CONVERGENCE THEOREMS OF IMPLICIT ITERATION PROCESS FOR A FINITE FAMILY OF ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. We prove that an implicit iteration process with errors which is generated by a finite family of asymptotically quasi-nonexpansive mappings converges strongly to a common fixed point of the mappings in convex metric spaces. Our main theorems extend and improve the recent results of Sun, Wittmann and Xu-Ori.

1. Introduction and Preliminaries

Throughout this paper, we assume that $X$ is a metric space and let $F(T_i) (i \in \mathcal{N})$ be the set of all fixed points of mappings $T_i$ respectively, that is, $F(T_i) = \{x \in X : T_i x = x\}$, where $\mathcal{N} = \{1, 2, 3, \cdots, N\}$. The set of common fixed points of $T_i (i \in \mathcal{N})$ denotes by $\mathcal{F}$, that is, $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i)$.

Definition 1.1. ([2],[4],[5]) Let $T : X \to X$ be a mapping. (1) $T$ is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$
for all $x, y \in X$.

(2) $T$ is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p)$$

for all $x \in X$ and $p \in F(T)$.

(3) $T$ is said to be asymptotically nonexpansive if there exists a sequence $h_n \in [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that

$$d(T^n x, T^n y) \leq h_n d(x, y)$$

for all $x, y \in X$ and $n \geq 0$.

(4) $T$ is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $h_n \in [1, \infty)$ with $\lim_{n \to \infty} h_n = 1$ such that

$$d(T^n x, p) \leq h_n d(x, p) \quad (1.1)$$

for all $x \in X$, $p \in F(T)$ and $n \geq 0$.

**Remark 1.1.** From the Definition 1.1, we know that the following implications hold:

$$
\begin{align*}
(1) \quad &\Rightarrow \quad (3) \\
\Downarrow F(T) \neq \emptyset \quad &\Downarrow F(T) \neq \emptyset \\
(2) \quad &\Rightarrow \quad (4)
\end{align*}
$$

In 2001, Xu-Ori [16] have introduced an implicit iteration process for a finite family of nonexpansive mappings in a Hilbert space $H$. Let $C$ be a nonempty subset of $H$. Let $T_1, T_2, \ldots, T_N$ be self-mappings of $C$ and suppose that $\mathcal{F} = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, the set of common fixed points of $T_i$, $i = 1, 2, \ldots, N$.

An implicit iteration process for a finite family of nonexpansive mappings is
A finite family of asymptotically quasi-nonexpansive mappings defined as follows, with \( \{t_n\} \) a real sequence in \((0,1)\), \( x_0 \in C \):

\[
x_1 = t_1x_0 + (1 - t_1)T_1x_1, \\
x_2 = t_2x_1 + (1 - t_2)T_2x_2, \\
\vdots \\
x_N = t_Nx_{N-1} + (1 - t_N)T_Nx_N, \\
x_{N+1} = t_{N+1}x_N + (1 - t_{N+1})T_1x_{N+1}, \\
\vdots
\]

which can be written in the following compact form:

\[
x_n = t_nx_{n-1} + (1 - t_n)T_nx_n, \quad n \geq 1, \tag{1.2}
\]

where \( T_k = T_{k \mod N} \). (Here the mod \( N \) function takes values in \( N \).) And they proved the weak convergence of the process (1.2).

In 2003, Sun [12] extend the process (1.2) to a process for a finite family of asymptotically quasi-nonexpansive mappings, with \( \{\alpha_n\} \) a real sequence in \((0,1)\) and an initial point \( x_0 \in C \), which is defined as follows:

\[
x_1 = \alpha_1x_0 + (1 - \alpha_1)T_1x_1 \\
\vdots \\
x_N = \alpha_Nx_{N-1} + (1 - \alpha_N)T_Nx_N, \\
x_{N+1} = \alpha_{N+1}x_N + (1 - \alpha_{N+1})T_1^2x_{N+1}, \\
\vdots \\
x_{2N} = \alpha_{2N}x_{2N-1} + (1 - \alpha_{2N})T_N^2x_{2N}, \\
x_{2N+1} = \alpha_{2N+1}x_{2N} + (1 - \alpha_{2N+1})T_1^3x_{2N+1}, \\
\vdots
\]

which can be written in the following compact form:

\[
x_n = \alpha_nx_{n-1} + (1 - \alpha_n)T_n^kx_n, \quad n \geq 1, \tag{1.3}
\]
where $n = (k-1)N + i, i \in \mathbb{N}$.

Sun [12] proved the strong convergence of the process (1.3) to a common fixed point, requiring only one member $T$ in the family $\{T_i : i \in \mathbb{N}\}$ to be semi-compact. The result of Sun [12] generalized and extended the corresponding main results of Wittmann [15] and Xu-Ori [16].

The purpose of this paper is to introduce and study the convergence problem of an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings in convex metric spaces. The main result of this paper is also, an extension and improvement of the well-known corresponding results in [1]-[11].

For the sake of convenience, we recall some definitions and notations.

In 1970, Takahashi [13] introduced the concept of convexity in a metric space and the properties of the space.

**Definition 1.2.** ([13]) Let $(X, d)$ be a metric space and $I = [0, 1]$. A mapping $W : X \times X \times I \rightarrow X$ is said to be a convex structure on $X$ if for each $(x, y, \lambda) \in X \times X \times I$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1-\lambda)d(u, y).$$

$X$ together with a convex structure $W$ is called a convex metric space, denoted it by $(X, d, W)$. A nonempty subset $K$ of $X$ is said to be convex if $W(x, y, \lambda) \in K$ for all $(x, y, \lambda) \in K \times K \times I$.

**Remark 1.2.** Every normed space is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. In fact,

$$d(u, W(x, y, z; \alpha, \beta, \gamma)) = \|u - (\alpha x + \beta y + \gamma z)\|\
\leq \alpha \|u - x\| + \beta \|u - y\| + \gamma \|u - z\|\
= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z), \quad \forall u \in X.$$

But there exists some convex metric spaces which can not be embedded into normed space.
A finite family of asymptotically quasi-nonexpansive mappings

**Example 1.1.** Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0, x_2 > 0, x_3 > 0\}$. For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W : X^3 \times I^3 \to X$ by

$$W(x, y, z; \alpha, \beta, \gamma) = (\alpha x_1 + \beta y_1 + \gamma z_1, \alpha x_2 + \beta y_2 + \gamma z_2, \alpha x_3 + \beta y_3 + \gamma z_3)$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 y_1 + x_2 y_2 + x_3 y_3|.$$ 

Then we can show that $(X, d, W)$ is a convex metric space, but it is not a normed space.

**Example 1.2.** Let $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For each $x = (x_1, x_2), y = (y_1, y_2) \in Y$ and $\lambda \in I$, we define a mapping $W : Y^2 \times I \to Y$ by

$$W(x, y; \lambda) = (\lambda x_1 + (1 - \lambda)y_1, \frac{\lambda x_1 x_2 + (1 - \lambda)y_1 y_2}{\lambda x_1 + (1 - \lambda)y_1})$$

and define a metric $d : Y \times Y \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$ 

Then we can show that $(Y, d, W)$ is a convex metric space, but it is not a normed space.

**Definition 1.3.** Let $(X, d, W)$ be a convex metric space with a convex structure $W$ and let $T_i : X \to X$ ($i \in \mathcal{N}$) be asymptotically quasi-nonexpansive mappings. For any given $x_0 \in X$, the iteration process $\{x_n\}$ defined by

$$x_1 = W(x_0, T_1 x_1, u_1; \alpha_1, \beta_1, \gamma_1),$$

$$x_N = W(x_{N-1}, T_N x_N, u_N; \alpha_N, \beta_N, \gamma_N),$$

$$x_{N+1} = W(x_N, T_1^2 x_{N+1}, u_{N+1}; \alpha_{N+1}, \beta_{N+1}, \gamma_{N+1}),$$

$$x_{2N} = W(x_{2N-1}, T_N^2 x_{2N}, u_{2N}; \alpha_{2N}, \beta_{2N}, \gamma_{2N}),$$

$$x_{2N+1} = W(x_{2N}, T_1^3 x_{2N+1}, u_{2N+1}; \alpha_{2N+1}, \beta_{2N+1}, \gamma_{2N+1})$$

and so on.
which can be written in the following compact form:

\[ x_n = W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 1 \quad (1.4) \]

where \( n = (k-1)N + i, i \in \mathbb{N}, \{u_n\} \) is bounded sequence in \( X \), \( \{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \) be three sequences in \([0, 1]\) such that \( \alpha_n + \beta_n + \gamma_n = 1 \) for \( n = 1, 2, 3, \cdots \). Process (1.4) is called the implicit iteration process with error for a finite family of mappings \( T_i \) \( (i = 1, 2, \cdots, N) \).

If \( u_n = 0 \) in (1.4) then,

\[ x_n = W(x_{n-1}, T_i^k x_n; \alpha_n, \beta_n), \quad n \geq 1 \quad (1.5) \]

where \( n = (k-1)N + i, i \in \mathbb{N}, \{\alpha_n\}, \{\beta_n\} \) be two sequences in \([0, 1]\) such that \( \alpha_n + \beta_n = 1 \) for \( n = 1, 2, 3, \cdots \). Process (1.5) is called the implicit iteration process for a finite family of mappings \( T_i \) \( (i = 1, 2, \cdots, N) \).

2. MAIN RESULTS

In order to prove the main theorems of this paper, we need the following lemma:

**Lemma 2.1.** ([14]) Let \( \{\rho_n\}, \{\lambda_n\} \) and \( \{\delta_n\} \) be the nonnegative sequences satisfying

\[ \rho_{n+1} \leq (1 + \lambda_n)\rho_n + \mu_n, \quad \forall \ n \geq n_0, \]

and

\[ \sum_{n=n_0}^{\infty} \lambda_n < \infty, \quad \sum_{n=n_0}^{\infty} \mu_n < \infty. \]

Then \( \lim_{n \to \infty} \rho_n \) exists.

Now we state and prove the following main theorems of this paper.

**Theorem 2.1.** Let \( (X, d, W) \) be a complete convex metric space. Let \( \{T_i : i \in \mathbb{N}\} \) be a finite family of asymptotically quasi-nonexpansive mappings from \( X \) into \( X \), that is,

\[ d(T_i^n x, p_i) \leq (1 + h_n(i))d(x, p_i) \]

where \( n = (k-1)N + i, i \in \mathbb{N} \).
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for all $x \in X, p_i \in F(T_i), i \in \mathbb{N}$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X, \{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2}), \sum_{n=1}^{\infty} h_{n(i)} < \infty (i \in \mathbb{N}), \sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{u_n\}$ is arbitrary bounded sequence in $X$. Then the implicit iteration process with error $\{x_n\}$ generated by (1.4) converges to a common fixed point of $\{T_i : i \in \mathbb{N}\}$ if and only if

$$\lim_{n \to \infty} \inf_{x \in \mathcal{F}} d(x_n, \mathcal{F}) = 0,$$

where $D_d(x, \mathcal{F})$ denotes the distance from $x$ to the set $\mathcal{F}$, i.e., $D_d(x, \mathcal{F}) = \inf_{y \in \mathcal{F}} d(x, y)$.

Proof. The necessity is obvious. Thus we will only prove the sufficiency. For any $p \in \mathcal{F}$, from (1.4), where $n = (k-1)N+i, T_n = T_{n(\mod N)} = T_i, i \in \mathbb{N}$, it follows that

$$d(x_n, p) = d(W(x_{n-1}, T_i^k x_n, u_n; \alpha_n, \beta_n, \gamma_n), p)$$

$$\leq \alpha_n d(x_{n-1}, p) + \beta_n d(T_i^k x_n, p) + \gamma_n d(u_n, p)$$

$$\leq \alpha_n d(x_{n-1}, p) + \beta_n (1+h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p)$$

$$\leq \alpha_n d(x_{n-1}, p) + (\beta_n + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p)$$

$$\leq \alpha_n d(x_{n-1}, p) + (1 - \alpha_n + h_{k(i)}) d(x_n, p) + \gamma_n d(u_n, p),$$

(2.1)

for all $p \in \mathcal{F}$. Since $\lim_{n \to \infty} \gamma_n = 0$, there exists a natural number $n_1$, such that for $n > n_1$, $\gamma_n \leq \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1-s) - \frac{s}{2} = \frac{s}{2}$$

for $n > n_1$. Thus, we have by (2.1) that

$$\alpha_n d(x_n, p) \leq \alpha_n d(x_{n-1}, p) + h_{k(i)} d(x_n, p) + \gamma_n d(u_n, p)$$

and

$$d(x_n, p) \leq d(x_{n-1}, p) + \frac{h_{k(i)}}{\alpha_n} d(x_n, p) + \frac{\gamma_n}{\alpha_n} d(u_n, p)$$

$$\leq d(x_{n-1}, p) + \frac{2}{s} h_{k(i)} d(x_n, p) + \frac{2}{s} \gamma_n d(u_n, p).$$

(2.2)
Since $\sum_{n=1}^{\infty} h_{k(i)} < \infty$ for all $i \in \mathcal{N}$, $\lim_{n \to \infty} h_{n(i)} = 0$ for each $i \in \mathcal{N}$. Hence there exists a natural number $n_2$, as $n > \frac{n_2}{N} + 1$ i.e., $n > n_2$ such that

$$h_{n(i)} \leq \frac{s}{4}, \quad \forall i \in \mathcal{N}.$$ 

Then (2.2) becomes

$$d(x_n, p) \leq \frac{s}{s - 2h_{k(i)}} d(x_{n-1}, p) + \frac{2\gamma_n}{s - 2h_{k(i)}} d(u_n, p).$$

(2.3)

Let

$$1 + \Delta_{k(i)} = \frac{s}{s - 2h_{k(i)}} = 1 + \frac{2h_{k(i)}}{s - 2h_{k(i)}}.$$ 

Then

$$\Delta_{k(i)} = \frac{2h_{k(i)}}{s - 2h_{k(i)}} < \frac{4}{s} h_{k(i)}.$$ 

Therefore

$$\sum_{k=1}^{\infty} \Delta_{k(i)} < \frac{4}{s} \sum_{k=1}^{\infty} h_{k(i)} < \infty, \quad \forall i \in \mathcal{N}$$

and (2.3) becomes

$$d(x_n, p) \leq (1 + \Delta_{k(i)})d(x_{n-1}, p) + \frac{2}{s - 2h_{k(i)}}\gamma_n d(u_n, p)$$

(2.4)

$$\leq (1 + \Delta_{k(i)})d(x_{n-1}, p) + \frac{4}{s}\gamma_n M, \quad \forall p \in \mathcal{F},$$

where, $M = \sup_{n \geq 1} d(u_n, p)$. This implies that

$$D_d(x_n, \mathcal{F}) \leq (1 + \Delta_{k(i)})d(x_{n-1}, \mathcal{F}) + \frac{4M}{s}\gamma_n.$$ 

Since $\sum_{k=1}^{\infty} \Delta_{k(i)} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$, from Lemma 2.1, we have

$$\lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0.$$
A finite family of asymptotically quasi-nonexpansive mappings

Next, we will prove that the process \( \{x_n\} \) is Cauchy. Note that when \( a > 0 \), \( 1 + a \leq e^a \), from (2.4) we have

\[
d(x_{n+m}, p) \leq (1 + \Delta_{k(i)}) d(x_{n+m-1}, p) + \frac{4M}{s} \gamma_{n+m} \\
\leq (1 + \Delta_{k(i)}) \left[ (1 + \Delta_{k(i)}) d(x_{n+m-2}, p) + \frac{4M}{s} \gamma_{n+m-1} \right] \\
+ \frac{4M}{s} \gamma_{n+m} \\
\leq (1 + \Delta_{k(i)})^2 \left[ (1 + \Delta_{k(i)}) d(x_{n+m-3}, p) + \frac{4M}{s} \gamma_{n+m-2} \right] \\
+ \frac{4M}{s} (1 + \Delta_{k(i)}) (\gamma_{n+m-1} + \gamma_{n+m}) \\
\leq (1 + \Delta_{k(i)})^3 d(x_{n+m-3}, p) \\
+ \frac{4M}{s} (1 + \Delta_{k(i)})^2 (\gamma_{n+m-2} + \gamma_{n+m-1} + \gamma_{n+m}) \\
\leq \cdots \\
\leq \exp \left\{ \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Delta_{k(i)} \right\} d(x_n, p) \\
+ \frac{4M}{s} \exp \left\{ \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Delta_{k(i)} \right\} \sum_{j=n+1}^{n+m} \gamma_j \\
\leq M'd(x_n, p) + \frac{4MM'}{s} \sum_{j=n+1}^{n+m} \gamma_j,
\]

for all \( p \in \mathcal{F} \) and \( n, m \in \mathbb{N} \), where \( M' = \exp \left\{ \sum_{i=1}^{N} \sum_{k=1}^{\infty} \Delta_{k(i)} \right\} < \infty \). Since \( \lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0 \) and \( \sum_{n=1}^{\infty} h_{k(i)} < \infty \ (i \in \mathcal{N}) \), there exists a natural number \( n_1 \) such that for \( n \geq n_1 \),

\[
D_d(x_n, \mathcal{F}) < \frac{\varepsilon}{4M'} \quad \text{and} \quad \sum_{j=n_1+1}^{\infty} \gamma_j \leq \frac{s \cdot \varepsilon}{16MM'}.
\]
Thus there exists a point $p_1 \in \mathcal{F}$ such that $d(x_{n_1}, p_1) \leq \frac{\epsilon}{4M'}$ by the definition of $D_d(x_n, \mathcal{F})$. It follows, from (2.5) that for all $n \geq n_1$ and $m \geq 0$,

$$
d(x_{n+m}, x_n) \leq d(x_{n+m}, p_1) + d(x_n, p_1)
$$

$$
\leq M'd(x_{n_1}, p_1) + \frac{4MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j + M'd(x_{n_1}, p_1)
$$

$$
+ \frac{4MM'}{s} \sum_{j=n_1+1}^{n+m} \gamma_j
$$

$$
< M' \cdot \frac{\epsilon}{4M'} + \frac{4MM'}{s} \cdot \frac{s \cdot \epsilon}{16MM'} + M' \cdot \frac{\epsilon}{4M'}
$$

$$
+ \frac{4MM'}{s} \cdot \frac{s \cdot \epsilon}{16MM'}
$$

$$
= \epsilon.
$$

This implies that $\{x_n\}$ is Cauchy. Because the space is complete, the process $\{x_n\}$ is convergent. Let $\lim_{n \to \infty} x_n = p$. Moreover, since the set of fixed points of asymptotically quasi-nonexpansive mapping is closed, so is $\mathcal{F}$, thus $p \in \mathcal{F}$ from $\lim_{n \to \infty} D_d(x_n, \mathcal{F}) = 0$, i.e., $p$ is a common fixed point of $\{T_i : i \in \mathcal{N}\}$. This completes the proof. \(\square\)

If $u_n = 0$, in Theorem 2.1, we can easily obtain the following theorem.

**Theorem 2.2.** Let $(X, d, W)$ be a complete convex metric space. Let $\{T_i : i \in \mathcal{N}\}$ be a finite family of asymptotically quasi-nonexpansive mappings from $X$ into $X$, that is,

$$
d(T^n_i x, p_i) \leq (1 + h_{n(i)})d(x, p_i)
$$

for all $x \in X$, $p_i \in F(T_i)$, $i \in \mathcal{N}$. Suppose that $\mathcal{F} \neq \emptyset$ and that $x_0 \in X$, $\{\alpha_n\} \subset (s, 1 - s)$ for some $s \in (0, 1)$, $\sum_{n=1}^{\infty} h_{n(i)} < \infty$ $(i \in \mathcal{N})$. Then the implicit iteration process $\{x_n\}$ generated by (1.5) converges to a common fixed point of $\{T_i : i \in \mathcal{N}\}$ if and only if

$$
\liminf_{n \to \infty} D_d(x_n, \mathcal{F}) = 0.
$$

From Theorem 2.1, we can also easily obtain the following theorem.
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**Theorem 2.3.** Let \((X, d, W)\) be a complete convex metric space. Let \(\{T_i : i \in \mathcal{N}\}\) be a finite family of quasi-nonexpansive mappings from \(X\) into \(X\), that is,

\[d(T_i x, p_i) \leq d(x, p_i)\]

for all \(x \in X, p_i \in F(T_i), i \in \mathcal{N}\). Suppose that \(\mathcal{F} \neq \emptyset\) and that \(x_0 \in X, \{\alpha_n\} \subset (s, 1 - s)\) for some \(s \in (0,1), \sum_{n=1}^{\infty} \gamma_n < \infty\) and \(\{u_n\}\) is arbitrary bounded sequence in \(X\). Then the implicit iteration process with error \(\{x_n\}\) generated by (1.4) converges to a common fixed point \(\{T_i : i \in \mathcal{N}\}\) if and only if

\[\lim_{n \to \infty} \inf D_d(x_n, \mathcal{F}) = 0.\]

**Remark 2.1.** The results presented in this chapter are extensions and improvements of the corresponding results in Wittmann [15], Xu-Ori [16] and Sun [12].

**REFERENCES**

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