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Hénon的アトラクターに対する吸引領域の問題

高橋博樹*

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Abstract

The basin problem for a strange attractor asks the asymptotic distribution of Lebesgue almost every initial point in the basin of attraction. A solution to this problem for Hénon-like attractors was given by Benedicks-Viana, and later by Wang-Young, under certain conditions of the Jacobian of the map, which are used in a crucial way to control the volume growth under iteration. The purpose of this paper is to remove the assumption of the Jacobian in their solutions, in a hope that the argument can be extended to a broader class of Hénon-like maps which are not necessarily invertible and possess singularities.

1 Introduction

[1] In [10], Mora-Viana isolated a class of parameter families of diffeomorphisms which they call *Hénon-like*, as an abstract model of the renormarization in generic one-parameter families of surface diffeomorphisms unfolding homoclinic tangencies associated with dissipative saddles [11]. Recall that the Hénon-like family $(H_{a,b})$ is a two parameter family of planar diffeomorphisms such that

1. $(a,b,x,y) \to H_{a,b}(x,y)$ is continuous and $(a,x,y) \to H_{a,b}(x,y)$ is C^3 for any b.
2. there exists a constant J independent of b such that
   (a) $H_{a,b}$ has the following form:
   $$H_{a,b}(x,y) = (1-ax^2, 0) + R(x,y,a,b), \;\; \|R(x,y,a,b)\|_{C^3} \leq J\sqrt{b}.$$
   (b) for any (a,b), $b \neq 0$,
   $$J^{-1}b \leq |\det DH_{a,b}| \leq Jb \text{ and } \|D\log|\det DH_{a,b}|\| \leq J.$$

*Department of Mathematics, Kyoto University, takahasi@math.kyoto-u.ac.jp
[1]本稿は, [12] の冒頭部分の引用です.

They proved the abundance of strange attractors in this family around parameter values close to $(2,0)$, by extending the pioneering work of Benedicks-Carleson [2]. For this type of attractors Benedicks-Viana [3][2] solved the *basin problem*, that is, the asymptotic distribution of Lebesgue almost every initial point in the basin of attraction coincides with the ergodic SRB measure, which is proved to exist by Benedicks-Young [4] [5]. In their argument on the basin problem, the assumption (b) which we call *the homogeneity assumption* is used at two crucial metric estimates: deducing that *unstable sides are roughly parallel*, and obtaining *area distortion bounds* which stay bounded as b tends to zero. The comprehensive paper of Wang-Young [20] on strange attractors also contains another solution to the basin problem in a similar but not the same context assuming a similar condition on Jacobians for the same purpose. We remark that all they actually need is that the condition (b) holds in a small neighborhood in which strange attractors potentially exist, i.e. in a neighborhood of the set $\{(x,0):|x|\leq 1\}$.

Our ultimate goal is to generalize these results on the basin problem [3] [20] to cases for non-invertible maps possessing singularities which deny the homogeneity assumption. This paper is an impetus to this goal; namely, we solve the basin problem for "Hénon-like attractors" generated by planar diffeomorphisms without relying on the homogeneity assumption. We do this in a hope that our argument can be combined with further parameter exclusions and be extended to cases where fold singularities are present. The author is currently working on this subject by using Tsujii's reconstruction of the Benedicks-Carleson theory [16].

One may ask whether families of diffeomorphisms which do not satisfy the homogeneity assumption are naturally embedded in bifurcation mechanisms of dynamics. In a separate paper [13] we shall prove that such families bifurcate through critical saddle-node cycles [6].

1.1 The family.

Throughout this paper we consider a two parameter family of planar diffeomorphisms of the following form:

$$F_{a,b}\colon \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} G(x,y,a)+bu(x,y,a,b) \\ bv(x,y,a,b) \end{pmatrix},$$

where $(a,x,y)\to u(x,y,a,b)$, $v(x,y,a,b)$, $G(x,y,a)$ are C^3 with bounded C^3 norms for any b. Letting $g_a=G(x,0,a)$ we assume that g_a is a unimodal map defined on $[-1,1]$. By this we mean g_a has a unique critical point $c\in(-1,1)$, $g_a'(x)$ changes its sign at c, and sends the boundary $\{-1,1\}$ into itself. For simplicity assume that the critical point of g_a does not change with parameter and it is 0, and that -1 is a fixed point of g_a. The map g_0 is a preperiodic Misiurewicz map, i.e. there exists a repelling periodic point Q of g_0 and $m\geq 2$ such that $g_0^m(0)=Q$, and all periodic points of g_0 is repelling. Letting $D(a,n)=\frac{d(g_a^n(g_a(0)))}{da}$ we further assume the limit

$$\lim_{n\to\infty}\frac{D(0,n)}{g_0^n(g_0(0))}$$

[2]This paper appeared in 2001 but the result had been announced in 1995. See [19].

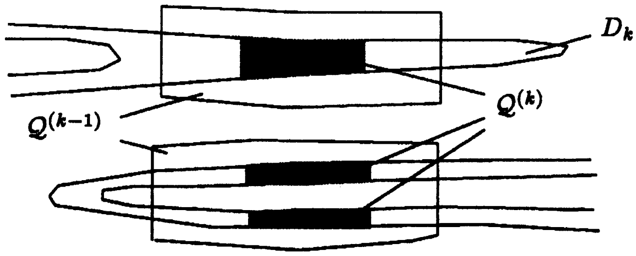

Figure 1: the geometry of the critical set.

which is known to exist [17] is nonzero. This assumption only concerns the parameter exclusion which we do not deal with in this paper. The point in the setting is that nothing particular is assumed on the Jacobians of the family $(F_{a,b})$.

We impose the following non-degeneracy conditions:

$$\partial_x v(0,0,0,0) \cdot g_0''(0) \neq 0. \tag{1}$$

(1) implies that if (a,b) is close to $(0,0)$ and $b \neq 0$, then $F_{a,b}$ maps a short segment in the x-axis containing $(0,0)$ to a curve which is C^2 close to the parabola $x = e \cdot y^2$ $(e \neq 0)$.

Denote by P the repelling fixed point of g_0 which is not $(-1,0)$. We use the same letters P, Q to denote their continuations for $F_{a,b}$ with (a,b) close to $(0,0)$. If there is no fear of confusion, we write $F = F_{a,b}$ and $z_i = F_i(z)$ for $z \in \mathbf{R}^2$ and $i \in \mathbf{Z}$, when it makes sense. We maintain the same convention for an arbitrary set $A \subset \mathbf{R}^2$, i.e. $A_i = F^i(A)$.

The properties of $(F_{a,b})$ imply the existence of an F-forward invariant closed rectangle $D = D(F)$ which contains P, and is bounded by two horizontal lines $\{(x,y) : |y| = 1/10\}$ and two vertical curves contained in $W^s(Q)$. The set D captures an important part of the dynamics of F. Put $\Omega = \bigcap_{n \geq 0} D_n$, where $D_n = F^n(D)$. The forward iterates of the horizontal boundaries of D are called *unstable sides*. The vertical boundaries of D play no role in our argument because they approach the fixed point Q under iteration.

1.2 The critical set of Wang-Young.

Following [20], we present a geometric model called *critical set* which lies at the heart of our argument. For all our purposes, we arrange things in a slightly different way from the original paper [20].

Regarding nonzero positive constants $\alpha_0, \beta_0, \delta_0, \gamma_0, \Delta_0$, we assume for the moment the relations $10\alpha_0 < \beta_0$, $\|g_0\|_{C^3} \leq e^{\Delta_0/2}$, $2.8\alpha_0/\Delta_0 < 1$, $\gamma_0 = \hat{\gamma}_0 - 5\alpha_0$, and $\delta_0 < 1$. The constant $\hat{\gamma}_0$ only depends on g_0, and will be specified later. Fix $\theta_0 > 0$ sufficiently small, say $< 10^{-4}$, depending on g_0. Denote by $C > 0$ any auxiliary constant which appears in many places of our estimates. Keep in mind that the values of C are different in different places.

For two nonzero vectors u and $\tilde{u}$, angle$(u, \tilde{u}) \in [0, \pi/2]$ denotes the smaller angle which they make. Put slope$(v) = \tan \text{angle}(v, \binom{1}{0})$. For a C^1 curve γ and $z \in \gamma$, $t_\gamma(z)$ denotes any unit tangent vector of γ at z. If γ is contained in the unstable sides, we simply write $t(z)$. A nonzero vector v is called *horizontal* if slope$(v) \leq 10\theta_0$ holds. A C^2 curve γ

is called *horizontal* if $\text{slope}(t_\gamma(z)) \leq 10\theta_0$ holds for all $z \in \gamma$, and the curvature of γ is smaller than θ_0^3 everywhere on γ.

1.2.1 Geometry of the critical set.

Fix a small neighborhood $\mathcal{N}$ of $(0,0)$ such that $\|DF_{a,b}\|_{C^3} \leq e^{\Delta_0}$ holds for all $(a,b) \in \mathcal{N}$. Fix $K > 0$ such that $|\det DF_{a,b}(z)| \leq Kb$ holds for all $z \in D$ and $(a,b) \in \mathcal{N}$. The critical set $\mathcal{C} \subset \Omega$ is given by $\mathcal{C} = \bigcap_{k=0}^{\infty} \mathcal{C}^{(k)}$, where $\{\mathcal{C}^{(k)}\}_{k\geq 0}$ is a decreasing sequence called *critical regions* such that;

1. $\mathcal{C}^{(0)} = \{(x,y) \in D : |x| \leq \delta_0\}$.
2. $\mathcal{C}^{(k)}$ is a subset of $D_k = F^k(D)$ and has a finite number of components called $\mathcal{Q}^{(k)}$ each of which is diffeomorphic to a rectangle. The set $\mathcal{Q}^{(k)}$ is bounded by two vertical lines, and by two horizontal curves in the unstable sides of D_k. The Hausdorff distance between the two horizontal curves is $\mathcal{O}(b^{k/4})$, and their projection on the x-axis are intervals with length $\min\{\delta_0, e^{-\beta_0 k}\}$.
3. $\mathcal{C}^{(k)}$ is related to $\mathcal{C}^{(k-1)}$ as follows: $\mathcal{Q}^{(k-1)} \bigcap D_k$ has at most finitely many components. Each of them is bounded by the two vertical boundaries of $\mathcal{Q}^{(k-1)}$, and by two horizontal curves in the unstable sides of D_k. Each component of $\mathcal{Q}^{(k-1)} \bigcap D_k$ contains exactly one component of $\mathcal{C}^{(k)}$. See Figure 1.

1.2.2 Critical points.

Around the mid point of each unstable side of $\mathcal{Q}^{(k)}$, there exists a unique point c such that

$$\|DF_{c_1}^n \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right))\| \geq e^{\gamma_0 n} \text{ and } \|DF_{c_1}^n t(c_1)\| \leq (Kb)^n$$

holds for all $n \geq 0$. The point c is called a *critical point of generation k*. By definition, $\mathcal{Q}^{(k)}$ contains infinitely many critical points. Letting $c = (c_x, c_y)$ be the critical point on the unstable side of $\mathcal{Q}^{(k)}$, we assume the relation $|c_x - c'_x| \leq (Kb/2)^k$ for any critical point $c' = (c'_x, c'_y) \in \mathcal{Q}^{(k)}$.

For $z = (x,y) \in D$, the distance to the critical set $d_{\mathcal{C}}(z)$ is defined as follows: $d_{\mathcal{C}}(z) = |x|$ for $z \notin \mathcal{C}^{(0)}$. Otherwise, letting $k_0 = \max\{k : z \in \mathcal{C}^{(k)}\}$ and $\mathcal{Q}^{(k_0)}$ be the component containing z, $d_{\mathcal{C}}(z)$ is defined to be the minimum of the horizontal distances between z and the two critical points on the unstable sides of $\mathcal{Q}^{(k_0)}$.

1.2.3 Dynamical assumptions.

For the critical set $\mathcal{C}$ we put two assumptions:

$(A1)$ for all critical point c and $n \geq 0$,

$$\sum_{1 \leq j \leq n+1, c_j \in \tilde{\mathcal{C}}^{(0)}} \log d_{\mathcal{C}}(c_j)^{-1} \leq \alpha_0 n,$$

where $\tilde{\mathcal{C}}^{(0)} := \{(x,y) \in D : |x| \leq \delta_0^{2.8\alpha_0/\Delta_0}\}$. Notice the relation $\mathcal{C}^{(0)} \subset \tilde{\mathcal{C}}^{(0)}$.

$(A2)$ For all critical point c and $n \geq 0$, there exists $\chi(n) \in [(1-10\alpha_0)n, n]$ such that

$$\mathrm{slope}(DF_{c_1}^{\chi(n)}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) \leq \theta_0.$$

The assumption $(A1)$ states two things on the orbits of the critical points: they do not come too close to the critical set, and do not enter the region $\tilde{C}^{(0)}$ so frequently. This formulation is inspired by the *bounded recurrence condition* introduced by Luzzatto [8] [3]. He proved that the assumption $(A1)$ in the one-dimensional situation is indeed realized with positive probability in parameter spaces. The reader should also refer to Luzzatto-Viana [9] in which a proof is given for the construction of a positive measure set of parameter values corresponding to the critical set[4] satisfying $(A1)$.

Wang-Young defined the critical set only for those parameters which were selected by the huge inductive parameter exclusion argument. In contrast, we define the critical set explicitly from the beginning, and develop arguments assuming the existence of the critical set.

The assumption $(A1)$ is stronger than the combination of the parameter exclusion rules (BA) and (FA), introduced by Benedicks-Carleson [2]. Wang-Young [20] proved the abundance of parameter values corresponding to the critical set satisfying (BA) and (FA). Thus, the existence, let alone the abundance, of the critical set with $(A1)$, $(A2)$ does not immediately follow from [20]. However, we remark that one can reconstruct arguments of [20] in light of [9], and can show the abundance of parameter values possessing the critical set satisfying $(A1)$. For these selected parameter values the assumption $(A2)$ is necessarily satisfied.

Theorem 1. (Wang-Young [20]) *Let $(F_{a,b})$ be as above. For any α_0 sufficiently small, there exist β_0, δ_0 such that for any $b \neq 0$ sufficiently close to 0, there exists a set of a-values Δ_b with* $\mathrm{Leb}(\Delta_b) > 0$ *such that for any $a \in \Delta_b$, the corresponding $F_{a,b}$ has the critical set $C(\alpha_0, \beta_0, \delta_0)$ satisfying $(A1)$ and $(A2)$, and admits an ergodic SRB measure $\mu_{a,b}$ supported on the closure of the unstable manifold of P.*

1.3 Statement of the result.

We now introduce a constant $\mu_0 := -10^{-2} \cdot \log b$ and the following terminology to state our main theorem. We say $z \in F(C^{(0)})$ is *controlled up to time n* if $d_C(z_j) \geq e^{-3\mu_0 j}$ holds for all $0 \leq j \leq n$. We say $z \in D$ is *eventually controlled* if there exists some n_0 such that $z_{n_0} \in F(C^{(0)})$ and z_{n_0} is controlled all the time.

Main theorem. *Let $(F_{a,b})$ be as above. For any α_0 sufficiently small, there exist β_0, δ_0 such that for any $b \neq 0$ sufficiently close to 0, if $F = F_{a,b}$ has the critical set $C(\alpha_0, \beta_0, \delta_0)$ satisfying* $(A1)$ *and* $(A2)$, *then Lebesgue almost every initial point $z \in D$ is eventually controlled. In particular,*

$$\limsup_{n\to\infty} \frac{1}{n} \log \|DF^n(z)\| \geq \frac{\gamma_0}{3}$$

[3] A similar condition implicitly appears in [14] [15]

[4] By this we mean the geometric structure in dynamical space which is constructed in [9]. The term "critical set" is not used there, so we have slightly abused a language.

holds for Lebesgue almost every $z \in D$.

Three remarks: the lower estimate of the upper Lyapunov exponent directly follows from Corollary 4.1. The main theorem should be understood in conjunction with Wang-Young's theorem to be explained in the next paragraph. The author suspects that extending the main theorem to higher dimensions [18] [21] presents a serious difficulty.

We say $z \in D$ is generic with respect to a probability measure μ if the asymptotic distribution of the orbit of z exists and coincides with μ, i.e. $\lim_{n\to\infty} n^{-1}\sum_{i=0}^{n-1}\delta_{z_i} = \mu$ holds. We claim that Wang-Young's theorem and the main theorem together imply that for any $(F_{a,b})$ as above and any (a,b) such that $a \in \Delta_b$, Lebesgue almost every initial point of D is generic with respect to the SRB measure $\mu_{a,b}$. Following [20], let us explain why this is so. We begin with taking a small horizontal curve of length $\sim \delta_0/(-\log\delta_0)^2$, denoted by Δ_+, located near one of the vertical boundaries of $\mathcal{C}^{(0)}$. Imitating the parameter exclusion argument in one-dimensional systems [1] [2], we construct a positive measure subset $\tilde\Delta_+$ of Δ_+ such that all points of $F(\tilde\Delta_+)$ is controlled all the time. We do the same thing with respect to Δ_- and construct $\tilde\Delta_-$, where Δ_- is also a small horizontal curve of the same length as Δ_+, located near the other vertical boundary of $\mathcal{C}^{(0)}$. From a point which is controlled all the time emanates a stable leaf; by this we roughly mean a sufficiently long C^1 vertical curve such that any two points lying on it are future asymptotic to each other. The collection of the stable leaves through $F(\tilde\Delta_-\cup\tilde\Delta_+)$ forms a lamination with absolutely continuous holonomies. We denote by $\mathcal{H}$ its pull back by F. The leaves of $\mathcal{H}$ are still horizontal, since $\Delta_+\cup\Delta_-$ is near the vertical boundaries of $\mathcal{C}^{(0)}$, and in particular they pass through the closure of $W^u(P)$. Suppose that there exists a positive Lebesgue measure set $B \subset D$ such that any point of B is not generic with respect to $\mu_{a,b}$. The SRB property of $\mu_{a,b}$, the Birkoff ergodic theorem, the absolute continuity of the holonomies along $\mathcal{H}$ altogether imply that the set $\{z \in B\colon \exists n \geq 0 \text{ s.t. } z_n \in \mathcal{H}\}$ has zero Lebesgue measure. Let $Y^{(i)}$ be the set of points $z \in D$ such that z_i is controlled all the time. According to the main theorem, there exists some i_0 such that $Y^{(i_0)}\bigcap B$ has positive Lebesgue measure. Let $\varepsilon > 0$ be an arbitrarily small number. By the Fubini theorem and the Lebesgue density theorem, one can take a horizontal curve γ in a way that $|\gamma\bigcap Y_{i_0}^{(i_0)}\bigcap B_{i_0}|_\gamma > 1-\varepsilon$ holds, where $|\cdot|_\gamma$ is the normalized arc length measure on γ. Let $\hat\Delta_+$ (resp. $\hat\Delta_-$) be a horizontal curve containing Δ_+ (resp. Δ_+) and extending to its both sides with length $\sim \delta_0/(-\log\delta_0)^2$. Define a return time function $R\colon \gamma\bigcap Y_{i_0}^{(i_0)}\bigcap B_{i_0} \to (0,\infty]$ in the following way; $R(z)$ is the first moment at which there exists a neighborhood V_z of z in γ such that $p_x(V_z) \supset p_x(\hat\Delta_+)$ or $p_x(V_z) \supset p_x(\hat\Delta_-)$ holds, where $p_x(x,y) = x$. Define $R(z) = \infty$ if no such $R(z)$ exists. By the main theorem, there exists a countable union of horizontal curves denoted by $\tilde\gamma$ such that $\tilde\gamma \subset \gamma$, $\tilde\gamma \supset \gamma\bigcap Y_{i_0}^{(i_0)}\bigcap B_{i_0}$, and R is well-defined on $\tilde\gamma$. The *return time estimate* of [2] or [5], including distortion estimates shows that the value of R is in fact finite for Lebesgue almost every $z \in \tilde\gamma$. Define a return map $T\colon \tilde\gamma \to \mathbf{R}^2$ by $T(z) = F^{R(z)}(z)$. By definition, $T(z)$ has a Markov-like structure with countably many branches with bounded distortions. Thus we obtain $|\{z \in \tilde\gamma\bigcap Y_{i_0}^{(i_0)}\bigcap B_{i_0} : \exists n \geq 0 \text{ s.t. } z_n \in \mathcal{H}\}|_{\tilde\gamma} \geq \min\{|\tilde\Delta_+|_{\Delta_+}, |\tilde\Delta_-|_{\Delta_-}\}/2$. Since the measure $|\tilde\Delta_\pm|_{\Delta_\pm}$ only depends on δ_0 and ε is arbitrary, this yields a contradiction if we choose $\varepsilon < \min\{|\tilde\Delta_+|_{\Delta_+}, |\tilde\Delta_-|_{\Delta_-}\}/2$ from the beginning. We lastly remark that the measure estimate of $\tilde{\Delta}_{\pm}$, and the return time estimate of course require a distortion argument which is not contained in this paper. For details, see [2] [10] [20].

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