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Kyoto University
A structure theorem for coupled balanced games without side payments

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The notion of 'core' for a game was defined as an independent solution concept by Gillies and Shapley[1][2]. The most basic issue is the core to be non-empty. The work of Bonderave[3] and Shapley[4] were on the balancedness condition for the non-emptiness of the core of a TU game and Scarf's[5] work was on balancedness in NTU games. In this report, we will study a general problem: Given n NTU games, is there a tight coupling between n NTU games so that the common core has non-empty?

Let \( N = \{1, 2, \ldots, n\}, \eta \) the collection of all non-empty subsets of \( N \), and for \( S \in \eta \),

\[
E^{S} = \{ (x_{1}, \ldots, x_{n}) \in \mathbb{R}^{n}; x_{i} = 0 \text{ if } i \notin S \}.
\]

Let \( X \subset E^{N}, S \in \eta, \alpha = (\alpha_{1}, \ldots, \alpha_{n}), \beta = (\beta_{1}, \ldots, \beta_{n}) \). Denote by \( \alpha^{S} \) the projection of \( \alpha \) to \( E^{S} \). Denote by "\( \leq \)" the natural order on \( E^{N} \). \( X \) is comprehensive if \( \alpha \in X, \beta \leq \alpha \) then \( \beta \in X \). Denote by \( \hat{X} \) the comprehensive hull of \( X \), that is, the smallest comprehensive set containing \( X \). An NTU game (game without side payments) is an ordered triple \((N, F, D)\). Here \( F \) is a closed subset of \( E^{N} \), and \( D \) is a function from \( \eta \) to open, comprehensive non-empty, proper subsets of \( E^{N} \) that satisfies

(i) \( D(N) \subset \hat{F} \),
(ii) if \( \alpha \in D(S) \) and \( \alpha^{S} = \beta^{S} \) then \( \beta \in D(S) \),
(iii) for each \( S \in \eta \),

\[
\{ \alpha^{S}; \alpha \in \overline{D(S)} \setminus \bigcup_{i \in S} D(\{i\}) \}
\]

is non-empty and bounded. Here \( \overline{D(S)} \) denotes the closure of \( D(S) \).
The core of the game \((N, F, D)\) is defined to be the set
\[
F \setminus \bigcup_{S \in \eta} D(S).
\]
The core represents the set of feasible outcomes that cannot be improved upon by any condition. A game \((N, F, D)\) is said to be balanced if
\[
\bigcap_{S \in \beta} D(S) \subset \hat{F}.
\]

**Theorem 1.** (Scarf) Every balanced game has a nonempty core.


To state multiple balanced KKM theorem proved by Shih and Lee[7], let us recall some notations. Let \(\{r_1, \ldots, r_n\}\) be affinely independent in \(E^N\). For \(X \subset E^N\), \(\text{conv}X\) denotes the convex hull of \(X\). Let
\[
A^S = \text{conv}\{r_i; i \in S\},
\]
\[
m_S = \frac{1}{\# S} \sum_{i \in S} r_i.
\]
We say that \(\{F_i; i \in N\}\) is a KKM covering of \(A^N\) if for each \(i \in N\), \(F_i\) is closed in \(A^N\), and
\[
A^S \subset \bigcup_{i \in S} F_i \text{ for all } S \in \eta.
\]
We say that \(\{C_S; S \in \eta\}\) is a Shapley covering of \(A^N\) if for each \(S \in \eta\), \(C_S\) is closed in \(A^N\), and
\[
A^S \subset \bigcup_{T \subset S, T \neq \emptyset} C_T.
\]
Let \(\pi: \eta \rightarrow A^N\) and \(\pi(S) \in A^S\) for all \(S \in \eta\). We say that \(\beta \subset \eta\) is balanced if
\[
m_N \in \text{conv}\{m_S; S \in \beta\};
\]
\(\beta\) is \(\pi\)-balanced if
\[
m_N \in \text{conv}\{\pi(S); S \in \beta\}.
\]

**Theorem 2.** (Shih and Lee) Let \(\{C_S; S \in \eta\}, i = 1, 2, \ldots, n\), be \(n\) Shapley coverings of \(A^N\), and \(\pi(S) \in A^S\) for all \(S \in \eta\). Then there exists a \(\pi\)-balanced family \(\{S_1, \ldots, S_n\}\) such that
\[
C_{S_1} \cap \ldots \cap C_{S_n}^n \neq \emptyset.
\]

Theorem 2 was built upon the following multiple balanced Sperner’s lemma.
Theorem 3. (Shih and Lee) Let $T$ be a simplicial subdivision of $A^N$, $T^0$ the vertex set of $T$. Let

$$\Phi = (\varphi^1, \ldots, \varphi^n) : T^0 \to 2^N \times \cdots \times 2^N$$

be such that

$$\Phi(T^0 \cap A^S) \subset 2^S \times \cdots \times 2^S$$

for all $S \in \eta$, and $\pi(S) \in A^S$ for all $S \in \eta$. Then there exist an $(n-1)$-simplex $\operatorname{conv}\{v_1, \ldots, v_n\} \in T$ and a $\pi$-balanced family $\{S_1, \ldots, S_n\}$ such that

$$\varphi^1(v_1) = S_1, \ldots, \varphi^n(v_n) = S_n.$$

There is a tight coupling between NTU games and Shapley coverings. Indeed, we have

Theorem 4. Let $\pi(S) \in A^S$ for all $S \in \eta$, and $V_i = (N, F, D_i)$ $n$ NTU games, $i = 1, \ldots, n$. Then there exist $n$ Shapley coverings $\{C_S^i; S \in \eta\}$ of $A^N$, $i = 1, \ldots, n$, induced by $n$ games $V_1, \ldots, V_n$, and there exists $\pi$-balanced family $\{S_1, \ldots, S_n\}$ such that

$$C^1_{S_1} \cap C^2_{S_2} \cap \cdots \cap C^n_{S_n} \neq \emptyset.$$

Furthermore, if for all $i = 1, 2, \ldots, n$, $V_i$ is $\pi$-balanced and there exists

$$\alpha \in C^1_{S_1} \cap C^2_{S_2} \cap \cdots \cap C^n_{S_n}$$

such that $\alpha \in C^i_{S_1} \cap C^i_{S_2} \cap \cdots \cap C^i_{S_n}$ for all $i = 1, 2, \ldots, n$, then

$$\operatorname{Core}(V_1) \cap \operatorname{Core}(V_2) \cap \cdots \cap \operatorname{Core}(V_n) \neq \emptyset.$$

References


