Renormalization for Near-Parabolic Fixed Points of Holomorphic Maps

Hiroyuki Inou and Mitsuhiko Shishikura

Consider a fixed point $z_0$ of a holomorphic function $f(z)$. We say $z_0$ is parabolic if its multiplier $\lambda = f'(z_0)$ is a root of unity. Here we consider the case $\lambda = 1$ and it is non-degenerate, i.e., $f''(z_0) \neq 0$. By a Möbius transformation sending $z_0$ to infinity, $f(z)$ is conjugate to $\bar{f}(z) = z + 1 + O(1/z)$. There exist univalent maps $\Phi_{\pm} : \{z; \pm \Re z > L\} \to \mathbb{C}$ for sufficiently large $L$ such that

$$\Phi_{\pm}(\bar{f}(z)) = \Phi_{\pm}(z) + 1$$

(1)

where both sides are defined. We call $\Phi_{+}$ (resp. $\Phi_{-}$) attracting (resp. repelling) Fatou coordinate for $\bar{f}(z)$ (or $f(z)$). We can extend the domain of definition of Fatou coordinates (or its inverses) by the functional equation (1), and then $\Phi_{\pm}$ are defined on $W_{\pm} = \{z; \pm \Im z > M + |\Re z|\}$ for sufficiently large $M$. Therefore, $E_f = \Phi_{+} \circ \Phi_{-}^{-1}$ is well-defined on $\Phi_{-}(W_{\pm})$. Since $E_f(z + 1) = E_f(z) + 1$, it can be extended to $\{z; |\Im z| > L'\}$ for sufficiently large $L'$. It is called a horn map. Since $\Phi_{+}(z) + c_+$ and $\Phi_{-}(z) + c_-$ ($c_{\pm} \in \mathbb{C}$) are also Fatou coordinates for $\bar{f}(z)$, we can replace $\Phi_{\pm}(z)$ by $\Phi_{\pm}(z) + c_{\pm}$ for some $c_{\pm}$ so that $E_f(z)$ is normalized as $E_f(z) = z + o(1)$ as $\Im z \to +\infty$.

Define $\Pi : \mathbb{C} \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ by $\Pi(z) = e^{2\pi i z}$. Then we can define a map $\hat{E}_f : \{0 < |z| < e^{-2\pi i L'}\} \cup \{|z| > e^{2\pi i L'}\} \to \mathbb{C}^*$ satisfying $\hat{E}_f \circ \Pi = \Pi \circ E_f$. We can extend $\hat{E}_f$ holomorphically at 0 and $\infty$. They are fixed points of $\hat{E}_f$ and by the normalization above, 0 is a parabolic fixed point for $\hat{E}_f$ of multiplier 1.

When we perturb $f(z)$ in an appropriate direction, $z_0$ bifurcates to two fixed points and return maps near each of the fixed points can be defined (cf. Yoccoz renormalization for a holomorphic germ of an indifferent fixed point). Consider the case $f(z) = e^{2\pi i z} + O(z^3)$ is an perturbation of $f_0(z) = z + z^2 + O(z^3)$ and assume $\alpha \neq 0$, $|\arg \alpha| < \pi/4$. For $f$ sufficiently close to $f_0$, we can still define Fatou coordinates and the horn map $E_f$. Then the return map around a fixed point 0 can be written as $\text{Return}(f)(z) = E_f(z) - 1/\alpha$. Taking a semiconjugacy by $\Pi$, we obtain a map $\text{Return}(f)(z) = e^{-2\pi i / \alpha} \bar{E}_f(z)$.

Since $E_f(z)$ converges to $E_{f_0}(z)$ locally uniformly on an appropriate domain, horn maps play an important role in studying bifurcations at parabolic fixed points (e.g. linearizability at a fixed point, existence of a Julia set of positive measure, satellite renormalizations,...). Furthermore, if $\alpha = 1/(n - \alpha_1)$ for $n \in \mathbb{Z}$ sufficiently large, we have $(\text{Return}(f))'(0) = \alpha_1$. If $\alpha_1$ is also small, $|\arg \alpha_1| < \pi/4$ and $(\text{Return}(f))''(0) \neq 0$, then we can again consider the return map $\text{Return}^2(f)(z)$ for $\text{Return}(f)(z)$. 
Hence, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ which has the continued fraction of the form

$$
\alpha = \frac{1}{a_1 - \frac{1}{a_2 - \cdots}} \quad a_i > \exists N \gg 1
$$

(2)

and appropriate $f_0(z) = z + z^2 + O(z^3)$, we can consider a sequence of return maps $e^{2\pi i \alpha} f_n(z) = \mathrm{Return}(e^{2\pi i \alpha_{n-1}} f_{n-1})$, where $\alpha_0 = \alpha$ and $\alpha_n \equiv -1/\alpha_{n-1} \mod \mathbb{Z}$. To obtain such an infinite sequence $\{f_n\}$, we must have some a priori estimate for $f_0$. Let us denote $\mathcal{R}_\alpha f = e^{-2\pi i / \alpha} \mathrm{Return}(e^{2\pi i \alpha} f)$ if it is defined.

Our aim here is to define a space of holomorphic maps which is invariant by $\mathcal{R}_\alpha$ and to obtain contraction property of $\mathcal{R}_\alpha$ on it. Since $\mathcal{R}_\alpha \to \mathcal{R}_0$ when $\alpha \to 0$, we first study the operator $\mathcal{R} = \mathcal{R}_0$. We call $f \sim \mathcal{R}(f)$ parabolic renormalization. Let

$$
\mathcal{F}_0 = \left\{ f : U_f \to \mathbb{C} \mid \begin{array}{l}
0 \in U_f: \text{open and connected in } \mathbb{C}, f: \text{holomorphic in } U_f, \\
f(0) = 0, f'(0) = 1, f : U_f \setminus \{0\} \to \mathbb{C}^* \text{ is a branched covering} \\
\text{with a unique critical value, all critical points have local degree 2}
\end{array} \right\} .
$$

We can define $\mathcal{R}$ on $\mathcal{F}_0$ and the following theorem is known:

**Theorem 1.**

(i) $\mathcal{R}(\mathcal{F}_0) \subset \mathcal{F}_0$.
(ii) Let $f_{\text{Koebe}} = z/(1-z)^2$ and $f_\star = \mathcal{R}(f_{\text{Koebe}})$. Then $f_\star$ is defined on $\mathbb{D}$ and $f_\star \in \mathcal{F}_0$.

Any $f \in \mathcal{R}(\mathcal{F}_0)$ can be written as $f = f_\star \circ \phi^{-1}$ where $\phi : \mathbb{D} \to U_f$ a conformal map with $\phi(0) = 0$, $\phi'(0) = 1$. Hence there exists a bijection between $\mathcal{R}(\mathcal{F}_0)$ and $\mathbb{S} = \{\phi : \mathbb{D} \to \mathbb{C} \mid \text{univalent, holomorphic, } \phi(0) = 0, \phi'(0) = 1\}$. We consider a topology on $\mathcal{R}(\mathcal{F}_0)$ which is induced from local uniform convergence topology on $\mathbb{S}$ by this bijection.

However, considering this space is not sufficient to study $\mathcal{R}_\alpha$ for $\alpha \neq 0$ (e.g. $\mathcal{R}_\alpha f$ have infinitely many critical values). Therefore, we need to relax covering property and we obtain the following:

**Main Theorem 1.** Let $P = z(1-z)^2$. There exist simply connected domains $V, V' \subset \mathbb{C}$ such that $V$ contains the fixed point $0$ and the critical point $-1/3$ for $P, V \subset V'$, $V$ is a quasidisk and

$$
\mathcal{F}_1 = \left\{ f = P \circ \phi^{-1} \mid \begin{array}{l}
\phi : V \to \mathbb{C} \text{ is conformal, } \phi(0) = 0, \phi'(0) = 1, \\
\text{and has a quasiconformal extension to } \mathbb{C}
\end{array} \right\} .
$$

satisfies the following:

(i) $\mathcal{F}_0 \setminus \{\text{quadratic polynomial}\} \subset \mathcal{F}_1$. In particular, $\mathcal{R}_0^0(z + z^2) \in \mathcal{F}_1$ ($n \geq 1$).
(ii) $\mathcal{R}_0 f \in \mathcal{F}_1$ for $f \in \mathcal{F}_1$. If $\mathcal{R}_0 f = P \circ \psi^{-1}$, then $\psi$ can be extended conformally on $V'$.
(iii) $f \sim \mathcal{R}_0$ is "holomorphic".
(iv) There exists $\alpha_0 > 0$ such that $\mathcal{R}_\alpha (0 < \alpha < \alpha_0)$ also satisfies (ii) and (iii).

**Main Theorem 2.** There exists a metric $d$ on $\mathcal{F}_1$ such that $\mathcal{R}_\alpha (0 \leq \alpha < \alpha_0)$ is a uniform contraction with respect to $d$.

**Corollary 2.** For $\alpha$ satisfying (2) and $f = f_0 \in \mathcal{F}_1$, the sequence of return maps $\{e^{2\pi i \alpha} f_n = \mathrm{Return}^n(e^{2\pi i \alpha} f)\}$ ($\alpha_n \equiv -1/\alpha_{n-1} \mod \mathbb{Z}$ and $f_n = \mathcal{R}_{\alpha_{n-1}} f_{n-1}$ ($n = 1, 2, \ldots$)) is defined and $f_n \in \mathcal{F}_1$. It is also true for $f(z) = z + z^2$.

**Corollary 3.** For $\alpha$ in Corollary 2, $g(z) = e^{2\pi i \alpha} z + z^2$ does not have dense critical orbit in its Julia set.