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Random Point Fields for Para-Particles of order 3

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概要
Random point fields which describe gases consist of para-particles of order three are given by means of the canonical ensemble approach. The analysis for the case of the para-fermion gases is discussed in full detail.

1 Introduction

The purpose of this note is to apply the method which we have developed in [TIa] to statistical mechanics of gases which consist of para-particles of order 3. We begin with quantum mechanical thermal systems of finite fixed numbers of para-bosons and/or para-fermions in the bounded boxes in $\mathbb{R}^d$. Taking the thermodynamic limits, random point fields on $\mathbb{R}^d$ are obtained. We will see that the point fields obtained in this way are those of $\alpha = \pm 1/3$ given in [ShTa03].

We use the representation theory of the symmetric group. (cf. e.g. [JK81, S91, Si96]) Its basic facts are reviewed briefly, in section 2, along the line on which the quantum theory of para-particles are formulated. We state the results in section 3. Section 4 devoted to the full detail of the discussion on the thermodynamic limits for para-fermion's case.
2 Brief review on Representation of the symmetric group

We say that \((\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{N}^n\) is a Young frame of length \(n\) for the symmetric group \(S_N\) if
\[
\sum_{j=1}^{n} \lambda_j = N, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0.
\]
We associate the Young frame \((\lambda_1, \lambda_2, \cdots, \lambda_n)\) with the diagram of \(\lambda_1\)-boxes in the first row, \(\lambda_2\)-boxes in the second row, ..., and \(\lambda_n\)-boxes in the \(n\)-th row. A Young tableau on a Young frame is a bijection from the numbers \(1, 2, \cdots, N\) to the \(N\) boxes of the frame.

Let \(M_p^N\) be the set of all the Young frames for \(S_N\) which have lengths less than or equal to \(p\). For each frame in \(M_p^N\), let us choose one tableau from those on the frame. The choices are arbitrary but fixed. \(T_p^N\) denotes the set of all the tableaux chosen in this way. The row stabilizer of tableau \(T\) is denoted by \(\mathcal{R}(T)\), i.e., the subgroup of \(S_N\) consists of those elements that keep all rows of \(T\) invariant, and \(C(T)\) the column stabilizer whose elements preserve all columns of \(T\).

Let us introduce the three elements
\[
a(T) = \frac{1}{\# \mathcal{R}(T)} \sum_{\sigma \in \mathcal{R}(T)} \sigma, \quad b(T) = \frac{1}{\# \mathcal{C}(T)} \sum_{\sigma \in \mathcal{C}(T)} \text{sgn}(\sigma)\sigma
\]
and
\[
e(T) = \frac{d_T}{N!} \sum_{\sigma \in \mathcal{R}(T)} \sum_{\tau \in \mathcal{C}(T)} \text{sgn}(\tau)\sigma\tau = c_T a(T)b(T)
\]
of the group algebra \(\mathbb{C}[S_N]\) for each \(T \in T_p^N\), where \(d_T\) is the dimension of the irreducible representation of \(S_N\) corresponding to \(T\) and \(c_T = d_T \# \mathcal{R}(T) \# \mathcal{C}(T) / N!\). As is known,
\[
a(T_1) b(T_2) = b(T_2) a(T_1) = 0 \quad (2.1)
\]
hold for any \(\sigma \in S_N\) if \(T_2 \rightarrow T_1\). The relations
\[
a(T)^2 = a(T), \quad b(T)^2 = b(T), \quad e(T)^2 = e(T), \quad e(T_1)e(T_2) = 0 \quad (T_1 \neq T_2) \quad (2.2)
\]
also hold for \(T, T_1, T_2 \in T_p^N\). For later use, let us introduce
\[
d(T) = e(T)a(T) = c_T a(T)b(T)a(T) \quad (2.3)
\]
for \(T \in T_p^N\). They satisfy
\[
d(T)^2 = d(T), \quad d(T_1)d(T_2) = 0 \quad (T_1 \neq T_2) \quad (2.4)
\]
which are shown readily from (2.2) and (2.1). The inner product $\langle \cdot, \cdot \rangle$ of $\mathbb{C}[S_N]$ is defined by

$$\langle \sigma, \tau > = \delta_{\sigma\tau} \quad \text{for} \quad \sigma, \tau \in S_N$$

and the sesqui-linearity.

The left representation $L$ and the right representation $R$ of $S_N$ on $\mathbb{C}[S_N]$ are defined by

$$L(\sigma)g = L(\sigma) \sum_{\tau \in S_N} g(\tau)\tau = \sum_{\tau \in S_N} g(\tau)\sigma\tau = \sum_{\tau \in S_N} g(\sigma^{-1}\tau)\tau$$

and

$$R(\sigma)g = R(\sigma) \sum_{\tau \in S_N} g(\tau)\tau = \sum_{\tau \in S_N} g(\tau)\sigma^{-1} = \sum_{\tau \in S_N} g(\tau\sigma),$$

respectively. Here and hereafter we identify $g : S_N \to \mathbb{C}$ and $\sum_{\tau \in S_N} g(\tau)\tau \in \mathbb{C}[S_N]$. They are extended to the representation of $\mathbb{C}[S_N]$ on $\mathbb{C}[S_N]$ as

$$L(f)g = fg = \sum_{\sigma, \tau} f(\sigma)g(\tau)\sigma\tau = \sum_{\sigma} \left( \sum_{\tau} f(\sigma\tau^{-1})g(\tau) \right)\sigma$$

and

$$R(f)g = g\hat{f} = \sum_{\sigma, \tau} g(\sigma)f(\tau)\sigma\tau^{-1} = \sum_{\sigma} \left( \sum_{\tau} g(\sigma\tau)f(\tau) \right)\sigma,$$

where $\hat{f} = \sum_{\tau} f(\tau^{-1})\tau = \sum_{\tau} f(\tau)^{-1}$. The character of the irreducible representation of $S_N$ corresponding to tableau $T \in \mathcal{T}_N^p$ is obtained by

$$\chi_T(\sigma) = \sum_{\tau \in S_N} (\tau, L(\sigma)R(\epsilon(T))\tau) = \sum_{\tau \in S_N} (\tau, \sigma\tau\epsilon(T)).$$

We introduce a tentative notation

$$\chi_g(\sigma) \equiv \sum_{\tau \in S_N} (\tau, L(\sigma)R(g)\tau) = \sum_{\tau, \gamma \in S_N} (\tau, \sigma\tau\gamma^{-1})g(\gamma) = \sum_{\tau \in S_N} g(\tau^{-1}\sigma\tau) \quad (2.5)$$

for $g = \sum_{\tau} g(\tau)\tau \in \mathbb{C}[S_N]$. Then $\chi_T = \chi_{\epsilon(T)}$ holds.

Now let us consider representations of $S_N$ on Hilbert spaces. Let $\mathcal{H}_L$ be a certain $L^2$ space which will be specified in the next section and $\otimes^N \mathcal{H}_L$ its $N$-fold Hilbert space tensor product. Let $U$ be the representation of $S_N$ on $\otimes^N \mathcal{H}_L$ defined by

$$U(\sigma)\varphi_1 \otimes \cdots \otimes \varphi_N = \varphi_{\sigma^{-1}(1)} \otimes \cdots \otimes \varphi_{\sigma^{-1}(N)}$$

for $\varphi_1, \cdots, \varphi_N \in \mathcal{H}_L$, or equivalently by

$$(U(\sigma)f)(x_1, \cdots, x_N) = f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) \quad \text{for} \quad f \in \otimes^N \mathcal{H}_L.$$

Obviously, $U$ is unitary: $U(\sigma)^* = U(\sigma^{-1}) = U(\sigma)^{-1}$. We extend $U$ for $\mathbb{C}[S_N]$ by linearity. Then $U(a(T))$ is an orthogonal projection because of $U(a(T))^* = U(a(T)) = U(a(T))$ and (2.2). So are $U(b(T))$, $U(d(T))$, and $P_{\mathcal{T}_N} = \sum_{T \in \mathcal{T}_N} U(d(T))$. Note that $\text{Ran} \ U(d(T)) = \text{Ran} \ U(e(T))$ because of $d(T)e(T) = e(T)$ and $e(T)d(T) = d(T)$. 


3 Para-statistics and Random point fields

3.1 Para-bosons of order 3

Let us consider a quantum system of \( N \) para-bosons of order \( p \) in the box \( \Lambda_L = [-L/2, L/2]^d \subset \mathbb{R}^d \). We refer the literatures [MeG64, HaT69, StT70] for quantum mechanics of para-particles. (See also [OK69].) The arguments of these literatures indicate that the state space of our system is given by \( \mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L \), where \( \mathcal{H}_L = L^2(\Lambda_L) \) with Lebesgue measure is the state space of one particle system in \( \Lambda_L \).

We need the heat operator \( G_L = e^{\beta \Delta_L} \) in \( \Lambda_L \), where \( \triangle_L \) is the Laplacian in \( \Lambda_L \) with periodic boundary conditions.

It is obvious that there is a CONS of \( \mathcal{H}_{L,N}^{pB} \) which consists of the vectors of the form \( U(d(T)) \varphi^{(L)}_{k_1} \otimes \cdots \otimes \varphi^{(L)}_{k_N} \), which are the eigenfunctions of \( \otimes^N G_L \). Then, we define the point field \( \mu_{L,N}^{pB} \) of \( N \) free para-bosons of order \( p \) as in section 2 of [Tia] and its generating functional is given by

\[
\int e^{-\langle f, \xi \rangle} d\mu_{L,N}^{pB}(\xi) = \frac{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)P_{pB}]}{\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)P_{pB}]}| \text{\footnote{Remark 1: \( \mathcal{H}_{L,N}^{pB} = P_{pB} \otimes^N \mathcal{H}_L \) is determined by the choice of the tableaux \( T \)'s. The spaces corresponding to different choices of tableaux are different subspaces of \( \otimes^N \mathcal{H}_L \). However, they are unitarily equivalent and the generating functional given above is not affected by the choice. In fact, \( \chi_T(\sigma) \) depends only on the frame on which the tableau \( T \) is defined.}}
\]

Lemma 3.1

\[
\int e^{-\langle f, \xi \rangle} d\mu_{L,N}^{pB}(\xi) = \frac{\sum_{T \in \mathcal{T}_{p}^N} \sum_{\sigma \in S_N} \chi_T(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N \tilde{G}_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_{p}^N} \sum_{\sigma \in S_N} \chi_T(\sigma) \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(\sigma)]} \tag{3.1}
\]

\[
= \frac{\sum_{T \in \mathcal{T}_{p}^N} \int_{\Lambda_L^N} \det_T \{\tilde{G}_L(x_i, x_j)\} 1_{\leq i, j \leq N} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_{p}^N} \int_{\Lambda_L^N} \det_T \{G_L(x_i, x_j)\} 1_{\leq i, j \leq N} dx_1 \cdots dx_N} \tag{3.2}
\]

Remark 2: \( \det T A = \sum_{\sigma \in S_N} \chi_T(\sigma) \prod_{i=1}^N A_{\sigma(i)} \) in (3.2) is called immanant.

Proof: Since \( \otimes^N G \) commutes with \( U(\sigma) \) and \( a(T)e(T) = e(T) \), we have

\[
\mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(d(T))] = \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))U(a(T))]
\]

\[
= \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[U(a(T))(\otimes^N G_L)U(e(T))] = \mathrm{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L)U(e(T))]. \tag{3.3}
\]
On the other hand, we get from (2.5) that
\[
\sum_{\sigma \in S_N} \chi_{g}(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)] = \sum_{\tau, \sigma} g(\tau^{-1} \sigma \tau) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\tau) U(\sigma) U(\tau^{-1})] \\
= N! \sum_{\sigma} g(\sigma) \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(\sigma)] = N! \text{Tr}_{\otimes^N \mathcal{H}_L}[(\otimes^N G_L) U(g)],
\]
where we have used the cyclicity of the trace and the commutativity of $U(\tau)$ with $\otimes^N G$. Putting $g = e(T)$ and using (3.3) and $P_{pB} = \sum_{T \in \mathcal{T}^N_{p}} U(d(T))$, we obtain the first equation. The second one is obvious.

Now, let us consider the thermodynamic limit
\[
L, N \to \infty, \quad N/L^d \to \rho > 0.
\]
We need the heat operator $G = e^{\beta \Delta}$ on $L^2(\mathbb{R}^d)$. In the following, $f$ is a nonnegative continuous function having a compact support. It is supposed to be fixed in the thermodynamic limit. Its support will be contained in $\Lambda_L$ for large enough $L$.

We get the limiting random point field $\mu_{\rho}^{3B}$ on $\mathbb{R}^d$ for the low density region.

**Theorem 3.2** The finite random point field for para-bosons of order 3 defined above converge weakly to the random point field whose Laplace transform is given by
\[
\int e^{-\langle f, \xi \rangle} d\mu_{\rho}^{3B}(\xi) = \text{Det}[1 + \sqrt{1 - e^{-f} r_* G(1 - r_* G)^{-1}} \sqrt{1 - e^{-f}}]^{-3}
\]
in the thermodynamic limit, where $r_* \in (0, 1)$ is determined by
\[
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-\beta |p|^2}}{1 - r_* e^{-\beta |p|^2}} = (r_* G(1 - r_* G)^{-1})(x, x),
\]
if
\[
\frac{\rho}{3} = \rho_c = \int_{\mathbb{R}^d} \frac{dp}{(2\pi)^d} \frac{e^{-\beta |p|^2}}{1 - e^{-\beta |p|^2}}.
\]

Remark: The high density region $\rho \geq 3 \rho_c$ is related to the Bose-Einstein condensation. We need a different analysis for the region. See [TIb] for the case of $p = 1$ and 2.

### 3.2 Para-fermions of order 3

For Young tableau $T$, $T'$ denotes the tableau obtained by exchanging the rows and the columns of $T$, i.e., $T'$ is the transpose of $T$. The transpose $\lambda'$ of the frame $\lambda$ can be defined similarly. Then, $T'$ lives on $\lambda'$ if $T$ lives on $\lambda$. It is obvious that
\[
\mathcal{R}(T') = \mathcal{C}(T), \quad \mathcal{C}(T') = \mathcal{R}(T).
\]
The generating functional of the point field $\mu_{L,N}^{pF}$ for $N$ para-fermions of order $p$ in the box $\Lambda_L$ is given by
\[
\int e^{-<j,\xi>}d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T \in \mathcal{T}_p^{N}} \text{Tr}_{\otimes^N H_L}[(\otimes^N G)U(d(T'))]}{\sum_{T \in \mathcal{T}_p^{N}} \text{Tr}_{\otimes^N H_L}[(\otimes^N G)U(d(T'))]}
\]
as in the case of para-bosons of order $p$. And the following expressions also hold.

**Lemma 3.3**
\[
\int e^{-<f,\xi>}d\mu_{L,N}^{pF}(\xi) = \frac{\sum_{T \in \mathcal{T}_p^{N}} \sum_{\sigma \in S_N} \chi_T'(\sigma) \text{Tr}_{\otimes^N H_L}[(\otimes^N G_L)U(\sigma)]}{\sum_{T \in \mathcal{T}_p^{N}} \int_{\Lambda_L^N} \det_{T'} \{\tilde{G}_L(x_i, x_j)\} dx_1 \cdots dx_N} 
\]
(3.6)
\[
= \frac{\sum_{T \in \mathcal{T}_p^{N}} \int_{\Lambda_L^N} \det_{T'} \{G_L(x_i, x_j)\} dx_1 \cdots dx_N}{\sum_{T \in \mathcal{T}_p^{N}} \int_{\Lambda_L^N} \det_{T'} \{G_L(x_j, x_j)\} dx_1 \cdots dx_N} 
\]
(3.7)

**Theorem 3.4** The finite random point fields for para-fermions of order 3 defined above converge weakly to the point field $\mu^{3F}_p$ whose Laplace transform is given by
\[
\int e^{-<j,\xi>}d\mu^{3F}_p(\xi) = \text{Det}[1 - \sqrt{1 - e^{-f}} r_* G(1 + r_* G)^{-1} \sqrt{1 - e^{-f}}]^3
\]
in the thermodynamic limit (3.5), where $r_* \in (0, \infty)$ is determined by
\[
\frac{\rho}{3} = \int \frac{dp}{(2\pi)^d} \frac{r_* e^{-|p|^2}}{1 + r_* e^{-|p|^2}} = (r_* G(1 + r_* G)^{-1})(x, x).
\]
(3.8)

4 Proof of Theorem 3.4

In the rest of this paper, we use results in [T1a] frequently. We refer them e.g., Lemma I.3.2 for Lemma 3.2 of [T1a]. Let $\psi_T$ be the character of the induced representation $\text{Ind}_{\mathcal{R}(T)}^{S_N}[1]$, where 1 is the one dimensional representation $\mathcal{R}(T) \ni \sigma \to 1$, i.e.,
\[
\psi_T(\sigma) = \sum_{\tau \in S_N} <\tau, L(\sigma)R(a(T))\tau> = \chi_{a(T)}(\sigma).
\]
Since the characters $\chi_T$ and $\psi_T$ depend only on the frame on which the tableau $T$ lives, not on $T$ itself, we use the notation $\chi_\lambda$ and $\psi_\lambda$ ($\lambda \in M_p^N$) instead of $\chi_T$ and $\psi_T$, respectively.
Let $\delta$ be the frame $(p-1, \cdots, 2, 1, 0) \in M^N_p$. Generalize $\psi_\mu$ to those $\mu = (\mu_1, \cdots, \mu_p) \in \mathbb{Z}^p$ which satisfies $\sum_{j=1}^{p} \mu_j = N$ by

$$\psi_\mu = 0 \quad \text{for} \quad \mu \in \mathbb{Z}^p - \mathbb{Z}_+^p$$

and

$$\psi_\mu = \psi_{\pi\mu} \quad \text{for} \quad \mu \in \mathbb{Z}_+^p \quad \text{and} \quad \pi \in S_p \quad \text{such that} \quad \pi\mu \in M^N_p,$$

where $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$. Then the determinantal form [JK81] can be written as

$$\chi_\lambda = \sum_{\pi \in S_p} \text{sgn}\pi \psi_{\lambda+\delta-\pi\delta}.$$ 

(4.1)

Let us recall the relations

$$\chi_{T'}(\sigma) = \text{sgn}\sigma \chi_T(\sigma), \quad \phi_{T'}(\sigma) = \text{sgn}\sigma \psi_T(\sigma),$$

where

$$\phi_{T'}(\sigma) = \sum_{\tau} \langle \tau, L(\sigma)R(b(T'))\tau \rangle = \chi_{b(T')}(\sigma)$$

denotes the character of the induced representation $\text{Ind}_{C(T')}^{S_N}[\text{sgn}]$, where $\text{sgn}$ is the representation $C(T') = \mathcal{R}(T') \ni \sigma \mapsto \text{sgn}\sigma$. Then we have a variant of (4.1)

$$\chi_{\lambda'} = \sum_{\pi \in S_p} \text{sgn}\pi \phi_{\lambda'+\delta'-\pi\delta'}. $$

(4.2)

Now we consider the denominator of (3.6). Let $T \in \mathcal{T}_p^N$ live on $\mu = (\mu_1, \cdots, \mu_p) \in M^N_p$. Thanks to (3.4) for $g = b(T')$, we have

$$\sum_{\sigma \in S_N} \phi_{T'}(\sigma)\text{Tr}_{\otimes^N\mathcal{H}}((\otimes^NG)U(\sigma)) = N!\text{Tr}_{\otimes^N\mathcal{H}}((\otimes^NG)U(b(T')))$$

$$= N! \prod_{j=1}^{p} \text{Tr}_{\otimes^{\mu_j}\mathcal{H}}((\otimes^{\mu_j}G)A_{\mu_j}),$$

where $A_n = \sum_{\tau \in S_n} \text{sgn}(\tau)U(\tau)/n!$ is the anti-symmetrization operator on $\otimes^n\mathcal{H}$. In the last step, we have used

$$b(T') = \prod_{j=1}^{p} \sum_{\sigma \in R_j} \frac{\text{sgn}\sigma}{\#R_j},$$

where $R_j$ is the symmetric group of $\mu_j$ numbers which lie on the $j$-th row of the tableau $T$. Then (4.2) yields

$$\sum_{\sigma \in S_N} \chi_{\lambda'}(\sigma)\text{Tr}_{\otimes^N\mathcal{H}}((\otimes^NG)U(\sigma)) = \sum_{\pi \in S_p} \text{sgn}\pi \sum_{\sigma \in S_N} \phi_{\lambda'+\delta'-\pi\delta'}(\sigma)\text{Tr}_{\otimes^N\mathcal{H}}((\otimes^NG)U(\sigma))$$
\[= N! \sum_{\pi \in S_p} \prod_{j=1}^{p} \text{Tr}_{\otimes^p \mathcal{H}_L} \left( (\otimes^\lambda G_L) \Lambda_{j-j+\pi(j)} \right). \]

Here we understand that \( \text{Tr}_{\otimes^p \mathcal{H}_L} (\otimes^n G) A_n \) = 1 if \( n = 0 \) and = 0 if \( n < 0 \) in the last expression. Let us recall the defining formula of Fredholm determinant

\[
\text{Det}(1 + J) = \sum_{n=0}^{\infty} \text{Tr}_{\otimes^n \mathcal{H}} [(\otimes^n J) A_n]
\]

for a trace class operator \( J \). We use it in the form

\[
\text{Tr}_{\otimes^n \mathcal{H}} [(\otimes^n G_L) A_n] = \oint_{S_r(0)} \frac{dz}{2\pi i z^{n+1}} \text{Det}(1 + z G_L), \tag{4.3}
\]

where \( r > 0 \) can be set arbitrary. Note that the right hand side equals to 1 for \( n = 1 \) and to 0 for \( n < 0 \). Then we have the following expression of the denominator of (3.6)

\[
\sum_{\lambda \in \mathcal{M}_p^N} \sum_{\pi \in S_N} \chi_{\lambda'}(\pi) \text{Tr}_{\otimes^N \mathcal{H}_L} [(\otimes^N G_L) U(\pi)] = N! \sum_{\lambda \in \mathcal{M}_p^N} \oint \cdots \oint_{S_r(0)} \frac{\prod_{1 \leq j < \ell} (z_j - z_{\ell})}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} dz_1 dz_2 dz_3. \tag{4.4}
\]

The similar formula for the numerator also holds.

Now we concentrate on the case of \( p = 3 \). To make the thermodynamic limit procedure explicit, let us take a sequence \( \{L_N\}_{N \in \mathbb{N}} \) which satisfies \( N/L_N^d \rightarrow \rho \) as \( N \rightarrow \infty \). In the followings, \( r = r_k \in [0, \infty) \) denotes the unique solution of

\[
\text{Tr} r G_{L_N} (1 + r G_{L_N})^{-1} = k \tag{4.5}
\]

for \( 0 \leq k \leq N \). We suppress the \( N \) dependence of \( r_k \). The existence and the uniqueness of the solution follow from the fact that the left-hand side of (4.5) is a continuous and monotone function of \( r \). See Lemma I.3.2, for details. We put

\[
v_k = \text{Tr} [r_k G_{L_N} (1 + r_k G_{L_N})^{-2}] \tag{4.6}
\]

and

\[
D_{k,l,m} = \oint \oint \oint_{S_r(0)^3} \frac{\prod_{j=1}^{3} \text{Det}(1 + z_j G_{L_N})}{(2\pi i)^3 z_1^{k+1} z_2^{l+1} z_3^{m+1}} dz_1 dz_2 dz_3,
\]
for $k, l, m \in \mathbb{Z}$. Note that $D_{k,l,m} = 0$ if at least one of $k, l, m$ is negative. Summing over $\lambda_1$ and $\lambda_3$ in (4.4) for $p = 3$, we get

$$
\sum_{\lambda \in M_N} \sum_{\sigma \in S_N} \chi_{\lambda}(\sigma) \text{Tr}_{\otimes^N H_{L_N}}[(\otimes^N G_{L_N}) U(\sigma)] = N! \left( \sum_{i=1}^{[N/3]+1} D_{N+3-2l,l-1,i} + \sum_{i=[N/3]+2}^{[N/2]+1} D_{i,i,N+2-2l} \right).
$$

Since $r > 0$ of the contour $S_r(0)$ is arbitrary, we may change the complex integral variables $z_j = r_j \eta_j$ with $\eta_j \in S_1(0)$ for $j = 1, 2, 3$. Thanks to the property of Fredholm determinant, we have

$$
\text{Det}[1 + z_j G_{L_N}] = \text{Det}[1 + r_j G_{L_N}] \text{Det}[1 + (\eta_j - 1)r_j G_{L_N}(1 + r_j G_{L_N})^{-1}]
$$

Now, we can put

$$
\mathcal{F}_{k,l,m} = \frac{r_0^{3k_0}v_0^{5/2}}{\text{Det}[1 + r_0 G_{L_N}]^3} D_{k,l,m} = R_{k_1,k_2,k_3} I_{k,l,m},
$$

where

$$
R_{k_1,k_2,k_3} = \prod_{j=1}^{3} \frac{r_0^{k_j} \text{Det}[1 + r_j G_{L_N}]}{r_j^{k_j} \text{Det}[1 + r_0 G_{L_N}]},
$$

and

$$
I_{k_1,k_2,k_3} = \oint \oint \oint_{S_1(0)^3} \left( \prod_{j=1}^{3} \text{Det}[1 + (\eta_j - 1)r_j G_{L_N}(1 + r_j G_{L_N})^{-1}] \right)
\times (r_1 \eta_1 - r_2 \eta_2)(r_2 \eta_2 - r_3 \eta_3)
\frac{d\eta_1 d\eta_2 d\eta_3}{(2\pi i)^3 \eta_1^{k_1+1} \eta_2^{k_2+1} \eta_3^{k_3+1}}.
$$

Here $k_0 = (N + 2)/3$ and $k_1, k_2, k_3 \in \mathbb{Z}_+$ satisfy $k_1 \geq k_2 \geq k_3$ and $k_1 + k_2 + k_3 = 3k_0$. We use the abbreviation $r_\nu$ for $r_{k_\nu}$ and $v_\nu$, for $v_{k_\nu}$, $\nu = 0, 1, 2, 3$, respectively. Here, let us recall that $r_0 \to r_*$ in the thermodynamic limit because of $k_0/L^d \to \rho/3$, (3.8) and Lemma I.3.5.

Define a sequence $\{f_N\}_{N \in \mathbb{N}}$ of nonnegative functions on $\mathbb{R}$ by

$$
f_N(x) = \begin{cases}
\mathcal{F}_{l,l,N+2-2l} & \text{for } \sqrt{N + 2} x \in [l - 1 - (N + 2)/3, l - (N + 2)/3] \text{ and } l = [N/3] + 2, \cdots, [N/2] + 1 \\
\mathcal{F}_{N+3-2l,l,l-1} & \text{for } \sqrt{N + 2} x \in [l - 1 - (N + 2)/3, l - (N + 2)/3] \text{ and } l = 1, 2, \cdots, [N/3] + 1 \\
0 & \text{otherwise}.
\end{cases}
$$

Then the denominator of (3.6) becomes

$$
N!\sqrt{N + 2} \frac{\text{Det}[1 + r_0 G_{L_N}]^3}{r_0^{3k_0}v_0^{5/2}} \int_{-\infty}^{\infty} f_N(x) \, dx
$$
We introduce $\tilde{D}_{k,l,m}, \tilde{\mathcal{F}}_{k,l,m}$ and $\tilde{f}_{N}$ using $\tilde{G}_{L_{N}}$ instead of $G_{L_{N}}$ in $D_{k,l,m}, \mathcal{F}_{k,l,m}$ and $f_{N}$ and so on, to get the expression

$$E_{L,N}^{3F}[e^{-\langle f, \xi \rangle}] = \frac{\text{Det}[1 + \tilde{r}_{0} G_{L_{N}}]^{3}}{\text{Det}[1 + G_{L_{N}}]^{3}} \frac{\tilde{v}_{0}^{5/2}}{v_{0}^{5/2}} \frac{\int_{-\infty}^{\infty} \tilde{f}_{N}(x) dx}{\int_{-\infty}^{\infty} f_{N}(x) dx}.$$ 

From Lemma I.3.6, we have

$$\frac{\tilde{v}_{0}}{v_{0}} \rightarrow 1 \quad (4.7)$$

in the thermodynamic limit. Similarly, we obtain

$$\frac{r_{0}^{k_{0}} \text{Det}[1 + \tilde{r}_{0} G_{L_{N}}]}{\tilde{r}_{0}^{k_{0}} \text{Det}[1 + G_{L_{N}}]} \rightarrow \text{Det}[1 - \sqrt{1 - e^{-f}} r_{*} G(1 + r_{*} G)^{-1} \sqrt{1 - e^{-f}}]$$

from the proof of Theorem I.3.1 (see Eq. (a-c), where we should read $N$ as $k_{0}, z_{N}$ as $r_{0}$ and $\alpha = -1$). Thus Theorem 3.4 is proved, if we get the following lemma:

**Lemma 4.1** Under the thermodynamic limit,

$$\int_{-\infty}^{\infty} \tilde{f}_{N}(x) dx, \int_{-\infty}^{\infty} f_{N}(x) dx \rightarrow \int_{-\infty}^{\infty} e^{-2\rho x^{2}/\kappa} \frac{dx}{(2\pi)^{3/2}}$$

hold, where

$$\kappa = \int \frac{dp}{(2\pi)^{d}} \frac{r_{*} e^{-\beta |p|^{2}}}{1 + r_{*} e^{-\beta |p|^{2}}}.$$ 

**Proof:** Let $k, r, v \in [0, \infty)$ satisfy the relations

$$k = \text{Tr} [r G_{L_{N}} (1 + r G_{L_{N}})^{-1}], \quad v = \text{Tr} [r G_{L_{N}} (1 + r G_{L_{N}})^{-2}]. \quad (4.8)$$

1° There exist positive constants $c_{1}$ and $c_{2}$ which depend only on the density $\rho$ such that

$$r_{j} \leq c_{1}, \quad r_{j} - r_{l} \leq c_{1} \frac{k_{j} - k_{l}}{k_{l}}, \quad c_{2} k_{j} \leq v_{j} \leq k_{j},$$

hold for $k_{j}, k_{l} > 0$ satisfying $k_{j} > k_{l}$.

We have $v \leq k$ and $r \leq r_{N}$ for $k \leq N$. Recall $r_{N}$ converges to the constant $r^{*}$ which determined by

$$\int \frac{dp}{(2\pi)^{d}} \frac{r^{*} e^{-\beta |p|^{2}}}{1 + r^{*} e^{-\beta |p|^{2}}} = \rho.$$ 

Then $\{r_{N}\}$ is bounded from above. Hence we have $r \leq r_{N} \leq c_{1}$ and $v \geq k/(1 + r_{N}) \geq k/(1 + c_{1})$ since $0 \leq G_{L_{N}} \leq 1$. Thanks to $dk/dr = v/r \geq k/c_{1}$, we get $c_{1} \int_{r_{l}}^{r_{j}} dk/k \geq \int_{r_{k}}^{r_{j}} dr$, which yields the second inequality. \qed
There exist positive constants $c'_0, c'_1$ and $c'_2$ which depend only on $\rho$ such that

$$A_{k,n} = \oint_{S_{1}(0)} \text{Det}[1 + (\eta - 1)r G_{L_{N}}(1 + r G_{L_{N}})^{-1}] \frac{(\eta - 1)^n d\eta}{2\pi i\eta^{k+1}} \quad (n = 0, 1, 2, k = 0, 1, \cdots, N)$$

satisfy

$$A_{k,0} = (1 + o(1))/\sqrt{2\pi v}, \quad A_{k,2} = (-1 + o(1))/\sqrt{2\pi v^3} \quad \text{for large } k \leq N$$

and

$$|A_{k,0}| \leq c'_0/\sqrt{1 + k}, \quad |A_{k,1}| \leq c'_1/\sqrt{1 + k^3},$$

$$|A_{k,2}| \leq c'_2/\sqrt{1 + k^3} \quad \text{for all } k = 0, 1, \cdots, N.$$

Put

$$h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\sqrt{v}/3]}(x) e^{-ix/\sqrt{v}} \text{Det}[1 + (e^{ix/\sqrt{v}} - 1)r G_{L_{N}}(1 + r G_{L_{N}})^{-1}],$$

as in the proof of Proposition I.A.2. Then, we have

$$|h_k(x)| \leq e^{-2x^2/\pi^2} \in L^1(\mathbb{R}) \quad (4.9)$$

and

$$h_k(x) = \chi_{[-\pi\sqrt{v}/3, \pi\gamma v/3]}(x) e^{-x^2/2} e^{5\text{arrow}} e^{-x^2/2} \quad \text{as } N \geq k \to \infty \quad (4.10)$$

where $|\delta| \leq 4|x^3|/9\sqrt{3v}$.

Setting $\eta = \exp(ix/\sqrt{v})$, we have

$$A_{k,n} = \int_{\infty}^{\infty} \frac{(e^{ix/\sqrt{v}} - 1)^n h_k(x)}{2\pi\sqrt{v}} dx.$$  

Then, $|A_{k,0}| \leq c'/\sqrt{v} \leq c''/\sqrt{k}$ for $k = 1, 2, \cdots, N$. On the other hand, Cauchy’s integral formula yields $A_{0,0} = 1$, readily. So we get the bound $|A_{k,0}| \leq c'_0/\sqrt{1 + k}$.

Now the asymptotic behavior of $A_{k,0}$ can be derived by the use of dominated convergence theorem and (4.10).

For $n = 1$, we have

$$A_{k,1} = \frac{i}{2\pi v} \int_{-\infty}^{\infty} x h_k(x) dx + R,$$

where

$$|R| \leq \int \frac{x^2}{4\pi\sqrt{v^3}} h_k(x) dx = O(1/\sqrt{v^3}).$$

The integrand of first term can be written as

$$x h_k(x) = \chi_{[-\pi\sqrt{3}/3, \pi\sqrt{3}/3]}(x) x e^{-x^2/2} + \chi_{[-\pi\sqrt{3}/3, \pi\sqrt{3}/3]}(x) x (e^\delta - 1) e^{-x^2/2}$$
The integral of the first term of the right hand side is 0. While the second term is bounded by $|x\delta|h(x)$, since $|e^{\delta}-1| \leq |\delta|e^{\delta\vee 0}$. For the third term, we use (4.9). Then we get the bound $|\int xh_k(x)\,dx| \leq c''/\sqrt{v}$ for $k \geq 1$. Together with $A_{0,1} = 0$, the bounds for $A_{k,1}$ are derived. Similarly, we get the formulae for $A_{k,2}$.

3° Let $(k_1, k_2, k_3) \in \mathbb{Z}_+$ satisfies

$$k_1 \geq k_2 \geq k_3, \quad k_1 + k_2 + k_3 = 3k_0 = N + 2$$

and

$$k_1 = k_2 \text{ or } k_2 = k_3 + 1.$$

Then the estimates

$$|v_0^{5/2}I_{k_1, k_2, k_3}| \leq c\left(\frac{k_0}{1+k_3}\right)^{5/2} \leq c' e^{(k_0-k_3)^2/4k_0}$$

hold for all such $(k_1, k_2, k_3)$ and

$$v_0^{5/2}I_{k_1, k_2, k_3} = \frac{v_0^{5/2}(1+o(1))}{(2\pi)^{3/2}v_1^{1/2}v_2^{3/2}v_3^{1/2}}$$

holds for large $N$ and $(k_1, k_2, k_3)$, where $c, c'$ are positive constants depending only on $\rho$.

In fact, expanding

$$(r_1\eta_1-r_2\eta_2)(r_2\eta_2-r_3\eta_3) = (r_1(\eta_1-1)-r_2(\eta_2-1)+r_1-r_2)(r_2(\eta_2-1)-r_3(\eta_3-1)+r_2-r_3)$$

in the integrand of $I_{k_1, k_2, k_3}$, we get the first inequality from 1° and 2°. The second inequality is obvious. Similarly, the asymptotic behavior follows.

4°

$$R_{k_1, k_2, k_3} = e^{-\Sigma_{j=1}^3(k_0-k_j)^2/2v_j}$$

holds where $v_j = \text{Tr} [r_j'G_L(1 + r_j'G_L)^{-2}]$ for a certain middle point $r_j'$ between $r_0$ and $r_j$. Especially, we have the bound

$$R_{k_1, k_2, k_3} \leq e^{-(k_0-k_3)^2/2k_0}.$$

Recall that $G_L$ is a non-negative trace class self-adjoint operator. If we put

$$\psi(t) = \log \text{det}[1 + e^tG_L] = \text{Tr} [\log(1 + e^tG_L)],$$

we have

$$\psi'(t) = \text{Tr} [e^tG_L(1 + e^tG_L)^{-1}], \quad \psi''(t) = \text{Tr} [e^tG_L(1 + e^tG_L)^{-2}].$$
In the equality
\[ \psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0) = \int_t^{t_0} (s - t_0)\psi''(s) \, ds + t_0(\psi'(t_0) - \psi'(t)), \]
apply
\[ \int_t^{t_0} (s - t_0)\psi''(s) \, ds = \int_t^{t_0} ds \int_{t_0}^{s} du \psi''(s), \frac{\psi''(u)}{\psi'(u)} = -\frac{(\psi'(t) - \psi'(t_0))^2}{2\psi''(u_c)} \]
where \(u_c\) is a middle point of \(t\) and \(t_0\). Then we obtain
\[ \frac{e^{t_0\psi'(t_0)}}{e^{t\psi(t)}}, \frac{\text{Det}[1 + e^tG_{L_{N}}]}{\text{Det}[1 + e^{t_0}G_{L_{N}}]} = e^{\psi(t) - t\psi'(t) - \psi(t_0) + t_0\psi'(t_0)} = e^{t_0(\psi'(t_0) - \psi'(t)) - \langle (\psi'(t) - \psi'(t_0))^2 \rangle / 2\psi''(u_c)} \]
Set \(e^t = r_j\) and \(e^{t_0} = r_0\). Then \(\psi'(t) = k_j, \psi'(t_0) = k_0, \psi''(t) = v_j\) and \(\psi''(t_0) = v_0\) hold. Taking the product of those equalities for \(j = 1, 2\) and \(3\), we get the desired expression, since \(3k_0 = k_1 + k_2 + k_3\).

5° Recall that the functions \(\varphi_k^{(L)}(x) = L^{-d/2} \exp(i2\pi k \cdot x/L)\) \((k \in \mathbb{Z}^d)\) constitute a C.O.N.S. of \(L^2(\Lambda_L)\), where \(G_{L} \varphi_k^{(L)} = e^{-\beta|2\pi k/L|^2} \varphi_k^{(L)}\) holds for all \(k \in \mathbb{Z}^d\). Then, we obtain
\[ \frac{v_0}{L^d} = \left(\frac{2\pi}{L}\right)^d \sum_{k \in \mathbb{Z}^d} \frac{r_0 e^{-\beta|2\pi k/L|^2}}{1 + r_0 e^{-\beta|2\pi k/L|^2}} \to \kappa, \]
in the thermodynamic limit, since \(k_0/L^d \to \rho/3\) and \(r_0 \to r_*\) hold.

From 3° and 4°, we have a bound
\[ |F_{k_1,k_2,k_3}| \leq c' e^{-(k_0-k_3)^2/4k_0} \] (4.11)
and
\[ F_{k_1,k_2,k_3} = \frac{v_0^{5/2} (1 + o(1))}{(2\pi)^{3/2}v_1^{1/2}v_2^{3/2}v_3^{1/2}} e^{-\sum_j (k_0-k_j)/2v_j} \] (4.12)
for large \(N, k_1, k_2, k_3\), where \(v_j\) is a mean value which we have written \(\psi''(u_c)\) in 4°.

For \(l = 1, 2, \cdots, [N/3] + 1, \sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3)\) implies \(|l - 1 - (N + 2)/3| \geq \sqrt{N + 2}|x|\), hence we get the bound
\[ f_N(x) = F_{N+3-2l,l-1} \leq c' e^{-(N+2)x^2/4k_0} \leq c'e^{-3x^2/4}. \]
We also get \(f_N(x) \leq c' \exp(-3x^2/4)\) for the other cases, similarly.

For fixed \(x \in \mathbb{R}\), we choose \(l \in \mathbb{Z}\) such that \(\sqrt{N + 2x} \in [l - 1 - (N + 2)/3, l - (N + 2)/3)\).
Then we have \(v_j/v_0 \to 1 (j = 1, 2, 3)\) and
\[ \sum_{j=1}^{3} \frac{(k_0 - k_j)^2}{v_j} = \frac{4N}{v_0} x^2 + o(1). \]
Hence, we obtain \( f_N(x) \to (2\pi)^{-3/2} \exp(-2\rho x^2/\kappa) \) in the thermodynamic limit. Thus the dominated convergence theorem yields the desired result for \( f_N \). Because of (4.7), the one for \( \tilde{f}_N \) can be proved similarly. 

参考文献


