<table>
<thead>
<tr>
<th>Title</th>
<th>Hanner type inequalities and duality (Nonlinear Analysis and Convex Analysis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KATO, Mikio; TAKAHASHI, Yasuji; YAMADA, Yasutaka</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2006, 1484: 98-104</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2006-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/58116">http://hdl.handle.net/2433/58116</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Affiliation</td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
We shall first discuss two kinds of Hanner type inequalities with a weight in a Banach space $X$ in connection with sharp uniform smoothness and convexity: the first kind of inequalities will characterize the $2$-uniform smoothness and $2$-uniform convexity of $X$, and the other the $p$-uniform smoothness and $q$-uniform convexity of $X$. Next we shall present a duality theorem on a "general" Hanner type inequality with "several weights", which is valid for both kinds of the above inequalities. Finally the best value of the weight constant in these inequalities for $L_p$-spaces will be determined.

Let $X$ be a Banach space and $X^*$ its dual space. Let $S_X$ be the unit sphere of $X$. Let $1 \leq p, q, r, \ldots \leq \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1/r + 1/r' = \ldots = 1$.

1. Hanner's inequalities for $L_p$ (Hanner [3], 1956)
   (i) If $1 < p \leq 2$, for all $f, g$ in $L_p$
   \[ ||f + g||_p^p + ||f - g||_p^p \geq \left( ||f||_p + ||g||_p \right)^p + ||f||_p - ||g||_p \right)^p \]  
   \[ (H1) \]
   (ii) If $2 \leq p < \infty$, for all $f, g$ in $L_p$
   \[ ||f + g||_p^p + ||f - g||_p^p \leq \left( ||f||_p + ||g||_p \right)^p + ||f||_p - ||g||_p \right)^p \]  
   \[ (H2) \]

2. Definition (i) The modulus of convexity of $X$:
   \[ \delta_X(\epsilon) := \inf \left\{ 1 - \left| \frac{x+y}{2} \right| : x, y \in S_X, ||x - y|| = \epsilon \right\} \]  
   for $0 \leq \epsilon \leq 2$.
   (ii) $X$ is uniformly convex if $\delta_X(\epsilon) > 0$ for all $\epsilon > 0$. 

(iii) $X$ is $q$-uniformly convex ($2 \leq q < \infty$) if there exists $C > 0$ such that $\delta_X(\epsilon) \geq C\epsilon^q$ for all $\epsilon > 0$.

3. Remark (i) If $1 \leq q < 2$ no Banach space is $q$-uniformly convex (cf. [2]; for a proof see e.g., [11, esp., p. 268]).

(ii) Let $2 \leq q \leq q_1 < \infty$. Then if $X$ is $q$-uniformly convex, $X$ is $q_1$-uniformly convex.

(iii) $L_q$ ($2 \leq q < \infty$) is $q$-uniformly convex (by Clarkson’s inequality of $(q, q)$-type).

(iv) $L_p$ ($1 < p \leq 2$) is $p'$-uniformly convex ($p' \geq 2$) (by Clarkson’s inequality of $(p, p')$-type). But in fact, $L_p$ ($1 < p \leq 2$) is 2-uniformly convex by Hanner’s inequality (H1).

For convenience of the reader we see (iii) and the latter statement of (iv) in the general Banach space setting.

Proof of (iii). Let $2 \leq q < \infty$. Assume that Clarkson’s inequality of $(q, q)$-type holds in $X$:  
\[ (\|x + y\|^q + \|x - y\|^q)^{1/q} \leq 2^{1/q'}(\|x\|^q + \|y\|^q)^{1/q}. \]

Let $x, y \in S_X$ and $\|x - y\| = \epsilon$. Then  
\[ \|x + y\|^q + \epsilon^q \leq 2^{1/q'}2^{q(1/q' + 1/q)} = 2^q, \]
whence  
\[ \left\| \frac{x + y}{2} \right\|^q + \left( \frac{\epsilon}{2} \right)^q \leq 1. \]

Therefore  
\[ \left( \frac{\epsilon}{2} \right)^q \leq 1 - \left\| \frac{x + y}{2} \right\|^q \leq q \left( 1 - \left\| \frac{x + y}{2} \right\| \right). \]

Consequently we have  
\[ 1 - \left\| \frac{x + y}{2} \right\| \geq \frac{1}{q} \left( \frac{\epsilon}{2} \right)^q, \]
from which it follows that  
\[ \delta_X(\epsilon) \geq \frac{1}{q2^q} \epsilon^q, \]
or $X$ is $q$-uniformly convex.

Proof of the latter assertion of (iv). Let $1 < p \leq 2$. We have to show the following: If Hanner’s inequality (H1),  
\[ \|x + y\|^p + \|x - y\|^p \geq \|x\|^p + \|y\|^p + \|x\| - \|y\|^p, \]

...
holds in $X$, then $X$ is 2-uniformly convex. Assume (H1). Then
\[
\left( \frac{\|x+y\|^2 + \|x-y\|^2}{2} \right)^{1/2} \geq \left( \frac{\|x+y\|^p + \|x-y\|^p}{2} \right)^{1/p} \geq \left( \frac{\|x\| + \|y\|^p + \|x\| - \|y\|^p}{2} \right)^{1/p} \geq \left( \frac{\|x\| + \gamma \|y\|^2 + \|x\| - \gamma \|y\|^2}{2} \right)^{1/2},
\]
where $\gamma = \sqrt{(p-1)/(2-1)} = \sqrt{p-1}$ (6, Corollary 1.e.15]). Therefore
\[
\|x+y\|^2 + \|x-y\|^2 \geq \|x\| + \gamma \|y\|^2 + \|x\| - \gamma \|y\|^2
= 2[\|x\|^2 + \gamma^2 \|y\|^2].
\]
Put here $x+y = u, x-y = v$. Then
\[
\|u\|^2 + \|v\|^2 \geq 2 \left[ \left\| \frac{u+v}{2} \right\|^2 + (p-1) \left\| \frac{u-v}{2} \right\|^2 \right].
\]
Now let $u, v \in S_X$ and $\|u-v\| = \epsilon$. Then
\[
2 \geq 2 \left[ \left\| \frac{u+v}{2} \right\|^2 + (p-1) \left( \frac{\epsilon}{2} \right)^2 \right],
\]
whence
\[
(p-1) \left( \frac{\epsilon}{2} \right)^2 \leq 1 - \left\| \frac{u+v}{2} \right\|^2 \leq 2 \left( 1 - \left\| \frac{u+v}{2} \right\| \right).
\]
Therefore
\[
\frac{p-1}{8} \epsilon^2 \leq 1 - \left\| \frac{u+v}{2} \right\|.
\]
Consequently we have
\[
\delta_X(\epsilon) \geq \frac{p-1}{8} \epsilon^2,
\]
or $X$ is 2-uniformly convex, as is desired.

4. Definition (i) The modulus of smoothness of $X$ is defined by
\[
\rho_X(\tau) := \sup \left\{ \left( \frac{\|x+\tau y\|^2 + \|x-\tau y\|^2}{2} - 1 \right) : x, y \in S_X \right\} \quad \text{for } \tau > 0
\]
(ii) $X$ is uniformly smooth if $\rho_X(\tau)/\tau \to 0$ as $\tau \to 0$.

(iii) $X$ is $p$-uniformly smooth $(1 < p \leq 2)$ if there exists $K > 0$ such that $\rho_X(\tau) \leq K\tau^p$ for all $\tau > 0$.

5. **Remark**

(i) No Banach space is $p$-uniformly smooth for $2 < p < \infty$.

(ii) Let $1 < p_1 \leq p \leq 2$. Then if $X$ is $p$-uniformly smooth, $X$ is $p_1$-uniformly smooth.

(iii) $L_p$ $(1 < p \leq 2)$ is $p$-uniformly smooth.

(iv) $L_q$ $(2 \leq q < \infty)$ is $2$-uniformly smooth.

The first kind of Hanner type inequalities

6. **Theorem** (Yamada-Takahashi-Kato [13]) Let $1 < p, s, t < \infty$. Then the following are equivalent.

(i) $X$ is $2$-uniformly convex.

(ii) There exists $\gamma > 0$ for which

$$
\|x + y\|^p + \|x - y\|^p \geq \left(\|x\| + \|\gamma y\|^p\right)^p + \left(\|x\| - \|\gamma y\|^p\right)^p
$$

holds in $X$.

(iii) There exists $\gamma > 0$ for which

$$
\left(\frac{\|x + y\|^s + \|x - y\|^s}{2}\right)^{1/s} \geq \left(\frac{\|x\| + \|\gamma y\|^t + \|x\| - \|\gamma y\|^t}{2}\right)^{1/t}
$$

holds in $X$.

According to Remark 3 (iv) the Hanner type inequalities (1) and (2) hold in $L_r$, $1 < r \leq 2$.

7. **Theorem** ([13]) Let $1 < p, s, t < \infty$. Then the following are equivalent.

(i) $X$ is $2$-uniformly smooth.

(ii) There exists $\gamma > 0$ for which

$$
\|x + y\|^p + \|x - y\|^p \leq \left(\|x\| + \|\gamma y\|^p\right)^p + \left(\|x\| - \|\gamma y\|^p\right)^p
$$

holds in $X$.

(iii) There exists $\gamma > 0$ for which

$$
\left(\frac{\|x + y\|^s + \|x - y\|^s}{2}\right)^{1/s} \leq \left(\frac{\|x\| + \|\gamma y\|^t + \|x\| - \|\gamma y\|^t}{2}\right)^{1/t}
$$

holds in $X$. 

The above Hanner type inequalities (3) and (4) hold in $L_r$, $2 \leq r < \infty$.

The second kind of Hanner type inequalities

8. Theorem ([13]) Let $2 \leq q < \infty$, $1 \leq t \leq q$. Then the following are equivalent.
   (i) $X$ is $q$-uniformly convex.
   (ii) There exists $\gamma > 0$ such that
        $$\left(\|x+y\|^q + \gamma \|x-y\|^q\right)^{1/q} \leq \left(\|x\|^t + \|x\|^{t'} + \|x\| - \|y\|\right)^{1/t}$$
        (5)
    for all $x, y \in X$.

   The Hanner type inequality (5) holds in $L_q$ ($2 \leq q < \infty$).

9. Theorem ([13]) Let $1 < p \leq 2$ and $p \leq s \leq \infty$. Then the following are equivalent.
   (i) $X$ is $p$-uniformly smooth.
   (ii) There exists $\gamma > 0$ such that
        $$\left(\|x+y\|^p + \gamma \|x-y\|^p\right)^{1/p} \geq \left(\|x\|^t + \|y\|^t + \|x\| - \|y\|\right)^{1/s}$$
        (6)
    for all $x, y \in X$.

   The Hanner type inequality (6) holds in $L_p$ ($1 < p \leq 2$).

Duality between Hanner type inequalities

According to Ball-Carlen-Lieb [1] Hanner's inequalities (H1) and (H2) are equivalent. This is extended as follows.

10. Theorem ([13]) Let $1 < s, t < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1$ and $\alpha, \beta, \gamma > 0$. Then the following are equivalent.
   (i) For all $x, y \in X$
        $$\left(\|\alpha(x+y)\|^s + \|\beta(x-y)\|^s\right)^{1/s} \geq \left(\|\|x\| + \|y\|^s\| + \|\|x\| - \|y\|^s\|\right)^{1/t}$$
        (7)
(ii) For all $x^*, y^* \in X^*$

$$
\left( \|\alpha^{-1}(x^* + y^*)\|^s' + \|\beta^{-1}(x^* - y^*)\|^s' \right)^{1/s'} \leq \left( \left( \|x^*\| + \|\gamma^{-1}y^*\| \right)^{t'} + \left( \|x^*\| - \|\gamma^{-1}y^*\| \right)^{t'} \right)^{1/t'}
$$

(8)

11. Corollary Let $1 < s, t, p < \infty$, $1/s + 1/s' = 1/t + 1/t' = 1/p + 1/q = 1$ and $\gamma > 0$.

(i) The inequality

$$
\left( \frac{\|x+y\|^s + \|x-y\|^s}{2} \right)^{1/s} \geq \left( \frac{\|\|x\| + \|\gamma y\|\|^t + \|\|x\| - \|\gamma y\|\|^t}{2} \right)^{1/t}
$$

(2)

holds in $X$ if and only if

$$
\left( \frac{\|x^* + y^*\|^s' + \|x^* - y^*\|^s'}{2} \right)^{1/s'} \leq \left( \left( \|x^*\| + \|\gamma^{-1}y^*\| \right)^{t'} + \left( \|x^*\| - \|\gamma^{-1}y^*\| \right)^{t'} \right)^{1/t'}
$$

(4*)

holds in $X^*$.

(ii) The inequality

$$
\|x+y\|^p + \|x-y\|^p \geq \|\|x\| + \|\gamma y\|\|^p + \|\|x\| - \|\gamma y\|\|^p
$$

(1)

holds in $X$ if and only if

$$
\|x^* + y^*\|^q + \|x^* - y^*\|^q \leq \|\|x^*\| + \|\gamma^{-1}y^*\|\|^q + \|\|x^*\| - \|\gamma^{-1}y^*\|\|^q
$$

(3*)

holds in $X^*$.

The best value of the weights for $L_p$-spaces

12. Theorem ([13]) Let $1 < p \leq 2$ and $1 < s, t < \infty$. Then the Hanner type inequality (2) holds in $L_p$:

$$
\left( \frac{\|x+y\|^s + \|x-y\|^s}{2} \right)^{1/s} \geq \left( \frac{\|\|x\| + \|\gamma y\|\|^t + \|\|x\| - \|\gamma y\|\|^t}{2} \right)^{1/t}
$$
The best value of $\gamma$ is

$$\gamma = \min \left\{ 1, \sqrt[\frac{p-1}{s-1}], \sqrt[\frac{s-1}{t-1}] \right\}$$

13. **Theorem** ([13]) Let $2 \leq p < \infty$ and $1 < s, t < \infty$. Then the Hanner type inequality (4) holds in $L_p$:

$$\left( \frac{\|x+y\|^s + \|x-y\|^s}{2} \right)^{1/s} \leq \left( \frac{\|x\| + \|\gamma y\|^t + \|x\| - \|\gamma y\|}{2} \right)^{1/t}$$

The best value of $\gamma$ is

$$\gamma = \max \left\{ 1, \sqrt[\frac{p-1}{s-1}], \sqrt[\frac{s-1}{t-1}] \right\}$$

**References**


