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Remarks on Pointwise Nonlinear ergodic theorems in L_p

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1. Introduction

In [1], [2] Baillon considered a class of nonexpansive self-mappings T of a bounded closed convex subset C of a Hilbert space H or L_p with $1 < p < \infty$, formed the Cesàro $(C, 1)$ mean value peocess

$$C_n^{(1)}[T]f = \frac{1}{n+1}\sum_{k=0}^{n} T^k f, \qquad n \geq 0$$

for $f \in C$ and established the weak nonlinear ergodic theorem for T. Then later, Krengel and Lin [5] considered another class of order preserving, L_∞ –norm decreasing and L_1 –nonexpansive operators in L_p and proved the following weak nonlinear ergodic theorem which can not be covered by Baillon's theorem: Let T be an operator in L_p $(1 < p < \infty)$ which is order preserving, L_∞ –norm decreasing and L_1 –nonexpansive. Then for any $f \in L_p$, $C_n^{(1)}[T]f$ converges weakly in L_p. If the basic measure is finite, $C_n^{(1)}[T]f$ converges weakly in L_1 for $f \in L_1$. The same result holds for operators in L_1^+.

In the settings of Baillon, Krengel and Lin, however, one can only expect weak convergence of the $(C, 1)$ process. Indeed, the example due to Krengel and Lin [5] shows that $C_n^{(1)}[T]f$ need not converge in the strong topology of L_p and the example given by Krengel [4] shows that the pointwise convergence of $C_n^{(1)}[T]f$ may fail to hold. Note here that the iteration process considered by Wittmann [7] has a different aspect. So, as suggested (implicitly) by Krengel and Lin, it seems to be a question of great significance to find those (extra) conditions under which $C_n^{(1)}[T]f$ converges almost everywhere or in the strong topology of L_p (cf. [7]). In [9] the author made an attempt to deal with this question in the strong topology of L_p. [By the way, in [4] Krengel dared to say: It therefore seems that the example essentially eliminates all hopes for general pointwise nonlinear ergodic theorems. Of course, the possibility of positive results for specific class of nonlinear operators remains.]

We are particularly interested in finding some conditions or in changing the settings under which the almost everywhere convergence of $C_n^{(1)}[T]f$ holds in both linear and nonlinear cases. In a forthcoming paper [10] the author proved the following theorem:

Theorem. Let T be an order preserving operator in L_p $(1 \leq p \leq \infty)$ with $T(0) = 0$. Let $0 < \alpha < \infty$ and $f, f^* \in L_p^+$. Assume that E is a measurable set of measure zero such that for any $\omega \in E^c$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}$ converges (absolutely) for each $z \in C$ with $\mathrm{Re}(z) > 1$. Here A_n^α, $n \geq 0$, denote the (C, α) coefficients of order α. Assume that for any $\omega \in E^c \cup \{f^* < \infty\}$, the analytic function

$$G_\omega(z) = \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z} - \frac{f^*(\omega)}{z-1}, \quad \mathrm{Re}(z) > 1$$

has an analytic or just continuous extension (also called $G_\omega(.)$) to the closed half-plane $\{\mathrm{Re}(z) \geq 1\}$. Finally assume that for each $\omega \in E^c \cup \{f^* < \infty\}$, there exists a constant $M_\omega \geq 1$ such that

$$(*) \qquad G_\omega(z) = O(|z|^{M_\omega}), \qquad \mathrm{Re}(z) > 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_k^{\alpha-1}(T^k f)(\omega) = f^*(\omega)$$

holds for almost all ω.

In this paper we will show that the above theorem is also valid even without assuming the growth condition (*). Our general approach to understanding analytic conditions will be via Dirichlet series concerning operators in L_p (cf. [10]). The proof will make heavy use of Landau's Tauberian technique in Landau-Wiener-Ikehara's Tauberian theorem for Dirichlet series (cf. [3]).

2. Pointwise nonlinear ergodic theorems in L_p

Let $L_p = L_p(\Omega, \Xi, \mu)$, $1 \leq p \leq \infty$, be the usual Lebesgue spaces, where (Ω, Ξ, μ) is a σ-finite measure space. A operator T in L_p is said to be L_p-norm decreasing if $\|Tf\|_p \leq \|f\|_p$ holds for all $f \in L_p$. T is called order preserving in L_p if $f, g \in L_p$ and $f \leq g$ imply $Tf \leq Tg$. T is called nonexpansive in L_p if $\|Tf - Tg\|_p \leq \|f - g\|_p$ holds for all f, $g \in L_p$. W say that T is positively homogeneous if $T(cf) = cTf$ for all $f \in L_p$ and any constant $c \geq 0$. For a real number $\alpha > -1$ and each integer $n \geq 0$, let A_n^α denote the (C, α) coefficient of order α, which is defined by the generating function

$$\frac{1}{(1-\lambda)^{\alpha+1}} = \sum_{n=0}^{\infty} A_n^\alpha \lambda^n, \quad 0 < \lambda < 1$$

with $A_0^\alpha = 1$. We also let $A_0^{-1} = 1$ and $A_n^{-1} = 0$ for all $n \geq 1$. Then for $\alpha > -1$, we have $A_n^\alpha > 0$, $A_n^0 = 1$, $A_n^\alpha \sim \frac{n^\alpha}{\Gamma(\alpha+1)}$, and

$$A_n^\alpha = \sum_{k=0}^{n} A_{n-k}^{\alpha-1} = \sum_{k=0}^{n} A_k^{\alpha-1} = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}{n!}.$$

Moreover, it follows that A_n^α is increasing in n for $\alpha > 0$ and decreasing in n for $-1 < \alpha < 0$. We will prove

Theorem 1. Let T be an order preserving operator in L_p $(1 \leq p \leq \infty)$ with $T(0) = 0$ and let $0 < \alpha < \infty$, $f, f^* \in L_p$. Assume that E is a set in Ξ with $\mu(E) = 0$ such that for any $\omega \in \Omega - E$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}$ converges (absolutely) for each $z \in C$ with $\mathrm{Re}(z) > 1$. Assume that for any $\omega \in \Omega - (E \cup E_0)$, where $E_0 = \{f^* = \infty\}$, the analytic function

$$G_\omega(z) = \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z} - \frac{f^*(\omega)}{z-1}, \qquad \mathrm{Re}(z) > 1$$

has an analytic or just continuous extension (also called $G_\omega(.)$) to the closed half-plane $\{\mathrm{Re}(z) \geq 1\}$. Then $\frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_k^{\alpha-1}(T^k f)(\omega)$ converges as $n \to \infty$ to $f^*(\omega)$ for almost all $\omega \in \Omega$.

We need some lemmas.

Lemma 1. Let T be an order preserving operator in L_p $(1 \leq p \leq \infty)$ with $T(0) = 0$ and let $0 < \alpha < \infty$, $f \in L_p^+$. Assume that E is a set in Ξ with $\mu(E) = 0$ such that for any $\omega \in \Omega - E$, (the abscissa of convergence)

$$a_\omega(\alpha; f) = \limsup_{n\to\infty} \frac{\log\left[\sum_{k=0}^{n} A_k^{\alpha-1}(T^n f)(\omega)\right]}{\log A_n^\alpha} \leq 1.$$

Then for each $\omega \in \Omega - E$, the generalized Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}$ converges (absolutely) for $z \in C$ with $\mathrm{Re}(z) > 1$.

Proof. Let $\omega \in \Omega - E$ be fixed and let $z \in C$, $\mathrm{Re}(z) > 1$. We choose some $\delta > 0$ (which may depends on (α, ω, z)) such that

$$a_\omega(\alpha; f) + \tfrac{\delta}{2} < a_\omega(\alpha; f) + \delta < \mathrm{Re}(z).$$

Then there exists a sufficiently large number $N_0 = N_0(\delta, a_\omega)$ (where $a_\omega = a_\omega(\alpha; f)$) such that

$$\sum_{n=0}^{m} A_n^{\alpha-1}(T^n f)(\omega) < (A_m^\alpha)^{a_\omega + \frac{\delta}{2}}, \quad m \geq N_0.$$

Thus, letting

$$D_{\omega,n}(s) = \sum_{k=0}^{n} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^s}, \qquad s \geq 0$$

and using the partial summation formula of Abel, we have, for $m \geq n+1 > N_0$

$$\begin{aligned}
\sum_{k=n+1}^{m} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{a_\omega+\delta}} &= \left(\sum_{k=0}^{m} - \sum_{k=0}^{n}\right) \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{a_\omega+\delta}} \\
&= \sum_{k=n}^{m-1} \left\{\frac{1}{(A_k^\alpha)^{a_\omega+\delta}} - \frac{1}{(A_{k+1}^\alpha)^{a_\omega+\delta}}\right\} D_{\omega,k}(0) + \frac{D_{\omega,m}(0)}{(A_m^\alpha)^{a_\omega+\delta}} - \frac{D_{\omega,n}(0)}{(A_n^\alpha)^{a_\omega+\delta}} \\
&\leq \sum_{k=n}^{m-1} (A_k^\alpha)^{a_\omega+\frac{\delta}{2}} \left\{\frac{1}{(A_k^\alpha)^{a_\omega+\delta}} - \frac{1}{(A_{k+1}^\alpha)^{a_\omega+\delta}}\right\} + \frac{1}{(A_m^\alpha)^{a_\omega+\delta-(a_\omega+\frac{\delta}{2})}} + \frac{1}{(A_n^\alpha)^{a_\omega+\delta-(a_\omega+\frac{\delta}{2})}}
\end{aligned}$$

$$\leq (a_\omega + \delta) \sum_{k=n}^{m-1} (A_k^\alpha)^{a_\omega + \frac{\delta}{2}} \int_{\log A_k^\alpha}^{\log A_{k+1}^\alpha} e^{-(a_\omega+\delta)u} du \;+\; \frac{2}{(A_n^\alpha)^{\frac{\delta}{2}}}$$

$$\leq (a_\omega + \delta) \sum_{k=n}^{m-1} \int_{\log A_k^\alpha}^{\log A_{k+1}^\alpha} e^{-\frac{\delta}{2}u} du \;+\; \frac{2}{(A_n^\alpha)^{\frac{\delta}{2}}}$$

$$\leq \frac{2(a_\omega+\delta)}{\delta} \left\{ \frac{1}{(A_n^\alpha)^{\frac{\delta}{2}}} - \frac{1}{(A_m^\alpha)^{\frac{\delta}{2}}} \right\} + \frac{2}{(A_n^\alpha)^{\frac{\delta}{2}}}.$$

This gives

$$0 \leq \lim_{n,m\to\infty} \sum_{k=n+1}^{m} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\mathrm{Re}(z)}} \leq \lim_{n,m\to\infty} \sum_{k=n+1}^{m} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{a_\omega+\delta}} = 0,$$

and the lemma follows.

Lemma 2. Let $\omega \in \Omega - E$ be fixed. Then

$$\int_1^\infty \frac{1}{v^{z+1}} \Big[\sum_{0\leq k\leq n;\, A_n^\alpha \leq v} A_k^{\alpha-1}(T^k f)(\omega) \Big] dv = \frac{1}{z} \sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}, \qquad \mathrm{Re}(z) > 1.$$

Proof. Let $\varepsilon > 0$ be fixed sufficiently small and let $\omega \in \Omega - E$, $z \in C$, $\mathrm{Re}(z) > 1$. By assumption there exists a number N_0 (which may depends on $(f, \varepsilon, \alpha, \omega, z)$) large enough so that

$$\sum_{k=N_0+1}^{\infty} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\mathrm{Re}(z)}} < \varepsilon.$$

We then have for sufficiently large v

$$\frac{1}{v^{\mathrm{Re}(z)}} \sum_{N_0+1\leq k\leq n;\, A_n^\alpha \leq v} A_k^{\alpha-1}(T^n f)(\omega) \leq \sum_{k=N_0+1}^{n} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_n^\alpha)^{\mathrm{Re}(z)}} \leq \sum_{k=N_0+1}^{\infty} \frac{A_k^{\alpha-1}(T^k f)(\omega)}{(A_k^\alpha)^{\mathrm{Re}(z)}} < \varepsilon.$$

This implies that $\lim_{N\to\infty} \frac{1}{(A_{N+1}^\alpha)^z} \sum_{k=0}^{N} A_k^{\alpha-1}(T^k f)(\omega) = 0$. Now let us define

$$S_\omega(v) = \sum_{0\leq k\leq n;\, A_n^\alpha\leq v} A_n^{\alpha-1}(T^n f)(\omega), \qquad v \geq 1, \;\; (n \geq 0)$$
$$= 0, \qquad v < 1.$$

Then it follows that

$$z \int_1^{A_{N+1}^\alpha} \frac{S_\omega(v)}{v^{z+1}} dv = z \sum_{j=0}^{N} \int_{A_j^\alpha}^{A_{j+1}^\alpha} \frac{S_\omega(v)}{v^{z+1}} dv$$

$$= z \sum_{j=0}^{N} S_\omega(A_j^\alpha) \int_{A_j^\alpha}^{A_{j+1}^\alpha} \frac{1}{v^{z+1}} dv$$

$$= \sum_{j=0}^{N} S_\omega(A_j^\alpha) \Big[\frac{1}{(A_j^\alpha)^z} - \frac{1}{(A_{j+1}^\alpha)^z} \Big]$$

$$= \frac{S_\omega(A_0^\alpha)}{(A_0^\alpha)^z} + \frac{S_\omega(A_1^\alpha)-S_\omega(A_0^\alpha)}{(A_1^\alpha)^z} + \cdots + \frac{S_\omega(A_N^\alpha)-S_\omega(A_{N-1}^\alpha)}{(A_N^\alpha)^z} - \frac{S_\omega(A_N^\alpha)}{(A_{N+1}^\alpha)^z}.$$

Thus, since $\lim_{N\to\infty} \frac{S_\omega(A_N^\alpha)}{(A_{N+1}^\alpha)^z} = 0$ for $\mathrm{Re}(z) > 1$, we obtain the desired equality

$$z\int_1^\infty \frac{S_\omega(v)}{v^{z+1}}dv = \sum_{n=0}^\infty \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}, \qquad \mathrm{Re}(z) > 1.$$

Lemma 3. Let $\omega \in \Omega - (E \cup E_0)$ be fixed and put

$$H_\omega(y) = e^{-y}S_\omega(e^y), \quad y \geq 0.$$

Then for some real $a > 0$,

$$\lim_{y\to\infty} \int_{-\infty}^{ay} H_\omega(y - \tfrac{v}{a})K_1(v)dv = f^*(\omega)\int_{-\infty}^\infty K_1(v)dv$$

holds with the Fejér kernel $K_\rho(t) = \frac{\sin^2(\rho t)}{\rho t^2}$, $\rho > 0$.

Proof. By Lemma 2 we see that

$$\int_0^\infty H_\omega(y)e^{-(z-1)y}dy = \frac{1}{z}\sum_{n=0}^\infty \frac{A_n^{\alpha-1}(T^n f)(\omega)}{(A_n^\alpha)^z}, \quad \mathrm{Re}(z) > 1,$$

and so

$$\int_0^\infty (H_\omega(y) - f^*(\omega))e^{-(z-1)y}dy = \frac{G_\omega(z) - f^*(\omega)}{z}, \qquad \mathrm{Re}(z) > 1.$$

Note that the Fourier transform of $K_a(t)$ becomes

$$\frac{1}{\pi}\int_{-\infty}^\infty K_a(t)e^{-iyt}dt = \begin{cases} 1 - \frac{|y|}{2a}, & \text{if } |y|\leq 2a, \\ 0, & \text{if } |y|>2a. \end{cases}$$

Using the Fejér kernel

$$\frac{1}{2}\int_{-2a}^{2a}\left(1 - \frac{|t|}{2a}\right)e^{i(y-u)t}dt = \frac{1}{4a}\int_0^{2a}\{\int_{-\tau}^{\tau} e^{it(y-u)}dt\}d\tau = \frac{1-\cos(2a(y-u))}{2a(y-u)^2} = K_a(y-u),$$

we have for $\sigma > 1$ and every $a > 0$

$$\frac{1}{2a}\int_{-2a}^{2a} e^{iyt}\left(1 - \frac{|t|}{2a}\right)\frac{G_\omega(\sigma+it) - f^*(\omega)}{\sigma+it}dt$$

$$= \frac{1}{2}\int_{-2a}^{2a}[\int_0^\infty (H_\omega(u) - f^*(\omega))e^{-(\sigma-1+it)u}du]\left(1 - \frac{|t|}{2a}\right)dt$$

$$= \frac{1}{2}\int_0^\infty (H_\omega(u) - f^*(\omega))e^{-(\sigma-1)u}du\int_{-2a}^{2a}\left(1 - \frac{|t|}{2a}\right)e^{i(y-u)t}dt$$

$$= \int_0^\infty H_\omega(u)e^{-(\sigma-1)u}K_a(y-u)du - f^*(\omega)\int_0^\infty e^{-(\sigma-1)u}K_a(y-u)du.$$

So, letting $\sigma \to 1$ gives

$$\frac{1}{2}\int_{-2a}^{2a} e^{iyt}\left(1-\frac{|t|}{2a}\right)\frac{G_\omega(1+it)-f^*(\omega)}{1+it}dt$$

$$= \int_0^\infty H_\omega(u)K_a(y-u)du - f^*(\omega)\int_0^\infty K_a(y-u)du$$

$$= \int_{-\infty}^{ay} H_\omega(y-\tfrac{v}{a})K_1(v)dv - f^*(\omega)\int_{-\infty}^{ay} K_1(v)dv.$$

Consequently, the desired conclusion follows immediately from this and the Riemann-Lebesgue theorem.

Proof of Theorem 1. After observing that $y_2 \geq y_1 > 0$ implies $H_\omega(y_2)e^{y_2} \geq H_\omega(y_1)e^{y_1}$, it follows from Lemma 3 that

$$f^*(\omega)\int_{-\infty}^\infty K_1(v)dv \geq \limsup_{y\to\infty}\int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y-\tfrac{v}{a})K_1(v)dv$$

$$\geq \limsup_{y\to\infty}\int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y-\tfrac{1}{\sqrt{a}})e^{-\frac{2}{\sqrt{a}}}K_1(v)dv$$

$$= \limsup_{y\to\infty} H_\omega(y-\tfrac{1}{\sqrt{a}})e^{-\frac{2}{\sqrt{a}}}\int_{-\sqrt{a}}^{\sqrt{a}} K_1(v)dv.$$

Therefore

$$\limsup_{y\to\infty} H_\omega(y) \leq \frac{f^*(\omega)e^{\frac{2}{\sqrt{a}}}}{\int_{-\sqrt{a}}^{\sqrt{a}}K_1(v)dv}\int_{-\infty}^\infty K_1(v)dv.$$

Moreover, letting $a \to \infty$ yields

$$\limsup_{y\to\infty} H_\omega(y) \leq f^*(\omega).$$

This also implies that $H_\omega(y)$ is bounded, so we may write $H_\omega(y) \leq C_\omega$ with a suitably chosen constant C_ω. On the other hand, for $y \geq \frac{1}{\sqrt{a}} > 0$,

$$\int_{-\infty}^{ay} H_\omega(y-\tfrac{v}{a})K_1(v)dv \leq \left(\int_{-\infty}^{-\sqrt{a}} + \int_{-\sqrt{a}}^{\sqrt{a}} + \int_{\sqrt{a}}^{ay}\right)H_\omega(y-\tfrac{v}{a})K_1(v)dv$$

$$\leq 2C_\omega\int_{\sqrt{a}}^\infty K_1(v)dv + \int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y+\tfrac{1}{\sqrt{a}})e^{\frac{2}{\sqrt{a}}}K_1(v)dv.$$

Thus by Lemma 3 again we have

$$f^*(\omega)\int_{-\infty}^\infty K_1(v)dv \leq 2C_\omega\int_{\sqrt{a}}^\infty K_1(v)dv + \liminf_{y\to\infty}\int_{-\sqrt{a}}^{\sqrt{a}} H_\omega(y+\tfrac{1}{\sqrt{a}})e^{\frac{2}{\sqrt{a}}}K_1(v)dv$$

$$= 2C_\omega \int_{\sqrt{a}}^{\infty} K_1(v)dv + \liminf_{y\to\infty} H_\omega(y) e^{\frac{2}{\sqrt{a}}} \int_{-\sqrt{a}}^{\sqrt{a}} K_1(v)dv,$$

and hence

$$\liminf_{y\to\infty} H_\omega(y) \geq \frac{f^*(\omega)\int_{-\infty}^{\infty} K_1(v)dv - 2C_\omega \int_{\sqrt{a}}^{\infty} K_1(v)dv}{\int_{-a}^{\sqrt{a}} K_1(v)dv} e^{-\frac{2}{\sqrt{a}}}.$$

Finally, let $a \to \infty$ to get

$$\liminf_{y\to\infty} H_\omega(y) \geq f^*(\omega).$$

The above two parts together shows that $\lim_{y\to\infty} H_\omega(y) = f^*(\omega)$. Hence we may take $y = \log A_n^\alpha$ to conclude that the theorem follows. The proof of the theorem has hereby completed.

Remarks. It should be noticed that an essential role in the proof of Theorem 1 is played by Wiener's general Tauberian theorem ([6], Theorem VIII) which guarantees that the following equation to hold for some $a > 0$

$$\lim_{y\to\infty} \frac{1}{\pi} \int_{-\infty}^{\infty} K_a(y-u) H_\omega(u)du = \frac{f^*(\omega)}{\pi} \int_{-\infty}^{\infty} K_a(u)du$$

is in fact valid for all real $a > 0$. We next demonstrate two cases realizing all the conditions of Theorem 1.

(1) We consider the function space $C[0,1]$ consisting of functions $f(t)$ continuous for $0 \leq t \leq 1$ such that $\|f\| = \max|f(t)|$. Let T be an order preserving, positively homogeneous and norm decreasing operator in $C[0,1]$ with $T(0) = 0$. Let $T_r = rT$ for some r, $0 < r < 1$. Let $0 < \alpha < \infty$ and $f \in C[0,1]^+$. Then one gets

$$\limsup_{n\to\infty} \frac{\log\left[\sum_{k=0}^{n} A_k^{\alpha-1}(T_r^k f)(t)\right]}{\log A_n^\alpha} = 0.$$

We can thus define the function $G_t(z)$ by the convergent Dirichlet series $\sum_{n=0}^{\infty} \frac{A_n^{\alpha-1}(T_r^n f)(t)}{(A_n^\alpha)^z}$ for $z \in \{\mathrm{Re}(z) \geq 1\}$. Clearly $G_t(z)$ is analytic in the closed half-plane $\{\mathrm{Re}(z) \geq 1\}$.

(2) Let $\beta > 0$ be fixed positive and define an operator T_β in $C[0,1]$ by the fractional integral

$$(T_\beta f)(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-u)^{\beta-1} f(u)du, \quad 0 \leq t \leq 1$$

for $f \in C[0,1]$. Let $0 < \alpha < \infty$ and $f \in C[0,1]^+$. Then we see that

$$\limsup_{n\to\infty} \frac{\log\left[\sum_{k=0}^{n} A_k^{\alpha-1}(T_\beta^k f)(t)\right]}{\log A_n^\alpha} = 0.$$

Thus we may obtain a function $G_t(z)$ analytic in $\{\mathrm{Re}(z) \geq 1\}$ which is defined by the convergent Dirichlet series $\sum_{n=0}^{\infty} \frac{A_k^{\alpha-1}(T_\beta^k f)(t)}{(A_n^\alpha)^z}$ for $z \in \{\mathrm{Re}(z) \geq 1\}$.

Applying a modified Karamata's argument for series to nonlinear operators (see [9]), we have

Theorem 2. Let T be an order preserving and L_∞ –norm decreasing operator in L_p $(1 \leq p \leq \infty)$ with $T(0) = 0$. Let $0 < \alpha < \infty$ and $f \in L_p^+ \cap L_\infty$. Define

$$\Psi_\alpha(t;f) = A_n^{\alpha-1} T^n f, \qquad n \leq t < n+1, \ n \geq 0.$$

If $\lambda^\alpha \int_0^\infty e^{-\lambda t} \Psi_\alpha(t;f)dt$ $(\lambda > 0)$ converges a.e. as $\lambda \to 0+$ to some $f_0 \in L_p^+$, then $\frac{1}{A_n^\alpha} \sum_{k=0}^n A_k^{\alpha-1} T^k f$ converges a.e. as $n \to \infty$ to the function f_0.

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