数理解析研究所講究録
1485 巻 2006 年 14-21

Second Julia sets of complex dynamical systems in C^2 – computer visualization –

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§0. Introduction

As most researchers in complex dynamical systems theory admit, computer generated pictures representing the Julia set, Madelbrot set, etc, have played a crucial role in its history. In this note, we consider complex dynamical systems in C^2 and try to visualize the so-called the second Julia sets in the case of polynomial or rational endomorphisms in C^2, or in CP^2, and the Julia sets for complex automorphism cases in C^2.

Such objects are fractal objects living in C^2, and hard to visualize. We tried to generate movies of stereo-graphic pictures. We are convinced that such visual understanding of these fractal objects leads us to a new horizon of complex dynamical systems. Precise definition of these objects requires many pages. In this note, we adopt a very naive concept of the Julia sets, without much justifications. We hope that some intuitive picture of these objects might be helpful for the understanding of mathematical precise definitions. We refer the readers to [10] and many articles cited in this survey, especially [7], [8] and to the series of papers [1], [2], [3]. The movies are posted on the following URL.

http://www.math.h.kyoto-u.ac.jp/˜ushiki/SecondJulia/index.html
http://www.math.h.kyoto-u.ac.jp/˜ushiki/HenonJulia/index.html

Readers are invited to visit these pages and download the movie files. Let us start with simple examples where the dynamics is almost one dimensional and relatively easy to understand. [Movie1-1] to [Movie3-12] are on the SecondJulia page. [Movie4-1] to [Movie6-3] are on the HenonJulia page.

§1. Symmetric product map

T.Ueda[11] discovered a simple way of constructing rational maps on the 2-dimensional complex projective space CP^2. The quotient space $(CP^1 \times CP^1)/\sim$ defined by identifying $(x,y) \sim (y,x)$, is isomorphic to CP^2. To each point $(x,y) \in CP^1 \times CP^1$ corresponds a point $[x+y : xy : 1] \in CP^2$. By using this fact, a rational function $f : CP^1 \to CP^1$ defines a rational map $\varphi : CP^2 \to CP^2$.

Let us take the Mandelbrot family $f(z) = z^2 + c$ of quadratic functions as an example. Consider a polynomial map $(x,y) \mapsto (X,Y)$ defined by

$$\begin{cases} X &= x^2 + c \\ Y &= y^2 + c \end{cases}.$$

By setting $u = x+y, v = xy$ and $U = X+Y, V = XY$, we obtain a rational map $(u,v) \mapsto (U,V)$ given by

$$\begin{cases} U &= u^2 - 2v + 2c \\ V &= v^2 + c(u^2 - 2v) + c^2 \end{cases}.$$

This map is called the symmetric product map of $f(z) = z^2 + c$. We denote this map by φ. The structure of Julia sets of such a rational mapping is relatively easy to understand, since the dynamics is almost the direct product of the one dimensional complex dynamical system except the identification $(x,y) \sim (y,x)$. The precise definition of the Julia sets and the second (or higher order) Julia sets requires the notion of currents, since they are "fractal dimensional" objects in C^2, and need to be described not only in terms of measures but also in terms of currents.

In our case of symmetric product map, the Julia sets and the second Julia sets are, roughly speaking, described as follows. Let $J(f)$ denote the usual Julia set of one dimensional complex dynamical system $f : CP^1 \to CP^1$, and let $F(f)$ denote the Fatou set of f. The Fatou set, $F(\varphi)$, of the symmetric product map φ is the direct product $F(f) \times F(f)$ considered as an open subset of CP^2 under the identification $(x,y) \sim (y,x)$. The (first) Julia set, say $J_1(\varphi)$ of the symmetric product map is the complement of the Fatou set. The second Julia set $J_2(\varphi)$ of the symmetric product map is the direct product $J(f) \times J(f)$ of f considered as a closed set of CP^2 under the identification $(x,y) \sim (y,x)$.

If $c = -2$, the Julia set of f is the line segment $[-2,2]$. The second Julia set of φ is isomorphic to a triangle, since it is obtained by folding a square along the diagonal and identifying the symmetric points $(x,y) \sim (y,x)$. In fact this triangular set is embedded in CP^2 via the 2 to 1 correspondence

$u = x + y, v = xy$. [Moviel-1] When the parameter c is varied to -2.2, for example, the Julia set of f becomes disconnected and the second Julia set becomes the symmetric product of Cantor sets.[Moviel-2] If $c = 0$, the Julia set of f is the unit circle. The second Julia set for this case is the Möbius band, since the direct product of unit circles is the real two dimensional torus and its image in CP^2 is obtained by identifying the symmetric points. The diagonal of the torus gives rise to the boundary of the Möbius band.

By varying the parameter c, we observe a series of various second Julia sets.[Moviel-3] In the movie, parameter c is varied from -2.0 to 0.05 linearly. The screen represents the real and imaginary x-coordinate. The real part of y is represented as the depth into the screen. Some rotation in CP^2 is used to view the fourth coordinate.

§2. Symmetric Polynomials

As Julia sets of CP^2 are very complicated and we are not used to describe the fractal object in higher dimensions, we look for simple but non trivial example of complex dynamical systems. Uchimura[12] studied a simple family of complex dynamical system in C^2. His family is defined as follows.

$$\begin{cases} X &= x^2 + cy \\ Y &= y^2 + cx \end{cases}.$$

He calls it a symmetric polynomial endomorphism. The advantage of this map is that it has a symmetry and has invariant line $\{x = y\}$ and periodic lines of period two $\{x = \omega y\}$ and $\{x = \omega^2 y\}$, where ω is a cubic root of the unity. When restricted to these complex lines, the dynamical system reduces to one-dimensional, and we can have some intuitive analogy between one dimensional dynamical systems and two dimensional ones.

When $c = 0$, the dynamical system becomes the direct product of simple maps and its second Julia set is the real two dimensional torus. When $c = -2$, the second Julia set is a hypocycloid homeomorphic to a triangle. It is included in a real 2-dimensional subspace $\{x = \bar{y}\}$. For real parameter value c, this real subspace is invariant under the dynamical system and behaves as a real polynomial map. The hypocycloid, has three cusp points. The dynamics on this subset is 4 to 1, except the critical locus. The cusp point in the diagonal line is a fixed point. and the other two cusp points are mapped to one another forming a periodic cycle of period two. Three critical points are located at the middle point of each edges of the hypocycloid. They are mapped to the cusp points. [Movie2-1] The hypocycloid is folded and mapped onto itself covering 4 times.

When parameter c is varied to $-2-\epsilon$, for small $\epsilon > 0$, the hypocycloid decomposes into a Cantor set. [SecondJulia/movie2-2] When parameter c is varied to $-2+\epsilon$, the second Julia set grows into a fractal set. [Movie2-3] [Movie2-4] In this symmetric polynomial maps, analogy with one dimensional complex dynamical system helps us to recognize its structure.

For parameter value of c which corresponds to the one dimensional map $f(z) = z^2 + i$ having a dendrite as its Julia set, the second Julia set is more complicated. [Movie2-5] This set is conjectured to be a dendrite by A.Kameyama. Second Julia sets for other parameters are rather complicated. Parameter values corresponding to the Douady's rabbit[Movie2-6], superattracting cycle of period two [Movie2-7], period doubling bifurcation [Movie2-8], and saddle-node bifurcation [Movie2-9] are used in the movies.

By varying the parameter, we observe a series of metamorphose of second Julia sets [Movie2-10]. A Cantor set (for parameter $c = -2.2$) is deformed into a various form of second Julia sets and finally approaches to a torus (final value in the movie is $c = -0.2$).

Family of one dimensional complex dynamical system $g(z) = z^2+cz$ is a family of quadratic polynomials. Each system is conjugate to an appropriate quadratic polynomial in the Mandelbrot family. This correspondence is 2 to 1(except at $c = 1$). Clearly, the origin, $z = 0$, is a fixed point of $g(z)$. Quadratic function of the Mandelbrot family has two fixed points (except the case of parabolic fixed point with multiplier 1). Usually, these fixed points are called the alpha and the beta fixed points. Under our correspondence the fixed point corresponding to the origin is the alpha fixed point for $\Re c < 1$ and the beta fixed point for $\Re c > 1$.

The "Mandelbrot set" of this family $g(z)$ contains two copies of the classical Mandelbrot set. These copies are simmetric with respect to $c = 1$, where these two copies intersect in a single point. This "Mandelbrot set" is called the Double Mandelbrot set. In the movies above, all parameters were taken from the left half part of the Double Mandelbrot set. The origin corresponded to the alpha fixed point.

For parameters in the right half part of the Double Mandelbrot set, the origin corresponds to the beta fixed point, which is a repelling fixed point located to the extremity point of Julia set. When $c = 2$, the dynamical system reduces to a simple dynamical system on the diagonal line, where the Julia set is a circle centered at $z = -1$ and passing the origin. Similar circles are found in the periodic line of period two $\{x = \omega y\}$ and $\{x = \omega^2 y\}$. The map is 2 to 1 in these complex lines. Hence each point in these circles has two inverse images in these lines. However, the dynamical system in C^2 is a polynomial map of degree two and each point must have 4 inverse images (counted with multiplicities). These points are not in these invariant lines, and their further inverse images do not belong to these lines neither. The circles in the invariant/periodic lines and its preimages together with the points in the closure of these points form the second Julia set [Movie2-11]. See [12] for detailed analysis of these cases. For some of such cases we refer to the movies. [Movie2-12] to [Movie2-20]

§3. Regular Polynomials

In this section, we try to describe some cases of regular polynomials. Let us consider the following family of polynomial endomorphisms.

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} x^2 & + & ax & + & by \\ y^2 & + & cx & + & dy \end{pmatrix}.$$

This family contains 8 complex parameters. If the 2×2matrix in the right hand side is invertible, this polynomial map extends to the line at infinity and defines a holomorphic map on the complex projective space CP^2. Such maps are called regular polynomial maps. Holomorphic map f defined on the complex projective space CP^2 can be lifted to a homogeneous polynomial map F on C^3(except the origin). The Green's potential function

$$G(z) = \lim_{n\to\infty} \frac{1}{d^n} \log |F^{\circ n}(z)|$$

defines a positive closed $(1,1)$-current T on the projective space such that

$$\pi^* T = dd^c G, \quad G(F(z)) = dG(z), \quad f^* T = dT.$$

The support of the exterior product $T \wedge T$ is called the second Julia set. By the theorem of Fornaess-Sibony and Friedland[4][5][6], for most points in CP^2, the backward images of the point gives an approximation of the maximal entropy measure $T \wedge T$ of the complex dynamical system defined by the holomorphic map.

Our family of regular polynomial maps is proposed since a 2×2matrix can be easily inverted, and the system of two quadratic polynomials can be solved explicitly, so that the backward orbits can be computed by using a computer. Moreover, the critical locus can be explicitly computed. Unfortunately, the family contains too many parameters and we do not have a good strategy to explore the dynamical systems. So, we made several series of movies for this family for randomly chosen parameters. [Movie3-1] to [Movie3-7]

Some families of such maps are also explored. [Movie3-8] to [Movie3-12]

§4. Julia sets of complex Hénon map

We use the following form for the Hénon map.

$$\begin{cases} X = x^2 + c + by \\ Y = x \end{cases}$$

Let $F(x,y) = (X,Y)$ denote the Hénon map defined by the above formula. We follow Hubbard[7] for a geometric characterization of Julia sets for the complex Hénon map. Let

$$K^+ = \{(x,y) \in C^2 | \{F^n(x,y) | n = 1,2,\cdots\} \text{ is bounded in } C^2\}$$

$$K^- = \{(x,y) \in C^2 | \{F^{-n}(x,y) | n = 1,2,\cdots\} \text{ is bounded in } C^2\}$$

denote the set of points whose forward/backward orbit is bounded. And let

$$J = \partial K^+ \cap \partial K^-$$

denote the Julia set of the complex Hénon map. Note that the movies we have here are not based on a rigorous theory. We computed many points which we suppose to be in or near the Julia set.

First, we compute a saddle fixed point and its unstable manifold. The computation procedure is essentially due to Poincaré [9], and extended to higher dimensional dynamical systems in [13], [14]. The unstable manifold of a saddle point is known to be dense in the Julia set. So if we take any point in the unstable manifold, its backward orbit is always bounded since the backward orbit converges to the saddle fixed point. And as there exist points, in any neighborhood of the saddle point, whose backward orbit is unbounded. Hence this point belongs to ∂K^-. Then iterate the point in the unstable manifold by the Hénon map and see if it diverges or not. As there is the so-called Green's function $G^+(x,y)$, which measures the "distance" to K^+. We look for points having very small value of Green's function. If it is non zero and very small, we guess it is near ∂K^+. As it is non zero, there are nearby points in the complement of K^+. And as it is very small, there must be some nearby point in K^+.

Here, the argument is heuristic. In fact, we see this method does not work very well in some cases. However, we chose this method so to have a glance of these fantastic objects. Movies on the WEB page, [Movie4-1] to [Movie4-4] represent some of the Julia set for some parameters.

Especially, [Movie4-1] visualizes the Julia set corresponding the famous complex Hénon attractor. Observe that the Hénon attractor in the real plane is embedded in the Julia set. Observe also that the Julia set is disconnected. [Movie4-5] shows a series of bifurcation by varying the parameter c while b is fixed. [Movie5-1] shows the case where the Julia set is a solenoid, and [Movie5-2] shows the Julia set which I called Phoenix in [15]. [Movie6-1] to [Movie6-3] are for volume preserving cases. We don't quite understand what we see, so far.

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