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Operator Space Theory via Numerical Radius Operator Spaces

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1 Introduction

In this article, we present the fundamental theory of operator spaces due to Ruan [R], Effros and Ruan [ER1], Blecher and Paulsen [BP] from the view point of the numerical radius operator space which is recently introduced in [IN4]. This is a joint work with M. Nagisa (Chiba Univ.). Most of the results related to this note are in [IN2], [IN3], [IN4].

The main ingredient can be described in the following figure.

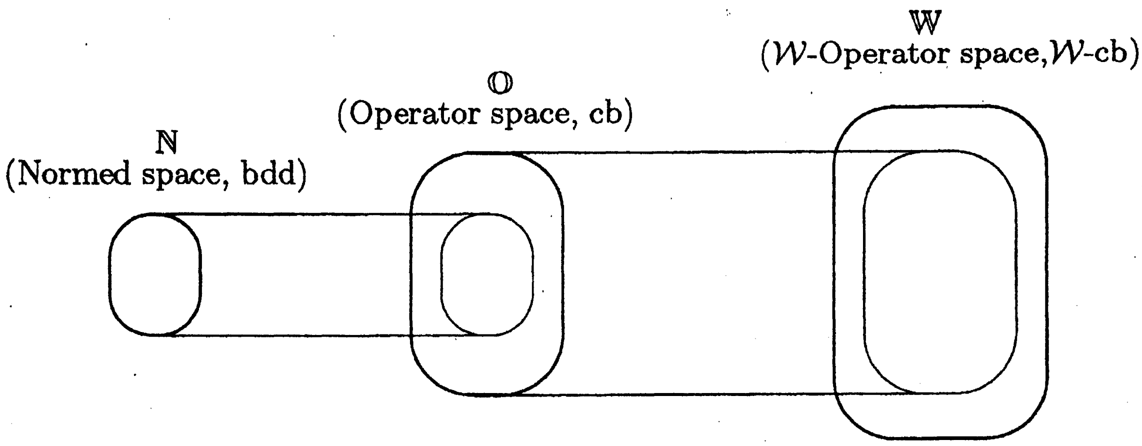

Figure: 1st and "2nd" Quantizations .

Let $\mathbb{N}$ denote the category of normed spaces, in which the objects are the normed spaces and the morphisms are the bounded maps (in short, bdd). We let $\mathbb{O}$ denote the category of operator spaces, in which the objects are the operator spaces and the morphisms are the completely bounded maps (in short, cb). As mentioned in the section 3.3 in [ER3], the category of normed spaces $\mathbb{N}$ is a subcategry of the category of operator spaces $\mathbb{O}$.

We also let $\mathbb{W}$ denote the category of numeical radius norm operator spaces (in short, $\mathcal{W}$-operator space) with the morphisms being the $\mathcal{W}$-completely bounded maps (in short, $\mathcal{W}$-cb). We will obtain a functor

$\mathcal{O}$: $\mathbb{W} \longrightarrow \mathbb{O}$ such that $\mathcal{O}(X) = 2\mathcal{W}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ symbolically. We will also find functors $\mathcal{W} : \mathbb{O} \longrightarrow \mathbb{W}$ which satisfy $\mathcal{O} \circ \mathcal{W}(X) = X$ for each operator space X. In other word, the category of operator spaces $\mathbb{O}$ is a subcategry of the category of numerical radius operator spaces $\mathbb{W}$.

2 Background

Before going to a numerical radius operator space, we will explain the background. Let $\mathcal{H}^n$ be the n-direct sum of a Hilbert space $\mathcal{H}$, and $\mathbb{B}(\mathcal{H}^n)$ the bounded operators on $\mathcal{H}^n$ which is identified with the $n \times n$ matrix space $\mathbb{M}_n(\mathbb{B}(\mathcal{H}))$. Recall that for $x \in \mathbb{B}(\mathcal{H})$, the numerical radius $w(x)$ is defined by $w(x) = \sup\{|(x\xi|\xi)| \mid \|\xi\| = 1, \xi \in \mathcal{H}\}$. We denote by $w_n(x)$ (resp. $\|x\|_n$) the numerical radius (resp. the operator norm) for $x \in \mathbb{M}_n(\mathbb{B}(\mathcal{H}))$. We let α be a bounded linear map from ℓ^1 to ℓ^∞, $\{e_i\}_{i=1}^\infty$ the standard basis of ℓ^1. We regard α as the infinite dimensional matrix $[\alpha_{ij}]$ where $\alpha_{ij} = \langle e_i, \alpha(e_j)\rangle$. The Schur multiplier S_α on $\mathbb{B}(\ell^2)$ is defined by $S_\alpha(x) = \alpha \circ x$ for $x = [x_{ij}] \in \mathbb{B}(\ell^2)$ where $\alpha \circ x$ is the Schur product $[\alpha_{ij}x_{ij}]$. In [IN2], it was shown that

$$\|S_\alpha\|_w = \sup_{x \neq 0} \frac{w(\alpha \circ x)}{w(x)} \leq 1$$

if and only if α has the following factorization $\alpha = a^t b a$ with $\|a\|^2\|b\| \leq 1$:

$$\begin{tikzcd} \ell^1 \arrow[r, "\alpha"] \arrow[d, "a"'] & \ell^\infty \\ \ell^2 \arrow[r, "b"'] & \ell^{2*} \arrow[u, "a^t"'] \end{tikzcd}$$

where a^t is the transposed map of a. This is an extension of Ando-Okubo's Theorem [AO].

Motivated by the above result, we proved a square factorization theorem of a bounded linear map through a pair of column Hilbert spaces $\mathcal{H}_c$ between an operator space and its dual space in [IN3]. More precisely, let us suppose that A is an operator space in $\mathbb{B}(\mathcal{H})$ and $A \otimes A$ is the algebraic tensor product. We defined the numerical radius Haagerup norm $\|u\|_{wh}$ of an element $u \in A \otimes A$ by

$$\|u\|_{wh} = \inf\{\frac{1}{2}\|[x_1, \ldots, x_n, y_1^*, \ldots, y_n^*]\|^2 \mid u = \sum_{i=1}^n x_i \otimes y_i\},$$

where the infimum runs over all representations of u as a finite sum $u = \sum_{i=1}^{n} x_i \otimes y_i$. The (original) Haagerup norm $\|u\|_h$ is definded in [EK] by

$$\|u\|_h = \inf\{\|[x_1,\ldots,x_n]\|\|[y_1,\ldots,y_n]^t\|^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i\},$$

where $[y_1,\ldots,y_n]^t$ is an $n \times 1$ column matrix over A. Let $T : A \longrightarrow A^*$ be a bounded linear map. We showed that $T : A \longrightarrow A^*$ has an extention T'' which factors through a pair of column Hilbert spaces $\mathcal{H}_c$ ($cf.$[ER3]) so that

$$\begin{tikzcd} C^*(A) \arrow[r, "T''"] \arrow[d, "a"'] & C^*(A)^* \\ \mathcal{H}_c \arrow[r, "b"'] & \mathcal{H}_c \arrow[u, "a^*"'] \end{tikzcd}$$

with $\inf\{\|a\|_{cb}^2\|b\|_{cb} \mid T'' = a^*ba\} \leq 1$ if and only if $T \in (A \otimes_{wh} A)^*$ with $\|T\|_{wh^*} \leq 1$ by the natural identification $\langle x, T(y)\rangle = T(x \otimes y)$ for $x, y \in A$.

As a consequence, the above result reads a square factorization of a bounded linear map through a pair of Hilbert spaces from a Banach space X to its dual space X^*.

On the other hand, we also proved in [IN2] that if A is a C^*-algebra on $\mathcal{H}$ and T a completely bounded A-bimodule map from the C^*-algebra of compact operators $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$, then there exist $\alpha = [\alpha_{ij}] \in \mathbb{B}(\ell^2(I))$ and $\{v_i \mid i \in I\} \subset A'$ such that

$$\sup\{w_n(T \otimes I_n(x)) \mid w_n(x) \leq 1, n \in \mathbb{N}\} = \|\alpha\|, \quad \sum_{i\in I} v_i v_i^* \leq 1$$

$$T(x) = \sum_{i,j\in I} v_i \alpha_{ij} x v_j^* \qquad x \in \mathbb{K}(\mathcal{H}).$$

From this point of view, we can define a norm $\|u\|_{wcb}$ for $u \in A \otimes A$ by

$$\|u\|_{wcb} = \inf\{\frac{1}{2}\|[\alpha_{ij}]\|\|[x_1,\ldots,x_n]\|^2\| \mid u = \sum_{i,j=1}^{n} \alpha_{ij} x_i \otimes x_j^*\}$$

where $[\alpha_{ij}]$ is an $n \times n$ complex matrix. Three above norms are mutually equivalent and satisfy the inequality

$$\frac{1}{2}\|u\|_h \leq \|u\|_{wh} \leq \|u\|_{wcb} \leq \|u\|_h$$

for $u \in A \otimes A$, if A is a selfadjoint operator space.

The completion of $A \otimes A$ by $\| \ \|_h$ (we denote it by $A \otimes_h A$) is an operator space by the natural way, but either $A \otimes_{wh} A$ or $A \otimes_{wcb} A$ is not an operator space. However both of $A \otimes_{wh} A$ and $A \otimes_{wcb} A$ have many similar properties of which $A \otimes_h A$ holds. We will show that these three tensor products are typical examples which describe the relation between operator spaces and numerical radius operator spaces in section 5.

3 Definitions

We give the definition of an operator space and a numerical radius operator space now.

Definition 3.1. (Ruan [R]) An (abstract) **operator space** is a complex linear space X together with a sequence of norms $\mathcal{O}_n(\cdot)$ on the $n \times n$ matrix space $\mathbb{M}_n(X)$ for each $n \in \mathbb{N}$, which satisfies the following Ruan's axioms OI, OII:

$$\text{OI.} \qquad \mathcal{O}_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\{\mathcal{O}_m(x), \mathcal{O}_n(y)\},$$

$$\text{OII.} \qquad \mathcal{O}_n(\alpha x \beta) \leq \|\alpha\| \mathcal{O}_m(x) \|\beta\|$$

for all $x \in \mathbb{M}_m(X), y \in \mathbb{M}_n(X)$ and $\alpha \in M_{n,m}(\mathbb{C}), \beta \in M_{m,n}(\mathbb{C})$.

Definition 3.2. (Itoh and Nagisa [IN4]) We call that X is an (abstract) **numerical radius operator space** if a complex linear space X admits a sequence of norms $\mathcal{W}_n(\cdot)$ on the $n \times n$ matrix space $\mathbb{M}_n(X)$ for each $n \in \mathbb{N}$, which satisfies a couple of conditions WI, WII, where WI is the same as OI, however WII is a slightly weaker condition than OII as follows:

$$\text{WI.} \qquad \mathcal{W}_{m+n}\left(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}\right) = \max\{\mathcal{W}_m(x), \mathcal{W}_n(y)\},$$

$$\text{WII.} \qquad \mathcal{W}_n(\alpha x \alpha^*) \leq \|\alpha\|^2 \mathcal{W}_m(x),$$

for all $x \in \mathbb{M}_m(X), y \in \mathbb{M}_n(X)$ and $\alpha \in M_{n,m}(\mathbb{C})$.

Given abstract numerical radius operator spaces (or operator spaces) X, Y and a linear map φ from X to Y, φ_n from $\mathbb{M}_n(X)$ to $\mathbb{M}_n(Y)$ is defined to be

$$\varphi_n([x_{ij}]) = [\varphi(x_{ij})] \qquad \text{for each } [x_{ij}] \in \mathbb{M}_n(X), \ n \in \mathbb{N}.$$

We use the notation $\mathcal{W}(x)$ (resp. $\mathcal{O}(x)$) for the norm of $x = [x_{ij}] \in \mathbb{M}_n(X)$ instead of $\mathcal{W}_n(x)$ (resp. $\mathcal{O}_n(x)$) without confusion. We denote the norm of φ_n by $\mathcal{W}(\varphi_n) = \sup\{\mathcal{W}(\varphi_n(x)) | x = [x_{ij}] \in \mathbb{M}_n(X), \mathcal{W}(x) \leq 1\}$ (resp. $\mathcal{O}(\varphi_n) = \sup\{\mathcal{O}(\varphi_n(x)) | x = [x_{ij}] \in \mathbb{M}_n(X), \mathcal{O}(x) \leq 1\}$. The $\mathcal{W}$-completely bounded norm (resp. completely bounded norm) of φ is defined by

$$\mathcal{W}(\varphi)_{cb} = \sup\{\mathcal{W}(\varphi_n) | n \in \mathbb{N}\}, \quad (\text{resp.} \quad \mathcal{O}(\varphi)_{cb} = \sup\{\mathcal{O}(\varphi_n) | n \in \mathbb{N}\}).$$

We say φ is $\mathcal{W}$-completely bounded (resp. completely bounded) if $\mathcal{W}(\varphi)_{cb} < \infty$ (resp.$\mathcal{O}(\varphi)_{cb} < \infty$). We call φ is a $\mathcal{W}$-complete isometry (resp. complete isometry) if $\mathcal{W}(\varphi_n(x)) = \mathcal{W}(x)$ (resp.$\mathcal{O}(\varphi_n(x)) = \mathcal{O}(x)$) for each $x \in \mathbb{M}_n(X)$, $n \in \mathbb{N}$.

4 Ruan's Theorem and Numerical Radius Operator Spaces

The next is fundamental in numerical radius operator spaces like the Ruan's Theorem in the operator space theory.

Theorem 4.1. *If X is an (abstract) numerical radius operator space with $\mathcal{W}_n$, then there exist a Hilbert space $\mathcal{H}$, a concrete numerical radius operator space $Y \subset \mathbb{B}(\mathcal{H})$ with the numerical radius $w(\cdot)$, and a $\mathcal{W}$-complete isometry Φ from $(X, \mathcal{W}_n)$ onto (Y, w_n).*

Theorem 4.1 leads to the following immediately by using the well-known equality for operators (See Holbrook [H]) between the operator norm and the numerical radius norm so that

$$\frac{1}{2}\|x\| = w\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \qquad \text{for } x \in \mathbb{B}(\mathcal{H}).$$

Corollary 4.2. (Ruan's Theorem [R]) If X is an operator space with $\mathcal{O}_n$, then there exist a Hilbert space $\mathcal{H}$, a concrete operator space $Y \subset \mathbb{B}(\mathcal{H})$, and a complete isometry Ψ from $(X, \mathcal{O}_n)$ onto $(Y, \| \ \|_n)$.

Proof. Since $(X, \mathcal{O}_n)$ is also a numerical radius operator space, we can find a $\mathcal{W}$-complete isometry Φ from $(X, \mathcal{O}_n)$ into $(B(\mathcal{H}), w_n)$ by Theorem 4.1.

We put $\Psi(x) = \frac{1}{2}\Phi(x)$. Then we have for $x \in \mathbb{M}_n(X)$,

$$\begin{aligned}
&\|\Psi_n(x)\|_n \le 2w_n(\Psi_n(x)) = w_n(\Phi_n(x)) \\
=&\mathcal{O}_n(x) = \mathcal{O}_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathcal{O}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) \\
\le&\mathcal{O}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2w_{2n}\left(\begin{bmatrix} 0 & \Psi_n(x) \\ 0 & 0 \end{bmatrix}\right) \\
=&\|\Psi_n(x)\|_n.
\end{aligned}$$

□

Corollary 4.3. *If X is a numerical radius operator space with $\mathcal{W}_n$, then there exist an operator space norm $\mathcal{O}_n$ on X and a complete & $\mathcal{W}$-complete isometry Φ from X into $\mathbb{B}(\mathcal{H})$.*

Proof. For given $\mathcal{W}_n$ and $x \in \mathbb{M}_n(X)$, we define $\mathcal{O}_n$ to be $\mathcal{O}_n(x) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)$. By Theorem 4.1, there exist a $\mathcal{W}$-complete isometry Φ from $(X, \mathcal{W}_n)$ into $(\mathbb{B}(\mathcal{H}), w_n)$. Since

$$\|\Phi_n(x)\|_n = 2w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \mathcal{O}_n(x),$$

Φ is also a complete isometry from $(X, \mathcal{O}_n)$ into $(\mathbb{B}(\mathcal{H}), \|\ \|_n)$.

□

Remark 4.4. We have to prepare a crucial inequality to show the Theorem 4.1. The difference between the condition OII and the condition WII essentially leads to the different inequalities as follows:

(1) Let X be an operator space. If $f \in \mathbb{M}_n(X)^*$ and $\mathcal{O}^*(f) \le 1$, then there exists a state p_0, q_0 on $\mathbb{M}_n(\mathbb{C})$ such that

$$|f(\alpha x \beta)| \le p_0(\alpha\alpha^*)^{\frac{1}{2}} q_0(\beta^*\beta)^{\frac{1}{2}} \mathcal{O}(x),$$

for all $\alpha \in \mathbb{M}_{n,r}(\mathbb{C}), x \in \mathbb{M}_r(X), \beta \in \mathbb{M}_{r,n}(\mathbb{C}), r \in \mathbb{N}$. [ER2]

(2) Let X be a numerical radius operator space. If $f \in \mathbb{M}_n(X)^*$ and $\mathcal{W}^*(f) \le 1$, then there exists a state p_0 on $\mathbb{M}_n(\mathbb{C})$ such that

$$|f(\alpha x \alpha^*)| \le p_0(\alpha\alpha^*)\mathcal{W}(x),$$

for all $\alpha \in \mathbb{M}_{n,r}(\mathbb{C}), x \in \mathbb{M}_r(X), r \in \mathbb{N}$.

As in the case of the operator space theory, we can see the basic operations are closed in numerical radius operator spaces X, Y as well. For $\varphi = [\varphi_{ij}] \in \mathbb{M}_n(\mathcal{W}CB(X,Y))$, we use the identification $\mathbb{M}_n(\mathcal{W}CB(X,Y)) = \mathcal{W}CB(X, \mathbb{M}_n(Y))$ by $\varphi(x) = [\varphi_{ij}(x)]$ for $x \in X$ with the norm $\mathcal{W}(\varphi)_{cb}$. Especially, $\mathbb{M}_n(X^*)$ is identified with $\mathcal{W}CB(X, \mathbb{M}_n(\mathbb{C}))$ where we give the numerical radius $w(\cdot)$ on $\mathbb{M}_n(\mathbb{C})$. If N is a closed subspace of X, we use the identification $\mathbb{M}_n(X/N) = \mathbb{M}_n(X)/\mathbb{M}_n(N)$. Here we state only the fundamental operations.

Proposition 4.5. *Suppose that X and Y are numerical radius operator spaces. Then*

(1) *$\mathcal{W}CB(X,Y)$ is a numerical radius operator space.*

(2) *The canonical inclusion $X \hookrightarrow X^{**}$ is $\mathcal{W}$-completely isometric.*

(3) *If N is a closed subspace of X, then X/N is a numerical radius operator space.*

5 Numerical Radius Norms and Operator Spaces

We note that if X is a numerical radius operator space with $\mathcal{W}_n$, then $\mathcal{W}_n$ induces a canonical operator space norm $\mathcal{O}_n^{\mathcal{W}}$ on X. We define $\mathcal{O}_n^{\mathcal{W}}$ by $\mathcal{O}_n^{\mathcal{W}}(x) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)$ for $x \in \mathbb{M}_n(X)$. By Theorem 4.1, there exists a $\mathcal{W}$-complete isometry Φ from $(X, \mathcal{W}_n)$ into $(\mathbb{B}(\mathcal{H}), w_n)$. Since

$$\|\Phi_n(x)\|_n = 2w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = \mathcal{O}_n^{\mathcal{W}}(x),$$

Φ is also a completely isometry from $(X, \mathcal{O}_n^{\mathcal{W}})$ into $(\mathbb{B}(\mathcal{H}), \| \ \|_n)$.

On the other hand, given an operator space X with $\mathcal{O}_n$, the numerical radius operator space which satisfies the equality

$$\text{(OW)} \qquad \frac{1}{2}\mathcal{O}_n(x) = \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \qquad \text{for } x \in \mathbb{M}_n(X).$$

is not unique (cf. Example 5.4 below). We call that a sequence of norms $\mathcal{W}_n$ is a **numerical radius norm affiliated with** $(X, \mathcal{O}_n)$ if $\mathcal{W}_n$ satisfies WI, WII and (OW).

We often write $\mathcal{W}$(resp. $\mathcal{O}$) instead of $\mathcal{W}_n$ (resp. $\mathcal{O}_n$).

Definition 5.1. We define a norm $\mathcal{W}_{\max}$ on an operator space X by

$$\mathcal{W}_{\max}(x) = \inf \frac{1}{2}\|aa^* + b^*b\| \quad \text{for } x \in \mathbb{M}_n(X),$$

where the infimum is taken over all $a \in \mathbb{M}_{n,r}(\mathbb{C}), y \in \mathbb{M}_r(X), b \in \mathbb{M}_{r,n}(\mathbb{C}), r \in \mathbb{N}$ such that $x = ayb$ and $\mathcal{O}(y) = 1$. We call $\mathcal{W}_{\max}$ is the maximal numerical radius norm affiliated with X.

It is easy to see that, for $x \in \mathbb{M}_n(X)$, we have

$$\mathcal{O}(x) = \inf \|a\|\|b\|$$

where the infimum is taken over all $x = ayb$ as in Definition 3.1. Then it follows that

$$\frac{1}{2}\mathcal{O}(x) \leq \mathcal{W}_{\max}(x) \leq \mathcal{O}(x) \quad \text{for } x \in \mathbb{M}_n(X).$$

Theorem 5.2. *Suppose that X is an operator space. Then $\mathcal{W}_{max}$ is a numerical radius norm affiliated with X and the maximal among all of numerical radius norms affiliated with X.*

Next we set $\mathcal{W}_{\min}(x) = \frac{1}{2}\mathcal{O}(x)$ for $x \in \mathbb{M}_n(X)$. It is clear that $\mathcal{W}_{\min}$ satisfies WI, WII and (OW). We can characterize numerical radius norms affiliated with an operator space X by using $\mathcal{W}_{\min}$ and $\mathcal{W}_{\max}$. We call $\mathcal{W}_{\min}$ is the minimal numerical radius norm affiliated with X.

Corollary 5.3. *Suppose that X is an operator space with $\mathcal{O}_n$, and $\mathcal{W}_n$ satisfies WI, WII. Then the following are equivalent:*

(1) (OW) $\frac{1}{2}\mathcal{O}_n(x) = \mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)$ for $x \in \mathbb{M}_n(X)$,

(2) There exists a complete and $\mathcal{W}$-complete isometry $\Phi : X \longrightarrow \mathbb{B}(\mathcal{H})$,

(3) $\mathcal{W}_{\min}(x) \leq \mathcal{W}(x) \leq \mathcal{W}_{\max}(x)$ for $x \in \mathbb{M}_n(X)$.

Example 5.4. Let X be an operator space. We present that there are uncountably many numerical radius norms affiliated with X.

From Corollary 5.3, there exists a complete and $\mathcal{W}$-complete isometry $\Phi_{\max} : X \longrightarrow \mathbb{B}(\mathcal{H})$ when we introduce the maximal numerical radius norm $\mathcal{W}_{\max}$ on X. Let $0 \le t \le 1$.

(a) We let

$$a_t = \begin{bmatrix} 0 & 1 & & & \\ & 0 & t & & \\ & & \ddots & \ddots & \\ & & & \ddots & t \\ & & & & 0 \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}), \quad n \ge 3.$$

Define that $\Phi_t(x) = \Phi_{\max}(x) \otimes a_t$ for $x \in X$. Since $\|a_t\| = 1$, then $\Phi_t : X \longrightarrow \mathbb{B}(\mathcal{H}) \otimes \mathbb{M}_n(\mathbb{C})$ is complete isometric. Set $\mathcal{W}^{(t)}(x) = w_m([\Phi_t(x_{ij})])$ for $x = [x_{ij}] \in \mathbb{M}_m(X)$. It is clear that $\mathcal{W}^{(t)}$ is a numerical radius norm affiliated with X. We can show that (in case $t = 1$ for $\mathcal{W}^{(t)}$)

$$\mathcal{W}_{\max}(x) \cos \frac{\pi}{n+1} \le \mathcal{W}^{(1)}(x) \le \mathcal{W}_{\max}(x) \quad \text{for } x \in \mathbb{M}_m(X),\ m \in \mathbb{N}. \text{ (cf.[HH])}$$

It turns out that $\mathcal{W}^{(1)}(x)$ is very close to $\mathcal{W}_{\max}(x)$ when n is sufficiently large. We note that $\mathcal{W}^{(0)} = \mathcal{W}_{\min}$ (in case $t = 0$ for $\mathcal{W}^{(t)}$). Since $[0,1] \ni t \longmapsto \mathcal{W}^{(t)}(x) \in \mathbb{C}$ is continuous, then there exist uncountably many distinct numerical radius norms $\mathcal{W}^{(t)}$ affiliated with X.

(b) We let

$$b_t = \begin{bmatrix} 0 & \sqrt{1-t} \\ 0 & \sqrt{t} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

Define that $\Psi_t(x) = \Phi_{\max}(x) \otimes b_t$ for $x \in X$. Set $\mathcal{V}^{(t)}(x) = w_m([\Psi_t(x_{ij})])$ for $x = [x_{ij}] \in \mathbb{M}_m(X)$. Then, by the same argument as a_t, $\{\mathcal{V}^{(t)}\}$ are uncountably many distinct numerical radius norms affiliated with X.

Example 5.5. Let $\mathbb{C}1$ be the one dimensinal operator space. Then for $\alpha = [\alpha_{ij}] \in \mathbb{M}_n(\mathbb{C}1)$, we have

$$\mathcal{W}_{\max}(\alpha) = w(\alpha).$$

Indeed, since $\mathcal{W}_{\max}(\alpha) = w([\alpha_{ij}z])$ for some $z \in \mathbb{B}(\mathcal{K})$ with $\|z\| = 1$, and α double commutes with $\begin{bmatrix} z & & \\ & \ddots & \\ & & z \end{bmatrix}$, we have $\mathcal{W}_{\max}(\alpha) \le w(\alpha)$. This and the maximality of $\mathcal{W}_{\max}$ imply that

$$w(\alpha) = \inf\{\frac{1}{2}\|\beta\beta^* + \gamma^*\gamma\| \mid \alpha = \beta y \gamma, \|y\| = 1,\ \beta, y, \gamma \in \mathbb{M}_n(\mathbb{C})\}.$$

We note that the above equality for $w(\alpha)$ is a special case of Ando's Theorem in [An] in case $\dim \mathcal{H} < \infty$.

In fact, Ando's Theorem [An] implies the next equality in general.

For every $a \in \mathbb{B}(\mathcal{H})$, we have

$$w(a) = \inf\{\frac{1}{2}\|xx^* + y^*y\| \mid a = xby, \|b\| = 1,\ x, b, y \in \mathbb{B}(\mathcal{H})\}. \qquad (*)$$

Moreover the infimum is attained in $(*)$.

Example 5.6. Let X, Y be operator spaces in $\mathbb{B}(\mathcal{H})$. For $x \in \mathbb{M}_{n,r}(X)$ and $y \in \mathbb{M}_{r,n}(Y)$, we denote by $x \odot y$ the element $[\sum_{k=1}^{r} x_{ik} \otimes y_{kj}] \in \mathbb{M}_n(X \otimes Y)$. We note that each element $u \in \mathbb{M}_n(X \otimes Y)$ has a form $x \odot y$ for some $x \in \mathbb{M}_{n,r}(X)$, $y \in \mathbb{M}_{r,n}(Y)$ and $r \in \mathbb{N}$.

(a)

We define

$$\|u\|_{wh} = \inf\{\frac{1}{2}\|xx^* + y^*y\| \ \mid \ u = x \odot y \in \mathbb{M}_n(X \otimes Y)\}$$

for $u \in \mathbb{M}_n(X \otimes Y)$ (cf. [IN3]). Then it is not hard to verify that $\|\ \|_{wh}$ satisfies the conditions WI and WII. Moreover $\|\ \|_{wh}$ is a numerical radius norm affiliated with the Haagerup norm $\|\ \|_h$, that is,

$$\frac{1}{2}\|u\|_h = \left\| \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \right\|_{wh} \qquad \text{for } u \in X \otimes Y.$$

(b)

We let denote $X^\dagger = \{x^* \in \mathbb{B}(\mathcal{H}) \mid x \in X\}$ and also define a norm $\|\ \|_{wcb}$ on $X \otimes X^\dagger$ by

$$\|u\|_{wcb} = \inf\{\frac{1}{2}\|\alpha\|\|x\|^2 \mid u = x\alpha\odot x^* \in \mathbb{M}_n(X\otimes X^\dagger), x \in \mathbb{M}_{n,r}(X), \alpha \in \mathbb{M}_r(\mathbb{C})\}$$

for $u \in \mathbb{M}_n(X \otimes X^\dagger)$ (cf. [Su2], [IN2]).

It is easy to see that $\|\ \|_{wcb}$ also satisfies WI and WII. Since $\|\ \|_{wh}$ has another form [IN3] on $X \otimes X^\dagger$ as

$$\|u\|_{wh} = \inf\{w(a)\|x\|^2 \mid u = xa\odot x^* \in \mathbb{M}_n(X\otimes X^\dagger), x \in \mathbb{M}_{n,r}(X), a \in \mathbb{M}_r(\mathbb{C})\},$$

we have

$$\frac{1}{2}\|u\|_h \leq \|u\|_{wcb} \leq \|u\|_{wh} \leq \|u\|_h \qquad u \in \mathbb{M}_n(X \otimes X^\dagger).$$

Thus it turns out from Corollary 5.3 that $\|\ \|_{wcb}$ is also a numerical radius norm affiliated with the operator space $X \otimes_h X^\dagger$ with the Haagerup norm $\|\ \|_h$, i.e.

$$\frac{1}{2}\|u\|_h = \left\| \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \right\|_{wcb} \qquad \text{for } u \in X \otimes X^\dagger.$$

We denote by $\mathcal{W}(X)$ the numerical radius operator space together with a numerical radius norm $\mathcal{W}$ affiliated with an operator space X. We call $\mathcal{W}(X)$ a numerical radius operator space affiliated with X. Let X, Y be operator spaces. It is clear that if $\varphi : X \longrightarrow Y$ is completely bounded, then $\varphi : \mathcal{W}(X) \longrightarrow \mathcal{W}(Y)$ is $\mathcal{W}$-completely bounded.

We have already obtained a functor $\mathcal{O} : \mathbb{W} \longrightarrow \mathbb{O}$ such that $\mathcal{O}(X) = 2\mathcal{W}\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ symbolically. We have also found functors $\mathcal{W} : \mathbb{O} \longrightarrow \mathbb{W}$ which satisfy $\mathcal{O} \circ \mathcal{W}(X) = X$ for each operator space X. $\mathcal{W}_{\max}$ and $\mathcal{W}_{\min}$ can be seen as the functors which embed $\mathbb{O}$ into $\mathbb{W}$ strictly. This is the reason why we named the figure 1st and "2nd" quantizations in Introduction.

Theorem 5.7. *Let X, Y be operator spaces. If $\varphi : X \longrightarrow Y$ is a linear map, then*

(1) $\mathcal{W}(\varphi : \mathcal{W}_{max}(X) \longrightarrow \mathcal{W}_{max}(Y))_{cb} = \mathcal{O}(\varphi : X \longrightarrow Y)_{cb}$,

(2) $\mathcal{W}(\varphi : \mathcal{W}_{min}(X) \longrightarrow \mathcal{W}_{min}(Y))_{cb} = \mathcal{O}(\varphi : X \longrightarrow Y)_{cb}$.

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