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A structure theorem for coupled balanced games without side payments

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The notion of 'core' for a game was defined as an independent solution concept by Gillies and Shapley[1][2]. The most basic issue is the core to be non-empty. The work of Bondereve[3] and Shapley[4] were on the balancedness condition for the non-emptiness of the core of a TU game and Scarf's[5] work was on balancedness in NTU games. In this report, we will study a general problem: *Given n NTU games, is there a tight coupling between n NTU games so that the common core has non-empty?*

Let $N = \{1, 2, \dots, n\}$, η the collection of all non-empty subsets of N, and for $S \in \eta$,

$$E^S = \{(x_1, \dots, x_n) \in \mathbb{R}^n;\ x_i = 0 \text{ if } i \notin S\}.$$

Let $X \subset E^N$, $S \in \eta$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$. Denote by α^S the projection of α to E^S. Denote by "$\leq$" the natural order on E^N. X is *comprehensive* if $\alpha \in X$, $\beta \leq \alpha$ then $\beta \in X$. Denote by $\hat{X}$ the *comprehensive hull* of X, that is, the smallest comprehensive set containing X. An NTU game (game without side payments) is an ordered triple (N, F, D). Here F is a closed subset of E^N, and D is a function from η to open, comprehensive non-empty, proper subsets of E^N that satisfies

(i) $D(N) \subset \hat{F}$,

(ii) if $\alpha \in D(S)$ and $\alpha^S = \beta^S$ then $\beta \in D(S)$,

(iii) for each $S \in \eta$,

$$\{\alpha^S;\ \alpha \in \overline{D(S)} \setminus \bigcup_{i \in S} D(\{i\})\}$$

is non-empty and bounded. Here $\overline{D(S)}$ denotes the closure of $D(S)$.

The *core* of the game (N, F, D) is defined to be the set

$$F \setminus \bigcup_{S \in \eta} D(S).$$

The core represents the set of feasible outcomes that cannot be improved upon by any condition. A game (N, F, D) is said to be *balanced* if

$$\bigcap_{S \in \beta} D(S) \subset \hat{F}.$$

Theorem 1. (Scarf) *Every balanced game has a nonempty core.*

Shapley[6] proved a balanced KKM theorem rooted in balanced Sperner's lemma.

To state *multiple balanced KKM theorem* proved by Shih and Lee[7], let us recall some notations. Let $\{r_1, \ldots, r_n\}$ be affinely independent in E^N. For $X \subset E^N$, $convX$ denotes the convex hull of X. Let

$$A^S = conv\{r_i;\ i \in S\},$$
$$m_S = \frac{1}{\sharp S} \sum_{i \in S} r_i.$$

We say that $\{F_i;\ i \in N\}$ is a *KKM covering* of A^N if for each $i \in N$, F_i is closed in A^N, and

$$A^S \subset \bigcup_{i \in S} F_i \text{ for all } S \in \eta.$$

We say that $\{C_S;\ S \in \eta\}$ is a *Shaply covering* of A^N if for each $S \in \eta$, C_S is closed in A^N, and

$$A^S \subset \bigcup_{T \subset S, T \neq \emptyset} C_T.$$

Let $\pi : \eta \longrightarrow A^N$ and $\pi(S) \in A^S$ for all $S \in \eta$. We say that $\beta \subset \eta$ is *balanced* if

$$m_N \in conv\{m_S;\ S \in \beta\};$$

β is π-balanced if

$$m_N \in conv\{\pi(S);\ S \in \beta\}.$$

Theorem 2. (Shih and Lee) *Let $\{C^i_S;\ S \in \eta\}$, $i = 1, 2, \ldots, n$, be n Shapley coverings of A^N, and $\pi(S) \in A^N$ for all $S \in \eta$. Then there exists a π-balanced family $\{S_1, \ldots, S_n\}$ such that*

$$C^1_{S_1} \cap \ldots \cap C^n_{S_n} \neq \emptyset.$$

Theorem 2 was built upon the following multiple balanced Sperner's lemma.

Theorem 3. (Shih and Lee) *Let T be a simplicial subdivision of A^N, T^0 the vertex set of T. Let*

$$\Phi = (\varphi^1, \dots, \varphi^n) : T^0 \longrightarrow 2^N \times \cdots \times 2^N$$

be such that

$$\Phi(T^0 \cap A^S) \subset 2^S \times \cdots \times 2^S \text{ for all } S \in \eta,$$

and $\pi(S) \in A^S$ for all $S \in \eta$. Then there exist an $(n-1)$-simplex $conv\{v_1, \dots, v_n\} \in T$ and a π-balanced family $\{S_1, \dots, S_n\}$ such that

$$\varphi^1(v_1) = S_1, \dots, \varphi^n(v_n) = S_n.$$

There is a tight coupling between NTU games and Shapley coverings. Indeed, we have

Theorem 4. *Let $\pi(S) \in A^S$ for all $S \in \eta$, and $V_i = (N, F, D_i)$ n NTU games, $i = 1, \dots, n$. Then there exist n Shapley coverings $\{C^i_S;\ S \in \eta\}$ of A^N, $i = 1, \dots, n$, induced by n games $V_1, \dots, V_n$, and there exists π-balanced family $\{S_1, \dots, S_n\}$ such that*

$$C^1_{S_1} \cap C^2_{S_2} \cap \ldots \cap C^n_{S_n} \neq \emptyset.$$

Furthermore, if for all $i = 1, 2, \dots, n$, V_i is π-balanced and there exists

$$\alpha \in C^1_{S_1} \cap C^2_{S_2} \cap \ldots C^n_{S_n}$$

such that $\alpha \in C^i_{S_1} \cap C^i_{S_2} \cap \ldots \cap C^i_{S_n}$ for all $i = 1, 2, \dots, n$, then

$$Core(V_1) \cap Core(V_2) \cap \ldots \cap Core(V_n) \neq \emptyset.$$

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