Operator Space Theory via Numerical Radius Operator Spaces

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1 Introduction

In this article, we present the fundamental theory of operator spaces due to Ruan [R], Effros and Ruan [ER1], Blecher and Paulsen [BP] from the view point of the numerical radius operator space which is recently introduced in [IN4]. This is a joint work with M. Nagisa (Chiba Univ.). Most of the results related to this note are in [IN2], [IN3], [IN4].

The main ingredient can be described in the following figure.

Let $\mathcal{N}$ denote the category of normed spaces, in which the objects are the normed spaces and the morphisms are the bounded maps (in short, $\text{bdd}$). We let $\mathcal{O}$ denote the category of operator spaces, in which the objects are the operator spaces and the morphisms are the completely bounded maps (in short, $\text{cb}$). As mentioned in the section 3.3 in [ER3], the category of normed spaces $\mathcal{N}$ is a subcategory of the category of operator spaces $\mathcal{O}$.

We also let $\mathcal{W}$ denote the category of numerical radius norm operator spaces (in short, $\mathcal{W}$-operator space) with the morphisms being the $\mathcal{W}$-completely bounded maps (in short, $\mathcal{W}$-$\text{cb}$). We will obtain a functor

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{1st and "2nd" Quantizations}
\end{figure}
\[ O : \mathcal{W} \to \mathcal{O} \text{ such that } O(X) = 2\mathcal{W} \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \text{ symbolically.} \] We will also find functors \( \mathcal{W} : \mathcal{O} \to \mathcal{W} \) which satisfy \( O \circ \mathcal{W}(X) = X \) for each operator space \( X \). In other word, the category of operator spaces \( \mathcal{O} \) is a subcategory of the category of numerical radius operator spaces \( \mathcal{W} \).

2 Background

Before going to a numerical radius operator space, we will explain the background. Let \( \mathcal{H}^n \) be the \( n \)-direct sum of a Hilbert space \( \mathcal{H} \), and \( \mathcal{B}(\mathcal{H}^n) \) the bounded operators on \( \mathcal{H}^n \) which is identified with the \( n \times n \) matrix space \( M_n(\mathcal{B}(\mathcal{H})) \). Recall that for \( x \in \mathcal{B}(\mathcal{H}) \), the numerical radius \( w(x) \) is defined by \( w(x) = \sup\{|(x\xi|\xi)| \ | \|\xi\| = 1, \xi \in \mathcal{H}\} \). We denote by \( w_n(x) \) (resp. \( \|x\|_n \)) the numerical radius (resp. the operator norm) for \( x \in M_n(\mathcal{B}(\mathcal{H})) \). We let \( a \) be a bounded linear map from \( \ell^1 \) to \( \ell^\infty \), \( \{e_i\}_{i=1}^\infty \) the standard basis of \( \ell^1 \). We regard \( \alpha \) as the infinite dimensional matrix \( [\alpha_{ij}] \) where \( \alpha_{ij} = (e_i, \alpha(e_j)) \). The Schur multiplier \( S_\alpha \) on \( \mathcal{B}(\ell^2) \) is defined by \( S_\alpha(x) = \alpha \circ x \) for \( x = [x_{ij}] \in \mathcal{B}(\ell^2) \) where \( \alpha \circ x \) is the Schur product \( [\alpha_{ij}x_{ij}] \). In [IN2], it was shown that

\[ \|S_\alpha\|_w = \sup_{x \neq 0} \frac{w(\alpha \circ x)}{w(x)} \leq 1 \]

if and only if \( \alpha \) has the following factorization \( \alpha = a^tba \) with \( \|a\|^2\|b\| \leq 1 \):

\[
\begin{array}{ccc}
\ell^1 & \xrightarrow{a} & \ell^\infty \\
\downarrow & & \uparrow a^t \\
\ell^2 & \xrightarrow{b} & \ell^{2*}
\end{array}
\]

where \( a^t \) is the transposed map of \( a \). This is an extension of Ando-Okubo's Theorem [AO].

Motivated by the above result, we proved a square factorization theorem of a bounded linear map through a pair of column Hilbert spaces \( \mathcal{H}_c \) between an operator space and its dual space in [IN3]. More precisely, let us suppose that \( A \) is an operator space in \( \mathcal{B}(\mathcal{H}) \) and \( A \otimes A \) is the algebraic tensor product. We defined the numerical radius Haagerup norm \( \|u\|_{wh} \) of an element \( u \in A \otimes A \) by

\[ \|u\|_{wh} = \inf\left\{ \frac{1}{2} \| [x_1, \ldots, x_n, y_1^*, \ldots, y_n^*] \|^2 \mid u = \sum_{i=1}^n x_i \otimes y_i \right\} , \]
where the infimum runs over all representations of $u$ as a finite sum $u = \sum_{i=1}^{n} x_i \otimes y_i$. The (original) Haagerup norm $\|u\|_h$ is defined in [EK] by

$$\|u\|_h = \inf\{\|[x_1, \ldots, x_n]|||y_1, \ldots, y_n]^t\|^2 \mid u = \sum_{i=1}^{n} x_i \otimes y_i\},$$

where $[y_1, \ldots, y_n]^t$ is an $n \times 1$ column matrix over $A$. Let $T : A \to A^*$ be a bounded linear map. We showed that $T : A \to A^*$ has an extension $T''$ which factors through a pair of column Hilbert spaces $\mathcal{H}_c$ (cf. [ER3]) so that

$$C^*(A) \xrightarrow{T''} C^*(A)^*$$

$$\downarrow a \hspace{1cm} \uparrow a^*$$

$$\mathcal{H}_c \xrightarrow{b} \mathcal{H}_c$$

with $\inf\{\|a\|^2_{cb}\|b\|_{cb} \mid T'' = a^*ba\} \leq 1$ if and only if $T' \in (A \otimes_{wh} A)^*$ with $\|T'\|_{wh}^* \leq 1$ by the natural identification $\langle x, T'(y) \rangle = T'(x \otimes y)$ for $x, y \in A$.

As a consequence, the above result reads a square factorization of a bounded linear map through a pair of Hilbert spaces from a Banach space $X$ to its dual space $X^*$.

On the other hand, we also proved in [IN2] that if $A$ is a $C^*$-algebra on $\mathcal{H}$ and $T$ a completely bounded $A$-bimodule map from the $C^*$-algebra of compact operators $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$, then there exist $\alpha = [\alpha_{ij}] \in \mathbb{B}(\ell^2(I))$ and $\{v_i \mid i \in I\} \subset A'$ such that

$$\sup\{w_n(T' \otimes I_n(x)) \mid w_n(x) \leq 1, n \in \mathbb{N}\} = \|\alpha\|, \quad \sum_{i \in I} v_i v_i^* \leq 1$$

$$T'(x) = \sum_{i,j \in I} v_i \alpha_{ij} x v_j^* \quad x \in \mathbb{K}(\mathcal{H}).$$

From this point of view, we can define a norm $\|u\|_{wcb}$ for $u \in A \otimes A$ by

$$\|u\|_{wcb} = \inf\{\frac{1}{2} \|[\alpha_{ij}]|||[x_1, \ldots, x_n]|^2 \mid u = \sum_{i,j=1}^{n} \alpha_{ij} x_i \otimes x_j^*\}$$

where $[\alpha_{ij}]$ is an $n \times n$ complex matrix. Three above norms are mutually equivalent and satisfy the inequality

$$\frac{1}{2} \|u\|_h \leq \|u\|_{wh} \leq \|u\|_{wcb} \leq \|u\|_h$$

for $u \in A \otimes A$, if $A$ is a selfadjoint operator space.
The completion of $A \otimes A$ by $\| \cdot \|_h$ (we denote it by $A \otimes_h A$) is an operator space by the natural way, but either $A \otimes_{wh} A$ or $A \otimes_{wcb} A$ is not an operator space. However both of $A \otimes_{wh} A$ and $A \otimes_{wcb} A$ have many similar properties of which $A \otimes_h A$ holds. We will show that these three tensor products are typical examples which describe the relation between operator spaces and numerical radius operator spaces in section 5.

3 Definitions

We give the definition of an operator space and a numerical radius operator space now.

**Definition 3.1.** (Ruan [R]) An (abstract) operator space is a complex linear space $X$ together with a sequence of norms $O_n(\cdot)$ on the $n \times n$ matrix space $M_n(X)$ for each $n \in \mathbb{N}$, which satisfies the following Ruan’s axioms OI, OII:

- **OI.** $O_{m+n}(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}) = \max\{O_m(x), O_n(y)\}$,
- **OII.** $O_n(\alpha x \beta) \leq ||\alpha|| O_m(x)||\beta||$

for all $x \in M_m(X), y \in M_n(X)$ and $\alpha \in M_{m,n}(\mathbb{C}), \beta \in M_{m,n}(\mathbb{C})$.

**Definition 3.2.** (Itoh and Nagisa [IN4]) We call that $X$ is an (abstract) numerical radius operator space if a complex linear space $X$ admits a sequence of norms $W_n(\cdot)$ on the $n \times n$ matrix space $M_n(X)$ for each $n \in \mathbb{N}$, which satisfies a couple of conditions WI, WII, where WI is the same as OI, however WII is a slightly weaker condition than OII as follows:

- **WI.** $W_{m+n}(\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}) = \max\{W_m(x), W_n(y)\}$,
- **WII.** $W_n(\alpha x \alpha^*) \leq ||\alpha||^2 W_m(x)$,

for all $x \in M_m(X), y \in M_n(X)$ and $\alpha \in M_{m,n}(\mathbb{C})$.

Given abstract numerical radius operator spaces (or operator spaces) $X$, $Y$ and a linear map $\varphi$ from $X$ to $Y$, $\varphi_n$ from $M_n(X)$ to $M_n(Y)$ is defined to be

$\varphi_n([x_{ij}]) = [\varphi(x_{ij})]$ for each $[x_{ij}] \in M_n(X), n \in \mathbb{N}$. 

We use the notation $\mathcal{W}(x)$ (resp. $\mathcal{O}(x)$) for the norm of $x = [x_{ij}] \in \mathbb{M}_{n}(X)$ instead of $\mathcal{W}_{n}(x)$ (resp. $\mathcal{O}_{n}(x)$) without confusion. We denote the norm of $\varphi_{n}$ by $\mathcal{W}(\varphi_{n}) = \sup\{\mathcal{W}(\varphi_{n}(x))|x = [x_{ij}] \in \mathbb{M}_{n}(X), \mathcal{W}(x) \leq 1\}$ (resp. $\mathcal{O}(\varphi_{n}) = \sup\{\mathcal{O}(\varphi_{n}(x))|x = [x_{ij}] \in \mathbb{M}_{n}(X), \mathcal{O}(x) \leq 1\}$). The $\mathcal{W}$-completely bounded norm (resp. completely bounded norm) of $\varphi$ is defined by

$$\mathcal{W}(\varphi)_{cb} = \sup\{\mathcal{W}(\varphi_{n})|n \in \mathbb{N}\}, \quad (\text{resp. } \mathcal{O}(\varphi)_{cb} = \sup\{\mathcal{O}(\varphi_{n})|n \in \mathbb{N}\}).$$

We say $\varphi$ is $\mathcal{W}$-completely bounded (resp. completely bounded) if $\mathcal{W}(\varphi)_{cb} < \infty$ (resp. $\mathcal{O}(\varphi)_{cb} < \infty$). We call $\varphi$ a $\mathcal{W}$-complete isometry (resp. complete isometry) if $\mathcal{W}(\varphi_{n}(x)) = \mathcal{W}(x)$ (resp. $\mathcal{O}(\varphi_{n}(x)) = \mathcal{O}(x)$) for each $x \in \mathbb{M}_{n}(X), n \in \mathbb{N}$.

4 Ruan's Theorem and Numerical Radius Operator Spaces

The next is fundamental in numerical radius operator spaces like the Ruan's Theorem in the operator space theory.

**Theorem 4.1.** If $X$ is an (abstract) numerical radius operator space with $\mathcal{W}_{n}$, then there exist a Hilbert space $\mathcal{H}$, a concrete numerical radius operator space $Y \subset \mathbb{B}(\mathcal{H})$ with the numerical radius $w(\cdot)$, and a $\mathcal{W}$-complete isometry $\Phi$ from $(X, \mathcal{W}_{n})$ onto $(Y, w_{n})$.

Theorem 4.1 leads to the following immediately by using the well-known equality for operators (See Holbrook [H]) between the operator norm and the numerical radius norm so that

$$\frac{1}{2}\|x\| = w\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) \quad \text{for } x \in \mathbb{B}(\mathcal{H}).$$

**Corollary 4.2.** (Ruan's Theorem [R]) If $X$ is an operator space with $\mathcal{O}_{n}$, then there exist a Hilbert space $\mathcal{H}$, a concrete operator space $Y \subset \mathbb{B}(\mathcal{H})$, and a complete isometry $\Psi$ from $(X, \mathcal{O}_{n})$ onto $(Y, \|\|_{n})$.

**Proof.** Since $(X, \mathcal{O}_{n})$ is also a numerical radius operator space, we can find a $\mathcal{W}$-complete isometry $\Phi$ from $(X, \mathcal{O}_{n})$ into $(B(\mathcal{H}), w_{n})$ by Theorem 4.1.
We put \( \Psi(x) = \frac{1}{2} \Phi(x) \). Then we have for \( x \in \mathbb{M}_n(X) \),

\[
\|\Psi_n(x)\|_n \leq 2w_n(\Psi_n(x)) = w_n(\Phi_n(x))
\]

\[
= O_n(x) = O_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) = O_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right)
\]

\[
\leq O_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2w_{2n}\left(\begin{bmatrix} 0 & \Psi_n(x) \\ 0 & 0 \end{bmatrix}\right)
\]

\[
= \|\Psi_n(x)\|_n.
\]

\[\square\]

**Corollary 4.3.** If \( X \) is a numerical radius operator space with \( \mathcal{W}_n \), then there exist an operator space norm \( O_n \) on \( X \) and a complete & \( \mathcal{W} \)-complete isometry \( \Phi \) from \( X \) into \( B(\mathcal{H}) \).

**Proof.** For given \( \mathcal{W}_n \) and \( x \in \mathbb{M}_n(X) \), we define \( O_n \) to be \( O_n(x) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix}\right) \).

By Theorem 4.1, there exist a \( \mathcal{W} \)-complete isometry \( \Phi \) from \( (X, \mathcal{W}_n) \) into \( (B(\mathcal{H}), \|\|_n) \). Since

\[
\|\Phi_n(x)\|_n = 2w_{2n}\left(\begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix}\right) = 2\mathcal{W}_{2n}\left(\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}\right) = O_n(x),
\]

\( \Phi \) is also a complete isometry from \( (X, O_n) \) into \( (B(\mathcal{H}), \|\|_n) \).

\[\square\]

**Remark 4.4.** We have to prepare a crucial inequality to show the Theorem 4.1. The difference between the condition OII and the condition WII essentially leads to the different inequalities as follows:

1. Let \( X \) be an operator space. If \( f \in \mathbb{M}_n(X)^* \) and \( O^*(f) \leq 1 \), then there exists a state \( p_0, q_0 \) on \( \mathbb{M}_n(\mathbb{C}) \) such that

\[
|f(\alpha x \beta)| \leq p_0(\alpha \alpha^*)^{\frac{1}{2}} q_0(\beta^* \beta)^{\frac{1}{2}} O(x),
\]

for all \( \alpha \in \mathbb{M}_{n,r}(\mathbb{C}), x \in \mathbb{M}_r(X), \beta \in \mathbb{M}_{r,n}(\mathbb{C}), r \in \mathbb{N} \). [ER2]

2. Let \( X \) be a numerical radius operator space. If \( f \in \mathbb{M}_n(X)^* \) and \( \mathcal{W}^*(f) \leq 1 \), then there exists a state \( p_0 \) on \( \mathbb{M}_n(\mathbb{C}) \) such that

\[
|f(\alpha x \alpha^*)| \leq p_0(\alpha \alpha^*) \mathcal{W}(x),
\]

for all \( \alpha \in \mathbb{M}_{n,r}(\mathbb{C}), x \in \mathbb{M}_r(X), r \in \mathbb{N} \).
As in the case of the operator space theory, we can see the basic operations are closed in numerical radius operator spaces $X, Y$ as well. For $\varphi = [\varphi_{ij}] \in M_n(WCB(X, Y))$, we use the identification $M_n(WCB(X, Y)) = WCB(X, M_n(Y))$ by $\varphi(x) = [\varphi_{ij}(x)]$ for $x \in X$ with the norm $W(\varphi)_{cb}$. Especially, $M_n(X^*)$ is identified with $WCB(X, M_n(C))$ where we give the numerical radius $w(\cdot)$ on $M_n(C)$. If $N$ is a closed subspace of $X$, we use the identification $M_n(X/N) = M_n(X)/M_n(N)$. Here we state only the fundamental operations.

**Proposition 4.5.** Suppose that $X$ and $Y$ are numerical radius operator spaces. Then

1. $WCB(X, Y)$ is a numerical radius operator space.
2. The canonical inclusion $X \hookrightarrow X^{**}$ is $W$-completely isometric.
3. If $N$ is a closed subspace of $X$, then $X/N$ is a numerical radius operator space.

## 5 Numerical Radius Norms and Operator Spaces

We note that if $X$ is a numerical radius operator space with $W_n$, then $W_n$ induces a canonical operator space norm $O_n^W$ on $X$. We define $O_n^W$ by $O_n^W(x) = 2W_{2n} \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right)$ for $x \in M_n(X)$. By Theorem 4.1, there exists a $W$-complete isometry $\Phi$ from $(X, W_n)$ into $(B(H), w_n)$. Since

$$||\Phi_n(x)|| = 2w_n \left( \begin{bmatrix} 0 & \Phi_n(x) \\ 0 & 0 \end{bmatrix} \right) = 2W_{2n} \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) = O_n^W(x),$$

$\Phi$ is also a completely isometry from $(X, O_n^W)$ into $(B(H), || ||_n)$.

On the other hand, given an operator space $X$ with $O_n$, the numerical radius operator space which satisfies the equality

$$(OW) \quad \frac{1}{2} O_n(x) = W_{2n} \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right) \quad \text{for } x \in M_n(X).$$

is not unique (cf. Example 5.4 below). We call that a sequence of norms $W_n$ is a **numerical radius norm affiliated with** $(X, O_n)$ if $W_n$ satisfies WI, WII and (OW).

We often write $W$(resp. $O$) instead of $W_n$ (resp. $O_n$).
Definition 5.1. We define a norm $W_{\text{max}}$ on an operator space $X$ by

$$W_{\text{max}}(x) = \inf \frac{1}{2} ||aa^* + b^*b||$$ for $x \in M_n(X),$$

where the infimum is taken over all $a \in M_{n,r}(\mathbb{C}), y \in M_r(X), b \in M_{r,n}(\mathbb{C}), r \in \mathbb{N}$ such that $x = ayb$ and $O(y) = 1.$ We call $W_{\text{max}}$ the maximal numerical radius norm affiliated with $X.$

It is easy to see that, for $x \in M_n(X),$ we have

$$O(x) = \inf \|a\|\|b\|$$

where the infimum is taken over all $x = ayb$ as in Definition 3.1. Then it follows that

$$\frac{1}{2} O(x) \leq W_{\text{max}}(x) \leq O(x) \quad \text{for} \quad x \in M_n(X).$$

Theorem 5.2. Suppose that $X$ is an operator space. Then $W_{\text{max}}$ is a numerical radius norm affiliated with $X$ and the maximal among all of numerical radius norms affiliated with $X.$

Next we set $W_{\text{min}}(x) = \frac{1}{2} O(x)$ for $x \in M_n(X).$ It is clear that $W_{\text{min}}$ satisfies WI, WII and (OW). We can characterize numerical radius norms affiliated with an operator space $X$ by using $W_{\text{min}}$ and $W_{\text{max}}.$ We call $W_{\text{min}}$ is the minimal numerical radius norm affiliated with $X.$

Corollary 5.3. Suppose that $X$ is an operator space with $O_n,$ and $W_n$ satisfies WI, WII. Then the following are equivalent:

1. (OW) $\frac{1}{2} O_n(x) = W_{2n} \left( \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \right)$ for $x \in M_n(X),$

2. There exists a complete and $W$-complete isometry $\Phi : X \longrightarrow B(H),$

3. $W_{\text{min}}(x) \leq W(x) \leq W_{\text{max}}(x)$ for $x \in M_n(X).$
Example 5.4. Let $X$ be an operator space. We present that there are uncountably many numerical radius norms affiliated with $X$.

From Corollary 5.3, there exists a complete and $\mathcal{W}$-complete isometry $\Phi_{\text{max}} : X \rightarrow \mathcal{B}(\mathcal{H})$ when we introduce the maximal numerical radius norm $\mathcal{W}_{\text{max}}$ on $X$. Let $0 \leq t \leq 1$.

(a) We let

$$a_t = \begin{bmatrix} 0 & 1 \\ 0 & t \\ \vdots & \ddots & \ddots \\ \vdots & \ddots & t \\ 0 & \end{bmatrix} \in \mathbb{M}_n(\mathbb{C}), \quad n \geq 3.$$ 

Define that $\Phi_t(x) = \Phi_{\text{max}}(x) \otimes a_t$ for $x \in X$. Since $\|a_t\| = 1$, then $\Phi : X \rightarrow \mathcal{B}(\mathcal{H}) \otimes \mathbb{M}_n(\mathbb{C})$ is complete isometric. Set $\mathcal{W}^{(t)}(x) = w_m([\Phi_t(x_{ij})])$ for $x = [x_{ij}] \in \mathbb{M}_m(X)$. It is clear that $\mathcal{W}^{(t)}$ is a numerical radius norm affiliated with $X$. We can show that (in case $t = 1$ for $\mathcal{W}^{(t)}$)

$$\mathcal{W}_{\text{max}}(x) \cos \frac{\pi}{n+1} \leq \mathcal{W}^{(1)}(x) \leq \mathcal{W}_{\text{max}}(x) \quad \text{for} \quad x \in \mathbb{M}_m(X), \ m \in \mathbb{N}. \ (\text{cf.}[\text{HH}])$$

It turns out that $\mathcal{W}^{(1)}(x)$ is very close to $\mathcal{W}_{\text{max}}(x)$ when $n$ is sufficiently large. We note that $\mathcal{W}^{(0)} = \mathcal{W}_{\text{min}}$ (in case $t = 0$ for $\mathcal{W}^{(t)}$). Since $[0,1] \ni t \mapsto \mathcal{W}^{(t)}(x) \in \mathbb{C}$ is continuous, then there exist uncountably many distinct numerical radius norms $\mathcal{W}^{(t)}$ affiliated with $X$.

(b) We let

$$b_t = \begin{bmatrix} 0 & \sqrt{1-t} \\ 0 & \sqrt{t} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C}).$$

Define that $\Psi_t(x) = \Phi_{\text{max}}(x) \otimes b_t$ for $x \in X$. Set $\mathcal{V}^{(t)}(x) = w_m([\Psi_t(x_{ij})])$ for $x = [x_{ij}] \in \mathbb{M}_m(X)$. Then, by the same argument as $a_t$, \{\mathcal{V}^{(t)}\} are uncountably many distinct numerical radius norms affiliated with $X$.

Example 5.5. Let $\mathbb{C}1$ be the one dimensional operator space. Then for $\alpha = [\alpha_{ij}] \in \mathbb{M}_n(\mathbb{C}1)$, we have

$$\mathcal{W}_{\text{max}}(\alpha) = w(\alpha).$$
Indeed, since $\mathcal{W}_{\max}(\alpha) = w([\alpha_{ij}z])$ for some $z \in \mathcal{B}(\mathcal{K})$ with $\|z\| = 1$, and $\alpha$ double commutes with $\begin{bmatrix} z & & \\ & \ddots & \\ & & z \end{bmatrix}$, we have $\mathcal{W}_{\max}(\alpha) \leq w(\alpha)$. This and the maximality of $\mathcal{W}_{\max}$ imply that

$$w(\alpha) = \inf\left\{ \frac{1}{2}\|\beta\beta^* + \gamma^*\gamma\| \mid \alpha = \beta y\gamma, \|y\| = 1, \beta, y, \gamma \in \mathbb{M}_n(\mathbb{C}) \right\}.$$ 

We note that the above equality for $w(\alpha)$ is a special case of Ando's Theorem in [An] in case $\dim \mathcal{H} < \infty$.

In fact, Ando's Theorem [An] implies the next equality in general.

For every $a \in \mathcal{B}(\mathcal{H})$, we have

$$w(a) = \inf\left\{ \frac{1}{2}\|xx^* + y^*y\| \mid a = xby, \|b\| = 1, x, b, y \in \mathcal{B}(\mathcal{H}) \right\}. \tag{\ast}$$

Moreover the infimum is attained in (\ast).

**Example 5.6.** Let $X, Y$ be operator spaces in $\mathcal{B}(\mathcal{H})$. For $x \in \mathbb{M}_{n,r}(X)$ and $y \in \mathbb{M}_{r,n}(Y)$, we denote by $x \odot y$ the element $\sum_{k=1}^{r} x_{ik} \otimes y_{kj} \in \mathbb{M}_n(X \otimes Y)$. We note that each element $u \in \mathbb{M}_n(X \otimes Y)$ has a form $x \odot y$ for some $x \in \mathbb{M}_{n,r}(X), y \in \mathbb{M}_{r,n}(Y)$ and $r \in \mathbb{N}$.

(a)

We define

$$\|u\|_{wh} = \inf\left\{ \frac{1}{2}\|xx^* + y^*y\| \mid u = x \odot y \in \mathbb{M}_n(X \otimes Y) \right\}$$

for $u \in \mathbb{M}_n(X \otimes Y)$ (cf. [IN3]). Then it is not hard to verify that $\|\|_{wh}$ satisfies the conditions WI and WII. Moreover $\|\|_{wh}$ is a numerical radius norm affiliated with the Haagerup norm $\|\|_h$, that is,

$$\frac{1}{2}\|u\|_h = \left\| \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \right\|_{wh} \text{ for } u \in X \otimes Y.$$ 

(b)
We let denote $X^\dagger = \{x^* \in \mathbb{B}(\mathcal{H}) \mid x \in X\}$ and also define a norm $\| \|_{wcb}$ on $X \otimes X^\dagger$ by

$$
\|u\|_{wcb} = \inf \{ \frac{1}{2} \|\alpha\| \|x\|^2 \mid u = x\alpha x^* \in \mathbb{M}_n(X \otimes X^\dagger), x \in \mathbb{M}_{n,r}(X), \alpha \in \mathbb{M}_r(\mathbb{C}) \}
$$

for $u \in \mathbb{M}_n(X \otimes X^\dagger)$ (cf. [Su2], [IN2]).

It is easy to see that $\| \|_{wcb}$ also satisfies WI and WII. Since $\| \|_{wh}$ has another form [IN3] on $X \otimes X^\dagger$ as

$$
\|u\|_{wh} = \inf \{ w(a)\|x\|^2 \mid u = xax^* \in \mathbb{M}_n(X \otimes X^\dagger), x \in \mathbb{M}_{n,r}(X), a \in \mathbb{M}_r(\mathbb{C}) \},
$$

we have

$$
\frac{1}{2}\|u\|_h \leq \|u\|_{wcb} \leq \|u\|_{wh} \leq \|u\|_h \quad u \in \mathbb{M}_n(X \otimes X^\dagger).
$$

Thus it turns out from Corollary 5.3 that $\| \|_{wcb}$ is also a numerical radius norm affiliated with the operator space $X \otimes_h X^\dagger$ with the Haagerup norm $\| \|_h$, i.e.

$$
\frac{1}{2}\|u\|_h = \| \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \|_{wcb} \quad \text{for} \quad u \in X \otimes X^\dagger.
$$

We denote by $\mathcal{W}(X)$ the numerical radius operator space together with a numerical radius norm $\mathcal{W}$ affiliated with an operator space $X$. We call $\mathcal{W}(X)$ a numerical radius operator space affiliated with $X$. Let $X,Y$ be operator spaces. It is clear that if $\varphi : X \rightarrow Y$ is completely bounded, then $\varphi : \mathcal{W}(X) \rightarrow \mathcal{W}(Y)$ is $\mathcal{W}$-completely bounded.

We have already obtained a functor $\mathcal{O} : \mathcal{W} \rightarrow \mathcal{O}$ such that $\mathcal{O}(X) = 2\mathcal{W} \left( \begin{array}{cc} 0 & X \\ 0 & 0 \end{array} \right)$ symbolically. We have also found functors $\mathcal{W} : \mathcal{O} \rightarrow \mathcal{W}$ which satisfy $\mathcal{O} \circ \mathcal{W}(X) = X$ for each operator space $X$. $\mathcal{W}_{\max}$ and $\mathcal{W}_{\min}$ can be seen as the functors which embed $\mathcal{O}$ into $\mathcal{W}$ strictly. This is the reason why we named the figure 1st and "2nd" quantizations in Introduction.

**Theorem 5.7.** Let $X,Y$ be operator spaces. If $\varphi : X \rightarrow Y$ is a linear map, then

1. $\mathcal{W}(\varphi : \mathcal{W}_{\max}(X) \rightarrow \mathcal{W}_{\max}(Y))_{cb} = \mathcal{O}(\varphi : X \rightarrow Y)_{cb},$
2. $\mathcal{W}(\varphi : \mathcal{W}_{\min}(X) \rightarrow \mathcal{W}_{\min}(Y))_{cb} = \mathcal{O}(\varphi : X \rightarrow Y)_{cb}.$
References


