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Kyoto University
A structure theorem for coupled balanced games without side payments

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The notion of 'core' for a game was defined as an independent solution concept by Gillies and Shapley[1][2]. The most basic issue is the core to be non-empty. The work of Bonderewe[3] and Shapley[4] were on the balancedness condition for the non-emptiness of the core of a TU game and Scarf's[5] work was on balancedness in NTU games. In this report, we will study a general problem: Given n NTU games, is there a tight coupling between n NTU games so that the common core has non-empty?

Let \( N = \{1, 2, \ldots, n\} \), \( \eta \) the collection of all non-empty subsets of \( N \), and for \( S \in \eta \),

\[
E^S = \{ (x_1, \ldots, x_n) \in \mathbb{R}^n; x_i = 0 \text{ if } i \notin S \}.
\]

Let \( X \subseteq E^N \), \( S \in \eta \), \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \beta = (\beta_1, \ldots, \beta_n) \). Denote by \( \alpha^S \) the projection of \( \alpha \) to \( E^S \). Denote by "\( \leq \)" the natural order on \( E^N \). \( X \) is comprehensive if \( \alpha \in X \), \( \beta \leq \alpha \) then \( \beta \in X \).

Denote by \( \hat{X} \) the comprehensive hull of \( X \), that is, the smallest comprehensive set containing \( X \). An NTU game (game without side payments) is an ordered triple \((N, F, D)\). Here \( F \) is a closed subset of \( E^N \), and \( D \) is a function from \( \eta \) to open, comprehensive non-empty, proper subsets of \( E^N \) that satisfies

(i) \( D(N) \subset \hat{F} \),
(ii) if \( \alpha \in D(S) \) and \( \alpha^S = \beta^S \) then \( \beta \in D(S) \),
(iii) for each \( S \in \eta \),

\[
\{ \alpha^S; \alpha \in \overline{D(S)} \setminus \bigcup_{i \in S} D(\{i\}) \}
\]

is non-empty and bounded. Here \( \overline{D(S)} \) denotes the closure of \( D(S) \).
The core of the game \((N, F, D)\) is defined to be the set

\[ F \setminus \bigcup_{S \in \eta} D(S). \]

The core represents the set of feasible outcomes that cannot be improved upon by any condition. A game \((N, F, D)\) is said to be balanced if

\[ \bigcap_{S \in \beta} D(S) \subset \hat{F}. \]

**Theorem 1.** (Scarf) Every balanced game has a nonempty core.


To state *multiple balanced KKM theorem* proved by Shih and Lee[7], let us recall some notations. Let \(\{r_1, \ldots, r_n\}\) be affinely independent in \(E^N\). For \(X \subset E^N\), \(\text{conv} X\) denotes the convex hull of \(X\). Let

\[
A^S = \text{conv}\{r_i; i \in S\}, \\
m_S = \frac{1}{\# S} \sum_{i \in S} r_i.
\]

We say that \(\{F_i; i \in N\}\) is a KKM covering of \(A^N\) if for each \(i \in N\), \(F_i\) is closed in \(A^N\), and

\[ A^S \subset \bigcup_{i \in S} F_i \text{ for all } S \in \eta. \]

We say that \(\{C_S; S \in \eta\}\) is a Shapley covering of \(A^N\) if for each \(S \in \eta\), \(C_S\) is closed in \(A^N\), and

\[ A^S \subset \bigcup_{T \subset S, T \neq \emptyset} C_T. \]

Let \(\pi: \eta \rightarrow A^N\) and \(\pi(S) \in A^S\) for all \(S \in \eta\). We say that \(\beta \subset \eta\) is balanced if

\[ m_N \in \text{conv}\{m_S; S \in \beta\}; \]

\(\beta\) is \(\pi\)-balanced if

\[ m_N \in \text{conv}\{\pi(S); S \in \beta\}. \]

**Theorem 2.** (Shih and Lee) Let \(\{C,S_i; S \in \eta, i = 1, 2, \ldots, n\}\) be \(n\) Shapley coverings of \(A^N\), and \(\pi(S) \in A^N\) for all \(S \in \eta\). Then there exists a \(\pi\)-balanced family \(\{S_1, \ldots, S_n\}\) such that

\[ C_{S_1} \cap \ldots \cap C_{S_n} \neq \emptyset. \]

Theorem 2 was built upon the following multiple balanced Sperner's lemma.
Theorem 3. (Shih and Lee) Let $T$ be a simplicial subdivision of $A^{N}$, $T^{0}$ the vertex set of $T$. Let
\[
\Phi = (\varphi^{1}, \ldots, \varphi^{n}) : T^{0} \rightarrow 2^{N} \times \cdots \times 2^{N}
\]
be such that
\[
\Phi(T^{0} \cap A^{S}) \subset 2^{S} \times \cdots \times 2^{S}
\]
and $\pi(S) \in A^{S}$ for all $S \in \eta$. Then there exist an $(n-1)$-simplex $\text{conv}\{v_{1}, \ldots, v_{n}\} \in T$ and a $\pi$-balanced family $\{S_{1}, \ldots, S_{n}\}$ such that
\[
\varphi^{1}(v_{1}) = S_{1}, \ldots, \varphi^{n}(v_{n}) = S_{n}.
\]

There is a tight coupling between NTU games and Shapley coverings. Indeed, we have

Theorem 4. Let $\pi(S) \in A^{S}$ for all $S \in \eta$, and $V_{i} = (N, F, D_{i})$ n NTU games, $i = 1, \ldots, n$. Then there exist $n$ Shapley coverings $\{C_{S}^{i}; S \in \eta\}$ of $A^{N}$, $i = 1, \ldots, n$, induced by $n$ games $V_{1}, \ldots, V_{n}$, and there exists $\pi$-balanced family $\{S_{1}, \ldots, S_{n}\}$ such that
\[
C_{S_{1}}^{1} \cap C_{S_{2}}^{2} \cap \cdots \cap C_{S_{n}}^{n} \neq \emptyset.
\]
Furthermore, if for all $i = 1, 2, \ldots, n$, $V_{i}$ is $\pi$-balanced and there exists
\[
\alpha \in C_{S_{1}}^{1} \cap C_{S_{2}}^{2} \cap \cdots C_{S_{n}}^{n}
\]
such that $\alpha \in C_{S_{1}}^{i} \cap C_{S_{2}}^{i} \cap \cdots C_{S_{n}}^{i}$ for all $i = 1, 2, \ldots, n$, then
\[
\text{Core}(V_{1}) \cap \text{Core}(V_{2}) \cap \cdots \cap \text{Core}(V_{n}) \neq \emptyset.
\]

References


