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Invariant sets associated with critical orbits for holomorphic endomorphisms of $\mathbb{P}^2$

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This note is the abstract of my talk in the conference held at RIMS, 20-24 June 2005. The study concerns the dynamics of rational self-maps of $\mathbb{P}^2$ (the complex projective plane), mainly focusing on the case of holomorphic maps. We proceed on the basis of [U] and refer to [S] for the general theory. A forthcoming paper [M2] will contain more details.

1 Steinness of Fatou sets

Let $f$ be a rational self-map of $\mathbb{P}^2$ of degree at least 2 which is dominant, i.e. $f(\mathbb{P}^2) = \mathbb{P}^2$. Denote by $I(f)$ the set of indeterminacy points for $f$, i.e. the set of points where $f$ cannot extend to be holomorphic. The set $I(f)$ is a finite set. In case of dimension 1, every rational map has no indeterminacy points, so, when we extend the Fatou-Julia theory to a higher dimensional setting, we must consider how we deal with $I(f)$. The following regularity condition was introduced by Fornaess-Sibony.

Definition 1.1. ([S]) We say that $f$ is algebraically stable (AS) if for all $n \geq 1$, the set $f^{-1}(I(f^n))$ contains no compact complex curve in $\mathbb{P}^2$. This is equivalent to $\deg(f^n) = (\deg(f))^n$ for all $n \geq 1$.

In the sequel, we suppose that $f$ is AS. For any $m \geq 1$, the map $f^m$ is holomorphic in $\mathbb{P}^2 \setminus \bigcup_{n \geq 1} I(f^n)$. From a dynamical viewpoint, we can define the Fatou set to be the set of Lyapunov stable points.

Definition 1.2. We define the Fatou set $\mathcal{F}$ to be the maximal open subset of $\mathbb{P}^2 \setminus \bigcup_{n \geq 1} I(f^n)$ in which $\{f^n\}_{n \geq 1}$ is locally equicontinuous. A connected component of $\mathcal{F}$ is called a Fatou component. The complement $\mathcal{J}$ of $\mathcal{F}$ is called the Julia set.
On the other hand, from a viewpoint of complex analysis, we can define Fatou sets using several notions of convergence for a sequence of meromorphic maps.

**Definition 1.3.** Let \( \{g_n\}_{n \geq 1} \) be a sequence of meromorphic maps from an open set \( D \subset \mathbb{P}^2 \) to \( \mathbb{P}^2 \). Let \( \Gamma_n \subset D \times \mathbb{P}^2 \) denote the graph of \( g_n \). Let \( g : D \rightarrow \mathbb{P}^2 \) be a meromorphic map and \( \Gamma \subset D \times \mathbb{P}^2 \) be the graph of \( g \).

(i) We say that \( \{g_n\}_{n \geq 1} \) **strongly converges** to \( g \) in \( D \) if for any compact set \( K \subset D \)

\[
\lim_{n \to \infty} \Gamma_n \cap (K \times \mathbb{P}^2) = \Gamma \cap (K \times \mathbb{P}^2)
\]

with respect to the Hausdorff metric.

(ii) We say that \( \{g_n\}_{n \geq 1} \) **weakly converges** to \( g \) in \( D \) if there is an analytic subset \( A \subset D \) of \( \text{codim}_\mathbb{C} A \geq 2 \) such that \( \{g_n\}_{n \geq 1} \) strongly converges to \( g \) in \( D \setminus A \).

By (i) and (ii) above, we may introduce notions of normality for a sequence of meromorphic maps in strong and weak senses. Thus, in case of the iterates \( \{f^n\}_{n \geq 1} \), we define the strong (resp. weak) Fatou set \( \mathcal{F}_s \) (resp. \( \mathcal{F}_w \)) as the maximal open subset of \( \mathbb{P}^2 \) in which \( \{f^n\}_{n \geq 1} \) is strongly (resp. weakly) normal.

By definition, it follows that \( \mathcal{F} \subset \mathcal{F}_s \subset \mathcal{F}_w \). By combining Ivashkovich's results on the convergence of meromorphic maps to a compact Kähler manifold and Sibony's results on Green currents, the following theorem is verified.

**Theorem A.** If \( f \) is a dominant AS rational self-map of \( \mathbb{P}^2 \) of degree at least 2,

\[
\mathcal{F} = \mathcal{F}_s = \mathcal{F}_w.
\]

In particular, each Fatou component is Stein, hence, the Julia set \( \mathcal{J} \) is connected.

Concerning the dynamics inside Fatou sets, we can find an interesting dynamical phenomenon which is related with indeterminacy points ([M1]).

## 2 Critically hyperbolic maps

Suppose that \( f \) is a holomorphic self-map of \( \mathbb{P}^2 \) of degree \( d \geq 2 \). Then, \( f \) is a \( d^2 \) to 1 branched covering. We denote by \( C = C(f) \) the critical set for \( f \). We define the critical limit set \( E = E(f) \) by

\[
E := \bigcap_{j \geq 1} \bigcup_{i \geq j} f^i(C).
\]
We denote the Green (1,1) current for $f$ by $T$. Since $f$ is holomorphic, it follows that
\[ J = J_1 := \text{supp}(T). \]
Further, it is known that $T \wedge T$ is a unique invariant probability measure of maximal entropy. We set $J_2 := \text{supp}(T \wedge T)$.

Throughout this section, we consider a set $\Lambda = \Lambda(f)$ defined by
\[ \Lambda := \bigcap_{n \geq 0} f^n(J_1 \cap E \cap \Omega), \]
where $\Omega$ is the nonwandering set for $f$. Since $\mathcal{F}$ is Stein, the critical set $C$ always intersects $J_1$. This implies that $\Lambda$ is nonempty.

**Proposition 2.1.** The set $\Lambda$ is a nonempty compact set such that $f(\Lambda) = \Lambda$. All saddle periodic points for $f$ are contained in $\Lambda$.

We consider the situation in which $f$ is hyperbolic on $\Lambda$. (Concerning hyperbolic sets for non-invertible maps, see [BJ] for instance.) We are going to study the global dynamics assuming some condition on the critical orbit. Critically finite maps have been studied by several authors (Fornæss-Sibony, Ueda, Jonsson, de Thelin, ...), so here we introduce a new condition. Let $\hat{\Lambda}$ denote the space of histories of points in $\Lambda$ for $f|_A : \Lambda \to \Lambda$.

**Definition 2.2.** We say that $f$ is critically hyperbolic if $\Lambda$ is a hyperbolic set for $f$ and $\hat{\Lambda}$ has local product structure.

We find examples of critically hyperbolic maps in the class of Axiom A. In case when $f$ satisfies Axiom A, we denote by
\[ \Omega = S_0 \cup S_1 \cup S_2 \]
the decomposition of the nonwandering set $\Omega$ for $f$ according to the unstable dimensions.

**Proposition 2.3.** Let $f$ be a holomorphic self-map of $\mathbb{P}^2$ of degree at least 2. If $f$ satisfies Axiom A and $f^{-1}(S_2) = S_2$, then $f$ is a critically hyperbolic map such that
\[ S_1 = \Lambda, \ S_2 = J_2. \]

**Remark 2.4.** If $f$ is a direct product of two hyperbolic polynomials in one variable, then $f$ and its perturbed maps satisfy this condition.
For critically hyperbolic maps, we can establish the following theorems.

Theorem B says that each Fatou component is eventually mapped to the immediate basin of an attracting periodic orbit and the number of attracting periodic orbits is finite.

**Theorem B.** Suppose $f$ is critically hyperbolic. Then, the Fatou set $\mathcal{F}$ for $f$ consists of the basins of attraction for finitely many attracting periodic orbits.

For a history $\hat{p} \in \hat{\Lambda}$, we denote the unstable manifold by $W^u(\hat{p})$. Then, the critical limit set $E$ can be described as follows.

**Theorem C.** Suppose that $f$ is critically hyperbolic and $\Lambda$ has pure unstable dimension 1. Then,

$$E = \{\text{attracting periodic points}\} \cup \bigcup_{\hat{p} \in \hat{\Lambda}} W^u(\hat{p}).$$

The dynamics inside the Julia sets for critically hyperbolic maps will be investigated in a future article.

**References**


[S] N.Sibony, Dynamique des applications rationnelles de $\mathbb{P}^k$ *Panor.Syntheses*, 8, 1999, 97-185