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Nondifferentiable Multiobjective Fractional Programming Problems under Generalized Convexity[1]

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Abstract. In this paper, we consider a class of nondifferentiable multiobjective fractional programs in which each component of the objective function contains a term involving the support function of a compact convex set. We present optimality conditions and duality results for a weakly efficient solution of nondifferentiable multiobjective fractional programming problems under generalized convexity.

1 Introduction and Preliminaries

The various concepts of generalized convexity and duality results for a fractional programming problem was introduced by many authors [1]-[14]. Duality and optimality for nondifferentiable multiobjective programming problems, in which the objective function contains a support function was studied by Mond and Schechter [15]. Bector *et al.* [1] , derived optimality conditions for a class of nondifferentiable convex multiobjective fractional programming problems and established some duality theorems. Recently, Kuk *et al.* [7] defined the concept of V-ρ-invexity for vector valued functions, which is generalization of the V-invex function [4],[13], and they proved the generalized Karush-Kuhn-Tucker sufficient optimality theorem, weak and strong duality for nonsmooth multiobjective programs under the V-ρ-invexity assumptions. Subsequently, Kuk *et al.* [8] extend their results to nonsmooth multiobjective fractional programs and Liang *et al.* [11] introduced (F, α, ρ, d)-convexity and obtained some corresponding optimality conditions and duality results for the single-objective fractional problem. Also, Liang *et al.* [12] extend their results to the multiobjective fractional programs. Very recently, Kim *et al.* [6] proved Fritz John and Kuhn-Tucker necessary and sufficient optimality conditions for nondifferentiable multiobjective fractional programming problems and obtained some duality results for a weakly efficient solution under V-ρ-invexity assumptions that was given by Kuk *et al.* [7].

In this paper, we consider a nondifferentiable multiobjective fractional programs in which each component of the objective function contains a term involving the support function of a compact convex set. We present necessary and sufficient optimality conditions, which is given by Kim *et al.* [6] and formulate

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a general dual problem. Also we establish duality theorems for weakly efficient solutions of nondifferentiable multiobjective fractional programming problems and introduce special cases of our duality results.

Now we consider the following multiobjective fractional programming problem,

$$\text{(MFP)} \quad \text{Minimize} \quad \left(\frac{f_1(x)+s(x|C_1)}{g_1(x)}, \ldots, \frac{f_p(x)+s(x|C_p)}{g_p(x)}\right)$$

$$\text{subject to} \quad h(x) \leqq 0, \quad x \in X_0,$$

where X_0 is an open set of R^n, $f := (f_1, \ldots, f_p) : X_0 \to R^p$, $g := (g_1, \ldots, g_p) : X_0 \to R^p$, and $h := (h_1, \ldots, h_m) : X_0 \to R^m$ are continuously differentiable over X_0; C_i, for each $i \in P = \{1, 2, \ldots, p\}$, is a compact convex set of R^n and $s(x|C_i) = \max\{\langle x, y\rangle \mid y \in C_i\}$. Further let, $S = \{x \in X_0 \ : \ h(x) \leqq 0\}$ be the set of all feasible solutions and $I(x) := \{i \ : \ h_i(x) = 0\}$ for any $x \in X_0$. Let $k_i(x) = s(x|C_i)$, $i = 1, \ldots, p$. Then k_i is a convex function and $\partial k_i(x) = \{w \in C_i | \ \langle w, x\rangle = s(x|C_i)\}$ [15], where ∂k_i is the subdifferential of k_i. We assume that $f(x) \geqq 0$ for all $x \in X_0$ and $g(x) > 0$ for all $x \in X_0$ whenever g is not linear.

We introduce the following definition due to Kuk *et al* [7].

Definition 1.1. A vector function $f : X_0 \to R^p$ is said to be (V, ρ)-invex at $u \in X_0$ with respect to functions η and $\theta : X_0 \times X_0 \to R^n$ if there exists $\alpha_i : X_0 \times X_0 \to \mathbb{R}_+ \backslash \{0\}$ and $\rho_i \in \mathbb{R}$, $i = 1, \ldots p$ such that for any $x \in X_0$, and for $i = 1, 2, \ldots, p$,

$$\alpha_i(x,u)\Big[f_i(x) - f_i(u)\Big] \geqq \nabla f_i(u)\eta(x,u) + \rho_i \|\theta(x,u)\|^2.$$

The function f is (V, ρ)-invex on X_0 if it is (V, ρ)-invex at every point in X_0.

We shall use the following theorem.

Theorem 1.1.[6] Assume that f and g are vector-valued differentiable functions defined on X_0 and $f(x) + \langle w, x\rangle \geqq 0$, $g(x) > 0$ for all $x \in X_0$. If $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$ are (V, ρ)-invex at $x_0 \in X_0$, then $\frac{f(\cdot)+\langle w,\cdot\rangle}{g(\cdot)}$ is (V, ρ)-invex at x_0, where

$$\bar{\alpha}_i(x, x_0) = \frac{g_i(x)}{g_i(x_0)}\alpha_i(x, x_0), \quad \bar{\theta}_i(x, x_0) = \left(\frac{1}{g_i(x_0)}\right)^{1/2}\theta_i(x, x_0).$$

2 Optimality Conditions

We present Fritz John and Kuhn-Tucker necessary and sufficient conditions, that were proved by Kim *et al.* [6] for weakly efficient solutions of (MFP).

Theorem 2.1. Fritz John Necessary Optimality Conditions
If $x_0 \in S$ is a weakly efficient solution of (MFP), then there exists $\lambda_i, i = 1, \ldots, p$, μ_j, $j = 1, \ldots, m$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)} \Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,$$

$$\langle w_i, x_0 \rangle = s(x_0 | C_i), \ w_i \in C_i, \ i = 1, \ldots, p,$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_m) \geqq 0, \ (\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_m) \neq 0.$$

Theorem 2.2. Kuhn-Tucker Necessary Optimality Conditions
Let $x_0 \in S$ is a weakly efficient solution of (MFP) and assume that there exists $z^* \in \mathrm{R}^n$ such that $\langle \nabla h_j(x_0), z^* \rangle > 0$, $j \in I(x_0)$. Then there exist $\lambda_i \geqq 0$, $i = 1, \ldots, p$, $\mu_j \geqq 0, j = 1, \ldots, m$ and $w_i \in C_i, i = 1, \ldots, p$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(x_0) + \langle w_i, x_0 \rangle}{g_i(x_0)} \Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,$$

$$\langle w_i, x_0 \rangle = s(x_0 | C_i), \ w_i \in C_i, \ i = 1, \ldots, p,$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0,$$

$$(\lambda_1, \ldots, \lambda_p) \neq (0, \ldots, 0).$$

Theorem 2.3. Kuhn-Tucker Sufficient Optimality Conditions
Let x_0 be a feasible solution of (MFP). Suppose that there exists $\lambda = (\lambda_1, \dots, \lambda_p) \in R^p_+$, $\lambda > 0$, $\sum_{i=1}^p \lambda_i = 1$ and $\mu = (\mu_1, \dots, \mu_m) \in R^m_+$ such that

$$\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(x_0) + \langle w_i, x_0\rangle}{g_i(x_0)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(x_0) = 0,$$

$$\langle w_i, x_0\rangle = s(x_0|C_i),\ w_i \in C_i,\ i = 1, \dots, p,$$

$$\sum_{j=1}^{m} \mu_j h_j(x_0) = 0.$$

If $f(\cdot) + \langle w, \cdot\rangle$ and $-g(\cdot)$ are (V, ρ)-invex at x_0 and h is (V, σ)-invex at x_0 with respect to the same η with $\sum_{i=1}^p \lambda_i \rho_i \geqq 0$ and $\sum_{j=1}^m \sigma_j \geqq 0$,then x_0 is a weakly efficient solution of (MFP).

3 Duality Theorems

We consider the following general dual problem to primal problem (MFP).

(MFD)$_G$ Maximize
$$\Big(\frac{f_1(u) + \langle w_1, u\rangle}{g_1(u)} + \mu_I h_I(u),\ \dots,\ \frac{f_p(u) + \langle w_p, u\rangle}{g_p(u)} + \mu_I h_I(u)\Big)$$

subject to
$$\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0, \quad (1)$$

$$\mu_J h_J(u) \geqq 0,$$
$$w_i \in C_i\ i = 1, \dots, p,$$
$$(\mu_1, \dots, \mu_m) \geqq 0,\ \lambda = (\lambda_1, \dots, \lambda_p) \in \Lambda^+,$$

where $I \cup J = \{1, \cdots, m\} = M$ and $I \cap J = \emptyset$. Let $\Lambda^+ = \{\lambda \in R^p : \lambda \geqq 0, \lambda^T e = 1, e = (1, \dots, 1) \in R^p\}$.

Theorem 3.1. Weak Duality Let $x \in S$ be a feasible for (MFP) and (u, λ, w, μ) be a feasible for (MFD)$_G$. Assume that the functions $f(\cdot) +$

$\langle w, \cdot\rangle, -g(\cdot)$ and h are (V, ρ)-invex functions over S with respect to the same η with $\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$.

Then the following cannot hold;

$$\frac{f(x) + s(x|C)}{g(x)} < \frac{f(u) + \langle w, u\rangle}{g(u)} + \sum_{j\in I} \mu_j h_j(u)e.$$

Proof. Assume that the result does not hold. Since $\langle w_i, x\rangle \leqq s(x|C_i)$, we have for all $i \in \{1, \ldots, p\}$

$$\begin{aligned}\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} &\leqq \frac{f_i(x) + s(x|C_i)}{g_i(x)} \\ &< \frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)} + \sum_{j\in I} \mu_j h_j(u).\end{aligned}$$

Since $\sum_{j\in J} \mu_j h_j(x) \leqq 0$ and $\sum_{j\in J} \mu_j h_j(u) \geqq 0$, for $i = 1, \ldots, p$,

$$\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} + \sum_{j=1}^{m} \mu_j h_j(x) < \frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)} + \sum_{j=1}^{m} \mu_j h_j(u).$$

By using (V, ρ)-invexity of h at u and Theorem 1.1, it follows that

$$\bar{\alpha}_i(x,u)\Big[\frac{f_i(x) + \langle w_i, x\rangle}{g_i(x)} + \sum_{j=1}^{m} \mu_j h_j(u) - \frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)} - \sum_{j=1}^{m} \mu_j h_j(u)\Big]$$

$$\geqq \Big[\nabla\Big(\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u)\Big]\eta(x,u) + \rho_i \|\bar{\theta}_i(x,u)\|^2.$$

Since $\lambda \in \Lambda^+$, we have

$$\Big[\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(u)\,\langle w_i, u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u)\Big]\eta(x,u) < (-\sum_{i=1}^{p} \lambda_i \rho_i)\|\bar{\theta}_i(x,u)\|^2. \quad (2)$$

Since $\sum_{i=1}^{p} \lambda_i \rho_i \|\bar{\theta}_i(x,u)\|^2 \geqq 0$, it follows from (2) that

$$\Big[\sum_{i=1}^{p} \lambda_i \nabla\Big(\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u)\Big]\eta(x,u) < 0,$$

which contradicts (1). □

Remark. If we replace $\lambda \in \Lambda^+$ in $(\text{MFD})_G$ by $\lambda > 0$, then above weak duality theorem holds in the sense of efficient solutions.

Theorem 3.2. Strong Duality If $\bar{x}$ is a weakly efficient solution of (MFP), and assume that there exists $z^* \in \mathrm{R}^n$ such that $\langle \nabla h_j(\bar{x}), z^*\rangle > 0,\ j \in I(\bar{x})$, then there exists $\bar{\lambda} \in \mathrm{R}^p,\ \bar{\mu} \in \mathrm{R}^m$ and $\bar{w} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is feasible for $(\text{MFD})_G$ and $\langle \bar{w}, \bar{x}\rangle = s(\bar{x}|C)$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $(\text{MFD})_G$.

Proof. Since $\bar{x}$ is a weakly efficient solution of (MFP) and there exists $z^* \in \mathrm{R}^n$ such that $\langle \nabla h_j(\bar{x}), z^*\rangle > 0,\ j \in I(\bar{x})$, there exists $\bar{\lambda} \in \mathrm{R}^p,\ \bar{\mu} \in \mathrm{R}^m$ and $w_i \in C_i,\ i = 1, \ldots, p$ such that $\sum_{i=1}^{p} \bar{\lambda}_i \nabla\Big(\frac{f_i(\bar{x}) + \langle w_i, \bar{x}\rangle}{g_i(\bar{x})}\Big) + \sum_{j=1}^{m} \bar{\mu}_j \nabla h_j(\bar{x}) = 0,\ \langle \bar{w}_i, \bar{x}\rangle = s(\bar{x}|C_i),\ \bar{w}_i \in C_i,\ i = 1, \ldots, p$ and $\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}) = 0$. Since $\sum_{j\in I} \bar{\mu}_j h_j(\bar{x}) + \sum_{j\in J} \bar{\mu}_j h_j(\bar{x}) = 0$ and $\bar{x}$ is a weakly efficient solution of (MFP), we can obtain $\sum_{j\in J} \bar{\mu}_j h_j(\bar{x}) \geqq 0$. Thus $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a feasible for $(\text{MFD})_G$, $\langle \bar{w}_i, \bar{x}\rangle = s(\bar{x}|C_i),\ i = 1, \ldots, p$. Since $\bar{x}$ is feasible for (MFP), it follows from weak duality that $\frac{f(\bar{x}) + s(\bar{x}|C)}{g(\bar{x})} \not< \frac{f(u) + \langle w, u\rangle}{g(u)} + \sum_{j\in I} \mu_j h_j(u)e$ for any $(\text{MFD})_G$ feasible solution (u, λ, w, μ). Hence $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $(\text{MFD})_G$. □

4 Special Cases

We introduce some special cases in [6] as our duality results.

If $I = M$ and $J = \emptyset$, then $(\text{MFD})_G$ is reduced to the following Mond-Weir type dual problem for (MFP):

$$(\text{MFD})_M \quad \text{Maximize} \quad \left(\frac{f_1(u) + \langle w_1, u\rangle}{g_1(u)}, \ldots, \frac{f_p(u) + \langle w_p, u\rangle}{g_p(u)}\right)$$

$$\text{subject to} \quad \sum_{i=1}^{p} \lambda_i \nabla\left(\frac{f_i(u) + \langle w_i, u\rangle}{g_i(u)}\right) + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0,$$

$$\sum_{j=1}^{m} \mu_j h_j(u) \geqq 0,$$

$$w_i \in C_i,\ i = 1, \ldots, p,$$
$$(\mu_1, \ldots, \mu_m) \geqq 0,\ \lambda = (\lambda_1, \ldots, \lambda_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathrm{R}^p : \lambda \geqq 0, \lambda^T e = 1, e = (1, \ldots, 1) \in \mathrm{R}^p\}$.

Theorem 4.1. Weak Duality Let $x \in S$ be a feasible for (MFP) and (u, λ, w, μ) be a feasible for $(\text{MFD})_M$. Assume that the functions $f(\cdot) + \langle w, \cdot\rangle, -g(\cdot)$ are (V, ρ)-invex functions over S and h is (V, σ)-invex at u with respect to the same η with $\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$ and $\sum_{j=1}^{m} \sigma_j \geqq 0$.
Then the following cannot hold,

$$\frac{f(x) + s(x|C)}{g(x)} < \frac{f(u) + \langle w, u\rangle}{g(u)}.$$

Theorem 4.2. Strong Duality If $\bar{x}$ is a weakly efficient solution of (MFP), and assume that there exists $z^* \in \mathrm{R}^n$ such that $\langle \nabla h_j(\bar{x}), z^*\rangle > 0,\ j \in I(\bar{x})$, then there exists $\bar{\lambda} \in \mathrm{R}^p$, $\bar{\mu} \in \mathrm{R}^m$ and $\bar{w} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is feasible for $(\text{MFD})_M$ and $\langle \bar{w}, \bar{x}\rangle = s(\bar{x}|C)$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $(\text{MFD})_M$.

If $I = \emptyset$ and $J = M$, then $(\text{MFD})_G$ is reduced to the following Wolfe type dual problem for (MFP):

$$(\text{MFD})_W \qquad \text{Maximize} \qquad \left(\frac{f_1(u) + \langle w_1, u \rangle}{g_1(u)} + \sum_{j=1}^{m} \mu_j h_j(u), \ \ldots, \right.$$

$$\left. \frac{f_p(u) + \langle w_p, u \rangle}{g_p(u)} + \sum_{j=1}^{m} \mu_j h_j(u) \right)$$

$$\text{subject to} \qquad \sum_{i=1}^{p} \lambda_i \nabla \Big(\frac{f_i(u) + \langle w_i, u \rangle}{g_i(u)} \Big) + \sum_{j=1}^{m} \mu_j \nabla h_j(u) = 0,$$

$$w_i \in C_i \ i = 1, \ldots, p,$$
$$(\mu_1, \ldots, \mu_m) \geqq 0, \quad \lambda = (\lambda_1, \ldots, \lambda_p) \in \Lambda^+,$$

where $\Lambda^+ = \{\lambda \in \mathrm{R}^p : \lambda \geqq 0, \lambda^T e = 1, e = (1, \ldots, 1) \in \mathrm{R}^p\}$.

Theorem 4.3. Weak Duality Let $x \in S$ be a feasible for (MFP) and (u, λ, w, μ) be a feasible for $(\text{MFD})_W$. Assume that the functions $f(\cdot) + \langle w, \cdot \rangle$, $-g(\cdot)$ and $h(\cdot)$ are (V, ρ)-invex functions over S with respect to the same η with $\sum_{i=1}^{p} \lambda_i \rho_i \geqq 0$.

Then the following cannot hold;

$$\frac{f(x) + s(x|C)}{g(x)} < \frac{f(u) + \langle w, u \rangle}{g(u)} + \sum_{j=1}^{m} \mu_j h_j(u) e.$$

Theorem 4.4. Strong Duality If $\bar{x}$ is a weakly efficient solution of (MFP), and assume that there exists $z^* \in \mathrm{R}^n$ such that $\langle \nabla h_j(\bar{x}), z^* \rangle > 0$, $j \in I(\bar{x})$, then there exists $\bar{\lambda} \in \mathrm{R}^p$, $\bar{\mu} \in \mathrm{R}^m$ and $\bar{w} \in C$ such that $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is feasible for $(\text{MFD})_W$ and $\langle \bar{w}, \bar{x} \rangle = s(\bar{x}|C)$. Moreover, if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{w}, \bar{\mu})$ is a weakly efficient solution of $(\text{MFD})_W$.

5 Conclusions

We introduce a class of nondifferentiable multiobjective fractional programming problem (MFP) with (V,ρ)-invexity. We present the concept of (V,ρ)-invexity for vector valued functions and give Fritz John and Kuhn-Tucker necessary, sufficient optimality conditions for weakly efficient solutions of our problem, in which each component of the objective function contains a term involving the support function of a compact convex set.

Also we formulate a general dual problem $(\text{MFD})_G$ to the primal problem (MFP) and prove the weak and strong duality theorems. Furthermore, we obtain some special cases of our duality results. Our results may serve as a framework for further reserch in this growing area of multiobjective fractional programming problems.

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