sqrt{v}} dx.$$

Then, $|A_{k,0}| \leq c'/\sqrt{v} \leq c''/\sqrt{k}$ for $k = 1, 2, \ldots, N$. On the other hand, Cauchy's integral formula yields $A_{0,0} = 1$, readily. So we get the bound $|A_{k,0}| \leq c'_0/\sqrt{1 + k}$.

Now the asymptotic behavior of $A_{k,0}$ can be derived by the use of dominated convergence theorem and (4.10).

For $n = 1$, we have

$$A_{k,1} = \frac{i}{2\pi v} \int_{-\infty}^{\infty} x h_k(x) dx + R,$$

where

$$|R| \leq \int \frac{x^2}{4\pi \sqrt{v^3}} h_k(x) dx = O(1/\sqrt{v^3}).$$

The integrand of first term can be written as

$$x h_k(x) = \chi_{[-\pi\sqrt{3}, \pi\sqrt{3}]}(x) xe^{-x^2/2} + \chi_{[-\pi\sqrt{3}, \pi\sqrt{3}]}(x)(e^{\delta} - 1)e^{-x^2/2}$$
\[+x[-\sqrt{v},-\sqrt{v}/3]\cup[-\sqrt{v}/3,\sqrt{v}]h_{k}(x).\]

The integral of the first term of the right hand side is 0. While the second term is bounded by \(|x\delta|h(x)| \leq |\delta|\delta^{\lambda_{0}}\). For the third term, we use (4.9). Then we get the bound \(|\int xh_{k}(x)dx| \leq c''/\sqrt{v}\) for \(k \geq 1\). Together with \(A_{0,1} = 0\), the bounds for \(A_{k,1}\) are derived. Similarly, we get the formulae for \(A_{k,2}\).

3° Let \((k_{1},k_{2},k_{3}) \in \mathbb{Z}_{+}\) satisfies

\[k_{1} \geq k_{2} \geq k_{3}, \quad k_{1} + k_{2} + k_{3} = 3k_{0} = N + 2\]

and

\[k_{1} = k_{2} \quad \text{or} \quad k_{2} = k_{3} + 1.\]

Then the estimates

\[|\nu_{0}^{5/2}I_{k_{1,k_{2},k_{3}}}| \leq c\left(\frac{k_{0}}{1+k_{3}}\right)^{5/2} \leq c'\delta^{(k_{0}-k_{3})^2/4k_{0}}\]

hold for all such \((k_{1},k_{2},k_{3})\) and

\[\nu_{0}^{5/2}I_{k_{1,k_{2},k_{3}}} = \frac{\nu_{0}^{5/2}(1+o(1))}{(2\pi)^{3/2}v_{1}^{1/2}v_{2}^{3/2}v_{3}^{1/2}}\]

holds for large \(N\) and \((k_{1},k_{2},k_{3})\), where \(c, c'\) are positive constants depending only on \(\rho\).

In fact, expanding

\[(r_{1}\eta_{1}-r_{2}\eta_{2})(r_{2}\eta_{2}-r_{3}\eta_{3}) = (r_{1}(\eta_{1}-1)-r_{2}(\eta_{2}-1)+r_{1}-r_{2})(r_{2}(\eta_{2}-1)-r_{3}(\eta_{3}-1)+r_{2}-r_{3})\]

in the integrand of \(I_{k_{1,k_{2},k_{3}}}\), we get the first inequality from 1° and 2°. The second inequality is obvious. Similarly, the asymptotic behavior follows. \(\diamond\)

4°

\[R_{k_{1},k_{2},k_{3}} = e^{-\Sigma_{j=1}^{3}(k_{0}-k_{j})^2/2v_{j}'}\]

holds where \(v_{j}' = \text{Tr}[r_{j}'G_{L_N}(1 + r_{j}'G_{L_N})^{-2}]\) for a certain middle point \(r_{j}'\) between \(r_{0}\) and \(r_{j}\). Especially, we have the bound

\[R_{k_{1},k_{2},k_{3}} \leq e^{-(k_{0}-k_{3})^2/2k_{0}}.\]

Recall that \(G_{L_N}\) is a non-negative trace class self-adjoint operator. If we put

\[\psi(t) = \log \det[1 + e^{t}G_{L_N}] = \text{Tr}[\log(1 + e^{t}G_{L_N})],\]

we have

\[\psi'(t) = \text{Tr}[e^{t}G_{L_N}(1 + e^{t}G_{L_N})^{-1}], \quad \psi''(t) = \text{Tr}[e^{t}G_{L_N}(1 + e^{t}G_{L_N})^{-2}].\]
In the equality
\[ \psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0) = \int_t^{t_0} (s - t_0)\psi''(s) \, ds + t_0(\psi'(t_0) - \psi'(t)), \]
apply
\[ \int_t^{t_0} (s - t_0)\psi''(s) \, ds = \int_t^{t_0} ds \int_{t_0}^s du \psi''(s), \frac{\psi''(u)}{\psi'(u)} = -\frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)} \]
where \( u_c \) is a middle point of \( t \) and \( t_0 \). Then we obtain
\[ \frac{e^{t_0\psi'(t_0)}}{e^{t\psi(t)}}, \frac{\text{det}[1 + e^{t}G_{L,N}]}{\text{det}[1 + e^{t_0}G_{L,N}]} = e^{\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0)} = e^{t_0(\psi'(t_0) - \psi'(t)) - (\psi'(t) - \psi'(t_0))^2/2\psi''(u_c)}. \]
Set \( e^t = r_j \) and \( e^{t_0} = r_0 \). Then \( \psi'(t) = k_j, \psi'(t_0) = k_0, \psi''(t) = v_j \) and \( \psi''(t_0) = v_0 \) hold.
Taking the product of those equalities for \( j = 1, 2 \) and 3, we get the desired expression, since \( 3k_0 = k_1 + k_2 + k_3 \).

5° Recall that the functions \( \varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L) \) \((k \in \mathbb{Z}^d)\) constitute a C.O.N.S. of \( L^2(\Lambda_L) \), where \( G_{L} \varphi_k^{(L)} = e^{-\beta|2\pi k/L|^2} \varphi_k^{(L)} \) holds for all \( k \in \mathbb{Z}^d \). Then, we obtain
\[ \frac{v_0}{L^d} = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \left(\frac{2\pi}{L}\right)^d \frac{r_0 e^{-\beta|2\pi k/L|^2}}{1 + r_0 e^{-\beta|2\pi k/L|^2}} \rightarrow \kappa, \]
in the thermodynamic limit, since \( k_0/L^d \rightarrow \rho/3 \) and \( r_0 \rightarrow r_* \) hold.

From 3° and 4°, we have a bound
\[ |F_{k_1, k_2, k_3}| \leq c'e^{-(k_0 - k_3)^2/4k_0} \quad (4.11) \]
and
\[ F_{k_1, k_2, k_3} = \frac{v_0^{5/2}(1 + o(1))}{(2\pi)^{3/2}v_1^{1/2}v_2^{3/2}v_3^{1/2}e^{-\sum_{j}(k_0 - k_j)/2v_j'}} \quad (4.12) \]
for large \( N, k_1, k_2, k_3 \), where \( v_j' \) is a mean value which we have written \( \psi''(u_c) \) in 4°.
For \( l = 1, 2, \ldots, [N/3] + 1, \sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3] \) implies \(|l - 1 - (N + 2)/3| \geq \sqrt{N + 2}|x|\), hence we get the bound
\[ f_N(x) = F_{N+3-2l,l-1} \leq c'e^{-(N+2)x^2/4k_0} \leq c'e^{-3x^2/4}. \]
We also get \( f_N(x) \leq c' \exp(-3x^2/4) \) for the other cases, similarly.

For fixed \( x \in \mathbb{R}, \) we choose \( l \in \mathbb{Z} \) such that \( \sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3]. \)
Then we have \( v_j/v_0 \to 1 \) \((j = 1, 2, 3)\) and
\[ \sum_{j=1}^{3} \frac{(k_0 - k_j)^2}{v_j'} = \frac{4N}{v_0}x^2 + o(1). \]
Hence, we obtain $f_N(x) \to (2\pi)^{-3/2} \exp(-2x^2/\kappa)$ in the thermodynamic limit. Thus the dominated convergence theorem yields the desired result for $f_N$. Because of (4.7), the one for $\tilde{f}_N$ can be proved similarly.

参考文献


