CONVERGENCE THEOREMS OF IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES (Nonlinear Analysis and Convex Analysis)

Author(s)
KIM, J.K.; KIM, K.S.; KIM, S.M.

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CONVERGENCE THEOREMS OF IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

J. K. Kim, K. S. Kim and S. M. Kim

ABSTRACT. We prove that an implicit iteration process with errors which is generated by a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the mappings in convex metric spaces. Our main theorems extend and improve the recent results of Sun, Wittmann and Xu-Ori.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, we assume that $X$ is a metric space and let $F(T_i) \ (i \in \mathcal{N})$ be the set of all fixed points of mappings $T_i$ respectively, that is, $F(T_i) = \{x \in X : T_ix = x\}$, where $\mathcal{N} = \{1, 2, 3, \cdots, N\}$. The set of common fixed points of $T_i$ $(i \in \mathcal{N})$ denotes by $\mathcal{F}$, that is, $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$.

Definition 1.1. ([2],[4],[5]) Let $T : X \to X$ be a mapping.

(1) $T$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

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for all $x, y \in X$.

(2) $T$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p)$$

for all $x \in X$ and $p \in F(T)$.

(3) $T$ is said to be asymptotically nonexpansive if there exists a sequence $h_n \in [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that

$$d(T^nx, T^ny) \leq h_n d(x, y)$$

for all $x, y \in X$ and $n \geq 0$.

(4) $T$ is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $h_n \in [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that

$$d(T^n x, p) \leq h_n d(x, p)$$

for all $x \in X$, $p \in F(T)$ and $n \geq 0$.

**Remark 1.1.** From the Definition 1.1, we know that the following implications hold:

(1) $\Downarrow$ (3)

$\downarrow F(T) \neq \emptyset$ $\downarrow F(T) \neq \emptyset$

(2) $\implies$ (4)

In 2001, Xu-Ori [16] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space $H$. Let $C$ be a nonempty subset of $H$. Let $T_1, T_2, \ldots, T_N$ be self-mappings of $C$ and suppose that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i$, $i = 1, 2, \ldots, N$.

An implicit iteration process for a finite family of nonexpansive mappings is
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defined as follows, with \( \{t_n\} \) a real sequence in \((0,1),\ x_0 \in C :\)
\[
\begin{align*}
x_1 &= t_1 x_0 + (1 - t_1) T_1 x_1, \\
x_2 &= t_2 x_1 + (1 - t_2) T_2 x_2, \\
&\vdots \\
x_N &= t_N x_{N-1} + (1 - t_N) T_N x_N, \\
x_{N+1} &= t_{N+1} x_N + (1 - t_{N+1}) T_1 x_{N+1}, \\
&\vdots
\end{align*}
\]
which can be written in the following compact form:
\[
x_n = t_n x_{n-1} + (1 - t_n) T_n x_n, \quad n \geq 1, \tag{1.2}
\]
where \( T_k = T_{k \mod N}. \) (Here the mod \( N \) function takes values in \( \mathbb{N}. \)) And they proved the weak convergence of the process (1.2).

In 2003, Sun [12] extend the process (1.2) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with \( \{\alpha_n\} \) a real sequence in \((0,1)\) and an initial point \( x_0 \in C, \) which is defined as follows :
\[
\begin{align*}
x_1 &= \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1 \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \\
x_{N+1} &= \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1 - \alpha_{2N+1}) T_1^3 x_{2N+1}, \\
&\vdots
\end{align*}
\]
which can be written in the following compact form :
\[
x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_i^k x_n, \quad n \geq 1, \tag{1.3}
\]
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where \( n = (k - 1)N + i, \ i \in \mathcal{N} \).

Sun [12] proved the strong convergence of the process (1.3) to a common fixed point, requiring only one member \( T \) in the family \( \{T_i : i \in \mathcal{N}\} \) to be semi-compact. The result of Sun [12] generalized and extended the corresponding main results of Wittmann [15] and Xu-Ori [16].

The purpose of this paper is to introduce and study the convergence problem of an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. The main result of this paper is also, an extension and improvement of the well-known corresponding results in [1]–[11].

For the sake of convenience, we recall some definitions and notations.

In 1970, Takahashi [13] introduced the concept of convexity in a metric space and the properties of the space.

**Definition 1.2.** ([13]) Let \((X, d)\) be a metric space and \(I = [0, 1]\). A mapping \(W : X \times X \times I \to X\) is said to be a convex structure on \(X\) if for each \((x, y, \lambda) \in X \times X \times I\) and \(u \in X\),

\[
d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda)d(u, y).
\]

\(X\) together with a convex structure \(W\) is called a convex metric space, denoted it by \((X, d, W)\). A nonempty subset \(K\) of \(X\) is said to be convex if \(W(x, y, \lambda) \in K\) for all \((x, y, \lambda) \in K \times K \times I\).

**Remark 1.2.** Every normed space is a convex metric space, where a convex structure \(W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z\), for all \(x, y, z \in X\) and \(\alpha, \beta, \gamma \in I\) with \(\alpha + \beta + \gamma = 1\). In fact,

\[
d(u, W(x, y, z; \alpha, \beta, \gamma)) = \|u - (\alpha x + \beta y + \gamma z)\|
\leq \alpha\|u - x\| + \beta\|u - y\| + \gamma\|u - z\|
= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \ \forall \ u \in X.
\]

But there exists some convex metric spaces which can not be embedded into normed space.
A finite family of asymptotically quasi-nonexpansive mappings

**Example 1.1.** Let \( X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\} \). For \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X \) and \( \alpha, \beta, \gamma \in I \) with \( \alpha + \beta + \gamma = 1 \), we define a mapping \( W : X^3 \times I^3 \to X \) by
\[
W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)
\]
and define a metric \( d : X \times X \to [0, \infty) \) by
\[
d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.
\]
Then we can show that \((X, d, W)\) is a convex metric space, but it is not a normed space.

**Example 1.2.** Let \( Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \). For each \( x = (x_1, x_2), y = (y_1, y_2) \in Y \) and \( \lambda \in I \), we define a mapping \( W : Y^2 \times I \to Y \) by
\[
W(x, y; \lambda) = \left( \lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1} \right)
\]
and define a metric \( d : Y \times Y \to [0, \infty) \) by
\[
d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.
\]
Then we can show that \((Y, d, W)\) is a convex metric space, but it is not a normed space.

**Definition 1.3.** Let \((X, d, W)\) be a convex metric space with a convex structure \( W \) and let \( T_i : X \to X \) \( (i \in \mathcal{N}) \) be asymptotically quasi-nonexpansive mappings. For any given \( x_0 \in X \), the iteration process \( \{x_n\} \) defined by
\[
x_1 = W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1),
\]
\[
x_2 = W(x_1, T_2 x_2, u_2; \alpha_2, \beta_2, \gamma_2),
\]
\[
x_3 = W(x_2, T_3 x_3, u_3; \alpha_3, \beta_3, \gamma_3),
\]
and so on.

**Example 1.1.** Let \( X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\} \). For \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X \) and \( \alpha, \beta, \gamma \in I \) with \( \alpha + \beta + \gamma = 1 \), we define a mapping \( W : X^3 \times I^3 \to X \) by
\[
W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)
\]
and define a metric \( d : X \times X \to [0, \infty) \) by
\[
d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.
\]
Then we can show that \((X, d, W)\) is a convex metric space, but it is not a normed space.

**Example 1.2.** Let \( Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\} \). For each \( x = (x_1, x_2), y = (y_1, y_2) \in Y \) and \( \lambda \in I \), we define a mapping \( W : Y^2 \times I \to Y \) by
\[
W(x, y; \lambda) = \left( \lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1} \right)
\]
and define a metric \( d : Y \times Y \to [0, \infty) \) by
\[
d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.
\]
Then we can show that \((Y, d, W)\) is a convex metric space, but it is not a normed space.

**Definition 1.3.** Let \((X, d, W)\) be a convex metric space with a convex structure \( W \) and let \( T_i : X \to X \) \( (i \in \mathcal{N}) \) be asymptotically quasi-nonexpansive mappings. For any given \( x_0 \in X \), the iteration process \( \{x_n\} \) defined by
\[
x_1 = W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1),
\]
\[
x_2 = W(x_1, T_2 x_2, u_2; \alpha_2, \beta_2, \gamma_2),
\]
\[
x_3 = W(x_2, T_3 x_3, u_3; \alpha_3, \beta_3, \gamma_3),
\]
and so on.
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which can be written in the following compact form:

$$x_n = W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 1$$  \hspace{1cm} (1.4)

where $n = (k-1)N + i$, $i \in \mathcal{N}$, $\{u_n\}$ is bounded sequence in $X$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three sequences in $[0, 1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$ for $n = 1, 2, 3, \cdots$. Process (1.4) is called the implicit iteration process with error for a finite family of mappings $T_i$ ($i = 1, 2, \cdots, N$).

If $u_n = 0$ in (1.4) then,

$$x_n = W(x_{n-1}, T_i^k x_n; \alpha_n, \beta_n), \quad n \geq 1$$  \hspace{1cm} (1.5)

where $n = (k-1)N + i$, $i \in \mathcal{N}$, $\{\alpha_n\}$, $\{\beta_n\}$ be two sequences in $[0, 1]$ such that $\alpha_n + \beta_n = 1$ for $n = 1, 2, 3, \cdots$. Process (1.5) is called the implicit iteration process for a finite family of mappings $T_i$ ($i = 1, 2, \cdots, N$).

2. MAIN RESULTS

In order to prove the main theorems of this paper, we need the following lemma:

**Lemma 2.1.** ([14]) Let $\{\rho_n\}, \{\lambda_n\}$ and $\{\delta_n\}$ be the nonnegative sequences satisfying

$$\rho_{n+1} \leq (1 + \lambda_n)\rho_n + \mu_n, \quad \forall n \geq n_0,$$

and

$$\sum_{n=n_0}^{\infty} \lambda_n < \infty, \quad \sum_{n=n_0}^{\infty} \mu_n < \infty.$$

Then $\lim_{n \to \infty} \rho_n$ exists.

Now we state and prove the following main theorems of this paper.

**Theorem 2.1.** Let $(X, d, W)$ be a complete convex metric space. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of asymptotically quasi-nonexpansive mappings from $X$ into $X$, that is,

$$d(T_i^n x, p_i) \leq (1 + h_{n(i)})d(x, p_i)$$
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for all \( x \in X, p_i \in F(T_i), i \in \mathbb{N} \). Suppose that \( F \neq \emptyset \) and that \( x_0 \in X, \) \( \{\beta_n\} \subset (s, 1-s) \) for some \( s \in (0, \frac{1}{2}) \), \( \sum_{n=1}^{\infty} h_{n(i)} < \infty \) (i \( \in \mathbb{N} \)), \( \sum_{n=1}^{\infty} \gamma_n < \infty \) and \( \{u_n\} \) is arbitrary bounded sequence in \( X \). Then the implicit iteration process with error \( \{x_n\} \) generated by (1.4) converges to a common fixed point of \( \{T_i : i \in \mathbb{N}\} \) if and only if

\[
\liminf_{n \to \infty} D_d(x_n, F) = 0,
\]

where \( D_d(x, F) \) denotes the distance from \( x \) to the set \( F \), i.e., \( D_d(x, F) = \inf_{y \in F} d(x, y) \).

**Proof.** The necessity is obvious. Thus we will only prove the sufficiency. For any \( p \in F \), from (1.4), where \( n = (k-1)N + i, T_n = T_{n(\text{mod} N)} = T_i, i \in \mathbb{N} \), it follows that

\[
d(x_n, p) = d(W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), p)
\leq \alpha_n d(x_{n-1}, p) + \beta_n d(T_i^k x_n, p) + \gamma_n d(u_n, p)
\leq \alpha_n d(x_{n-1}, p) + \beta_n d(x_n, p) + \gamma_n d(u_n, p)
\leq \alpha_n d(x_{n-1}, p) + (\beta_n + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p)
\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p),
\]

for all \( p \in F \). Since \( \lim_{n \to \infty} \gamma_n = 0 \), there exists a natural number \( n_1 \), such that for \( n > n_1, \gamma_n \leq \frac{s}{2} \). Hence

\[
\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1-s) - \frac{s}{2} = \frac{s}{2}
\]

for \( n > n_1 \). Thus, we have by (2.1) that

\[
\alpha_n d(x_n, p) \leq \alpha_n d(x_{n-1}, p) + h_{k(i)} d(x_n, p) + \gamma_n d(u_n, p)
\]

and

\[
d(x_n, p) \leq d(x_{n-1}, p) + \frac{h_{k(i)}}{\alpha_n} d(x_n, p) + \frac{\gamma_n}{\alpha_n} d(u_n, p)
\leq d(x_{n-1}, p) + \frac{2}{s} h_{k(i)} d(x_n, p) + \frac{2}{s} \gamma_n d(u_n, p).\]
Since \( \sum_{n=1}^{\infty} h_{k(i)} < \infty \) for all \( i \in \mathcal{N} \), \( \lim_{n \to \infty} h_{n(i)} = 0 \) for each \( i \in \mathcal{N} \). Hence there exists a natural number \( n_2 \), as \( n > \frac{n_2}{N} + 1 \) i.e., \( n > n_2 \) such that

\[
h_{n(i)} \leq \frac{s}{4}, \quad \forall i \in \mathcal{N}.
\]

Then (2.2) becomes

\[
d(x_n, p) \leq \frac{s}{s-2h_{k(i)}}d(x_{n-1}, p) + \frac{2\gamma_n}{s-2h_{k(i)}}d(u_n, p). \tag{2.3}
\]

Let

\[
1 + \Delta_{k(i)} = \frac{s}{s-2h_{k(i)}} = 1 + \frac{2h_{k(i)}}{s-2h_{k(i)}}.
\]

Then

\[
\Delta_{k(i)} = \frac{2h_{k(i)}}{s-2h_{k(i)}} < \frac{4}{s}h_{k(i)}.
\]

Therefore

\[
\sum_{k=1}^{\infty} \Delta_{k(i)} < \frac{4}{s} \sum_{k=1}^{\infty} h_{k(i)} < \infty, \quad \forall i \in \mathcal{N}
\]

and (2.3) becomes

\[
d(x_n, p) \leq (1 + \Delta_{k(i)})d(x_{n-1}, p) + \frac{2}{s-2h_{k(i)}}\gamma_n d(u_n, p) \leq (1 + \Delta_{k(i)})d(x_{n-1}, p) + \frac{4}{s}\gamma_n M, \quad \forall p \in \mathcal{F},
\]

where, \( M = \sup_{n \geq 1} d(u_n, p) \). This implies that

\[
D_d(x_n, \mathcal{F}) \leq (1 + \Delta_{k(i)})d(x_{n-1}, \mathcal{F}) + \frac{4M}{s}\gamma_n.
\]

Since \( \sum_{k=1}^{\infty} \Delta_{k(i)} < \infty \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \), from Lemma 2.1, we have

\[
\lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0.
\]
A finite family of asymptotically quasi-nonexpansive mappings

Next, we will prove that the process \( \{x_n\} \) is Cauchy. Note that when \( a > 0 \), \( 1 + a \leq e^a \), from (2.4) we have

\[
d(x_{n+m}, p) \leq (1 + \Delta_k(i))d(x_{n+m-1}, p) + \frac{4M}{s} \gamma_{n+m}
\leq (1 + \Delta_k(i)) \left[ (1 + \Delta_k(i))d(x_{n+m-2}, p) + \frac{4M}{s} \gamma_{n+m-1} \right]
\leq (1 + \Delta_k(i))^2 \left[ (1 + \Delta_k(i))d(x_{n+m-3}, p) + \frac{4M}{s} \gamma_{n+m-2} \right]
\leq \cdots
\leq \exp \left\{ \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Delta_k(i) \right\} d(x_n, p)
+ \frac{4M}{s} \exp \left\{ \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Delta_k(i) \right\} \sum_{j=n+1}^{n+m} \gamma_j
\leq M'd(x_n, p) + \frac{4MM'}{s} \sum_{j=n+1}^{n+m} \gamma_j,
\]

for all \( p \in \mathcal{F} \) and \( n, m \in \mathbb{N} \), where \( M' = \exp \left\{ \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Delta_k(i) \right\} < \infty \). Since \( \lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0 \) and \( \sum_{n=1}^{\infty} h_{k(i)} < \infty \) (\( i \in \mathbb{N} \)), there exists a natural number \( n_1 \) such that for \( n \geq n_1 \),

\[
D_d(x_n, \mathcal{F}) < \frac{\varepsilon}{4M'} \quad \text{and} \quad \sum_{j=n_1+1}^{\infty} \gamma_j \leq \frac{s \cdot \varepsilon}{16MM'},
\]
Thus there exists a point $p_1 \in \mathcal{F}$ such that $d(x_{n_1}, p_1) \leq \frac{\epsilon}{4M'}$ by the definition of $D_d(x_n, \mathcal{F})$. It follows, from (2.5) that for all $n \geq n_1$ and $m \geq 0$,

$$d(x_{n+m}, x_n) \leq d(x_{n+m}, p_1) + d(x_n, p_1)$$

$$\leq M'd(x_{n_1}, p_1) + \frac{4MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j + M'd(x_{n_1}, p_1)$$

$$+ \frac{4MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j$$

$$< M' \cdot \frac{\epsilon}{4M'} + \frac{4MM'}{s} \cdot \frac{s \cdot \epsilon}{16MM'} + M' \cdot \frac{\epsilon}{4M'}$$

$$+ \frac{4MM'}{s} \cdot \frac{s \cdot \epsilon}{16MM'}$$

$$= \epsilon.$$}

This implies that $\{x_n\}$ is Cauchy. Because the space is complete, the process $\{x_n\}$ is convergent. Let $\lim_{n \to \infty} x_n = p$. Moreover, since the set of fixed points of asymptotically quasi-nonexpansive mapping is closed, so is $\mathcal{F}$, thus $p \in \mathcal{F}$ from $\lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0$, i.e., $p$ is a common fixed point of $\{T_i : i \in \mathcal{N}\}$. This completes the proof. \hfill $\square$

If $u_n = 0$, in Theorem 2.1, we can easily obtain the following theorem.

**Theorem 2.2.** Let $(X, d, W)$ be a complete convex metric space. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of asymptotically quasi-nonexpansive mappings from $X$ into $X$, that is,

$$d(T_i^n x, p_i) \leq (1 + h_{n(i)})d(x, p_i)$$

for all $x \in X$, $p_i \in F(T_i)$, $i \in \mathcal{N}$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$,

$\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} h_{n(i)} < \infty$ ($i \in \mathcal{N}$). Then the implicit iteration process $\{x_n\}$ generated by (1.5) converges to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if

$$\lim_{n \to \infty} \inf_{\mathcal{N}} D_d(x_n, \mathcal{F}) = 0.$$

From Theorem 2.1, we can also easily obtain the following theorem.
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**Theorem 2.3.** Let \((X, d, W)\) be a complete convex metric space. Let \(\{T_i : i \in \mathcal{N}\}\) be a finite family of quasi-nonexpansive mappings from \(X\) into \(X\), that is,

\[ d(T_i x, p_i) \leq d(x, p_i) \]

for all \(x \in X\), \(p_i \in F(T_i)\), \(i \in \mathcal{N}\). Suppose that \(\mathcal{F} \neq \emptyset\) and that \(x_0 \in X\), \(\{\alpha_n\} \subset (s, 1 - s)\) for some \(s \in (0, 1)\), \(\sum_{n=1}^{\infty} \gamma_n < \infty\) and \(\{u_n\}\) is arbitrary bounded sequence in \(X\). Then the implicit iteration process with error \(\{x_n\}\) generated by (1.4) converges to a common fixed point \(\{T_i : i \in \mathcal{N}\}\) if and only if

\[ \liminf_{n \to \infty} D_d(x_n, \mathcal{F}) = 0. \]

**Remark 2.1.** The results presented in this chapter are extensions and improvements of the corresponding results in Wittmann [15], Xu-Ori [16] and Sun [12].

**REFERENCES**


J. K. Kim, K. S. Kim and S. M. Kim


J. K. Kim
DEPARTMENT OF MATHEMATICS,
KYUNGNAM UNIVERSITY,
MASAN, KYUNGNAM, 631-701, KOREA
E-mail address: jongkyuk@kyungnam.ac.kr

K. S. Kim
DEPARTMENT OF MATHEMATICS,
KYUNGNAM UNIVERSITY,
MASAN, KYUNGNAM, 631-701, KOREA
E-mail address: kksmj@mail.kyungnam.ac.kr

S. M. Kim
DEPARTMENT OF MATHEMATICS,
KYUNGNAM UNIVERSITY,
MASAN, KYUNGNAM, 631-701, KOREA