A MEAN VALUE INEQUALITY FOR PLURISUBHARMONIC FUNCTIONS ON A COMPACT KÄHLER MANIFOLD

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1. Introduction.

Let $(\omega, M)$ be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$, where $\omega$ is a Kähler form in $2\pi c_1(M)$. Let $\varphi$ be a Kähler potential function and $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$ be its corresponding Kähler form on $M$. Since the Ricci form of $\omega_\varphi$ represents the class $2\pi c_1(M)$, then for any nonnegative number $\lambda \leq 1$, there is a uniform smooth function $h_{\lambda, \varphi}$ such that

$$ \begin{cases} 
\text{Ric}(\omega_\varphi) = \lambda \omega_\varphi + (1 - \lambda) \omega + \sqrt{-1} \partial \bar{\partial} h_{\lambda, \varphi} \\
\int_M e^{h_{\lambda, \varphi}} \omega_\varphi^n = \int_M \omega^n = V.
\end{cases} \quad (1.1) $$

When $\lambda = 1$, $h_{\lambda, \varphi} = h_\varphi$ is nothing, just is a Ricci potential function of $\omega_\varphi$. For a smooth function $h$ on $M$, we write

$$ \text{Ric}^h(\omega_\varphi) = \text{Ric}(\omega_\varphi) - \sqrt{-1} \partial \bar{\partial} h $$

as a modified Ricci curvature with respect to $h$. Then (1.1) implies

$$ \text{Ric}^h(\omega_\varphi) \geq \lambda \omega_\varphi. $$

In this note, we shall prove

**Theorem.** Let $M$ be a compact Kähler manifold with positive first Chern class $c_1(M) > 0$. Let $\omega$ and $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$ be two Kähler forms in $2\pi c_1(M)$. Suppose that the function $h_{\lambda, \varphi}$ defined by (1.1) satisfies

$$ |h_{\lambda, \varphi}| \leq A \quad (1.2) $$

for some constant $A$. Then for any $\delta > 0$, there is a uniform constant $C$ depending only on the numbers $\delta, A$ and the Kähler form $\omega$ such that

$$ \text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C (1 + I(\varphi))^{n+\delta}, \quad (1.3) $$

where

$$ I(\varphi) = \frac{1}{V} \int_M \varphi(\omega^n - \omega_\varphi^n). $$


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We note that equation (1.1) is just one studied by S.T. Yau for the Calabi’s problem when $\lambda = 0$ ([Ya]). He proved that the oscillation of $\varphi$ is bounded by the metric $\omega$ and the $C^0$-norm of function $h_{0,\varphi}$. We also note that for some class of functions $h_{\lambda,\varphi}$ inequality (1.3) can be improved as follow,

$$\text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C(1 + I(\varphi)), \quad (1.4)$$

where $C$ is a uniform constant depending only on the number $A$ and the Kähler form $\omega$ ([Ma], [CTZ]). In general we guess that (1.4) is also true. Inequality (1.4) can be regarded as a generalization of the mean value inequality on a compact Riemannian manifold with positive Ricci curvature ([CP], [BM]).

Some applications of inequality (1.3) have been found in the Kähler geometry, such as in the study of uniqueness of Kähler-Ricci solitons ([TZ1]), and in the proof of convergence of Kähler-Ricci flow ([TZ2]), etc. The proof of Theorem depends on a prior $C^0$-estimate for plurisubharmonic functions by using the relative capacity theory which was first found by Klodziej ([Ko]). For a self-containing, we give a brief describing for the relative capacity in the next section.

2. Relative capacity and $C^0$-estimate.

In this section, we will use the relative capacity theory for plurisubharmonic functions to derive a $C^0$-estimate on certain Monge-Ampère equation.

First we recall some notations which can be found in [BT]. For any compact subset $K$ of a strictly pseudoconvex domain $\Omega$ in $\mathbb{C}^n$, its relative capacity in $\Omega$ is defined as

$$\text{cap}(K, \Omega) = \sup \{ \int_K (\sqrt{-1}\partial\bar{\partial}u)^n \mid u \in \text{PSH}(\Omega), -1 \leq u < 0 \},$$

where $\text{PSH}(\Omega)$ denotes the space of plurisubharmonic functions (abbreviated as psh) in the weak sense. For any open set $U \subset \Omega$, we have

$$\text{cap}(U, \Omega) = \sup \{ \text{cap}(K, \Omega) \mid \text{for any compact } K \subset U \}.$$

The extremal function of $K$ relative to $\Omega$ is defined by

$$u_K(z) = \sup \{ u(z) \mid u \in \text{PSH}(\Omega) \cap L^\infty(\Omega), u < 0 \text{ and } u|_{K} \leq -1 \}.$$

One can show that $u_K^*(z) = \lim_{z' \to z} u_K(z')$ is a psh function. It is called the upper semicontinuous regularization of $u_K$. A compact set $K$ is said to be regular if $u_K^* = u_K$.

Here are some properties of $u_K^*$ (cf. [BT], [AT]):

- $u_K^* \in \text{PSH}(\Omega), -1 \leq u_K^* \leq 0, \lim_{z \to \partial \Omega} u_K^* = 0,$
- $(\sqrt{-1}\partial\bar{\partial}u^*)^n = 0 \text{ on } \Omega \setminus K,$
- $u_K^* = -1 \text{ on } K,$ except on a set of relative capacity zero,

moreover, we have

$$\text{cap}(K, \Omega) = \int_{\Omega} (\sqrt{-1}\partial\bar{\partial}u_K^*)^n = \int_K (\sqrt{-1}\partial\bar{\partial}u_K^*)^n. \quad (2.1)$$
Lemma 2.1. Let $\Omega$ be a strictly pseudoconvex domain in $\mathbb{C}^n$ and $u < 0$ be a smooth solution of the following complex Monge-Ampère equation on $\Omega$,

$$\det(u_{i\overline{j}}) = f.$$ 

Suppose that $u$ and $f$ satisfy:

$$u(p) > c \quad (p \in \Omega) \quad \text{and} \quad \int_{K}fdv \leq B \text{cap}(K, \Omega)\frac{\text{cap}(K, \Omega)^{\frac{1}{2}}}{1 + \text{cap}(K, \Omega)^{\frac{1}{2}}}$$ \hspace{1cm} (2.2)

for any compact subset $K$ of $\Omega$, where $B$ is a uniform constant. If the sets

$$U(s) = \{z | u(z) < s\} \cap \Omega'$$

are non-empty and relatively compact in $\Omega' \subset \Omega' \subset \subset \Omega$ for any $s \in [S, S+D]$, where $S$ is some number, then there is a uniform constant $C$, which depends only on $c, D, \delta, \Omega', \Omega$, such that

$$-\inf_{\Omega} u \leq CB^\delta + D.$$ \hspace{1cm} (2.3)

Proof. This lemma is essentially due to [Ko]. For readers' convenience, we will include a proof using an argument from [TZ1]. Put

$$a(s) = \text{cap}(U(s), \Omega) \quad \text{and} \quad b(s) = \int_{U(s)}(\sqrt{-1}\partial\overline{\partial}u)^n.$$ 

Then we define an increasing sequence $s_0, s_1, ..., s_N$ by setting $s_0 = S$ and

$$s_j = \sup\{s | a(s) \leq \lim_{t \to s_{j-1}^+} e a(t)\}$$

for $j = 1, ..., N$, where $N$ is chosen to be the greatest integer such that $s_N \leq S + D$. By using an argument in Lemma 4.1 of [TZ1], we can prove

$$S + D - s_N \leq (Be)^{\frac{1}{2}} a(S + D)^{\frac{1}{2}}.$$ \hspace{1cm} (2.4)

and

$$s_N - S \leq 2(Be)^{\frac{1}{2}} (1 + n\delta) a(S + D)^{\frac{1}{2}}.$$ \hspace{1cm} (2.5)

However, it was proved in [AT] (or Theorem 1.2.11 in [Ko]) that

$$\text{cap}(\{u < s\} \cap \Omega', \Omega) \leq \frac{c'}{|s|},$$

where $c'$ depends only on $c$ and $\Omega'$. It implies that

$$a(S + D) \leq \frac{c'}{|D - S|}.$$ \hspace{1cm} (2.6)
Combining (2.4)-(2.6), we get

$$D \leq 2(2 + n\delta)(Be)^{\frac{1}{n}} \left(\frac{c'}{-D - S}\right)^{\frac{1}{n}}.$$  

It follows

$$-S \leq c' \left(\frac{2(2 + n\delta)}{D}\right)^{n\delta} e^{\delta} B^{\delta} + D,$$

consequently, we have

$$-\inf_{\Omega} u \leq c' \left(\frac{2(1 + n\delta)}{D}\right)^{n\delta} e^{\delta} B^{\delta} + D,$$

so (2.3) is proved. \(\square\)

Lemma 2.2. Let \(\Omega\) be a strictly pseudoconvex domain in \(C^n\) and \(u < 0\) be a smooth solution of the following complex Monge-Ampère equation on \(\Omega\),

$$\det(u_{ij}) = f.$$

Suppose that \(u\) satisfies

$$u(p) > c \ (p \in \Omega).$$

Define \(U(s)\) as in last lemma. If \(U(s)\) are non-empty and relatively compact in \(\Omega''\) for any \(s \in [S, S + D]\) for some \(S\), then for any positive \(\delta \leq \delta_0\) and \(\epsilon \leq \epsilon_0\), there is a uniform constant \(C = C(c, D, \delta_0, \epsilon_0, \Omega', \Omega)\) such that

$$-\inf_{\Omega} u \leq C(\frac{1}{\delta\epsilon})^{n+\delta} \|f\|_{L^{1+\epsilon}(\Omega)}^{\delta} + D.$$

Proof. Let \(u_K\) be the relative extremal function of a regular set \(K\) with respect to \(\Omega\) and \(v = \text{cap}^{-\frac{1}{n}}(K, \Omega) u_K\). Then \(v\) is a psh function and satisfies

$$\int_{\Omega} (\sqrt{-1} \partial \bar{\partial} v)^n = 1, \quad \text{and} \quad \lim_{s \to \partial \Omega} v = 0.$$

By Lemma 2.5.1 in [Ko], we have

$$\lambda(U'(s)) \leq c' \exp\{-2\pi |s|\}$$

for some uniform constant \(c'\) independent of \(v\), where \(\lambda(U'(s))\) is the Lebesgue measure of \(U'(s) = \{v < s\}\). It follows that for any \(q \geq 1\),

$$\int_{\Omega} |v|^q d\mu \leq |\Omega| + \sum_{i=1}^\infty \int_{-s-i \leq v \leq -s} |v|^q d\mu$$

$$\leq |\Omega| + c' \sum_{i=1}^\infty (s + 1)^q e^{-2\pi s}$$

$$\leq |\Omega| + c' e^{4\pi} \int_2^{+\infty} s^q e^{-2\pi s} ds$$

$$\leq C_1 2^{q+2} ([q] + 2)! \leq C_1 2^{q+2} (q + 2)^{q+2}.$$
On the other hand, for any $\epsilon > 0$ we have
\[
\text{cap}(K,\Omega)^{-1}(1 + \text{cap}^{-1/\delta}(K,\Omega)) \int_K f d\mu \\
\leq \int_K |v|^n(1 + |v|^\frac{3}{2}) f d\mu \\
\leq \int_\Omega (|v|^n + |v|^n(1 + \frac{3}{2})) f d\mu \\
\leq [(\int_\Omega |v|^\frac{n(1+\delta)}{\delta} d\mu)^{\frac{1}{1+\delta}} + \int_\Omega |v|^\frac{n(1+\delta)(1+\epsilon)}{\delta\epsilon} d\mu]\|f\|_{L^{1+\epsilon}(\Omega)}.
\]
(2.8)

Combining (2.7) and (2.8), we get
\[
\int_\Omega f d\mu \leq B\text{cap}(K,\Omega)\frac{\text{cap}(K,\Omega)^{1/2}}{1 + \text{cap}(K,\Omega)^{1/2}},
\]
where
\[
B = 2C_1^2 \frac{n(1+\delta)}{\delta \epsilon} + 2\frac{n(1 + \delta)(1 + \epsilon)}{\delta \epsilon} + 2\|f\|_{L^{1+\epsilon}(\Omega)}
\leq C_2\frac{\|f\|_{L^{1+\epsilon}(\Omega)}}{\delta \epsilon}.
\]

Therefore, it follows from Lemma 2.1 that
\[
-\inf_M \varphi \leq C\frac{\|f\|_{L^{1+\epsilon}(\Omega)}}{\delta \epsilon} + D.
\]

Now the lemma follows from replacing $3n\delta$ by $\delta$. $\square$

**Proposition 2.1.** Let $(M, g)$ be a compact Kähler manifold and $\varphi$ be a smooth solution of the following complex Monge-Ampère equation on $M$,
\[
\det(g_{ij} + \varphi_{ij}) = \det(g_{ij})f,
\]
\[
\sup_M \varphi = 0.
\]

Then, for any positive $\delta \leq \delta_0$ and $\epsilon \leq \epsilon_0$, there are two uniform constants $C, C'$ which depending only on $g, \delta_0, \epsilon_0$ such that
\[
-\inf_M \varphi \leq C\frac{\|f\|_{L^{1+\epsilon}(M)}}{\delta \epsilon} + C'.
\]

**Proof.** This is a direct corollary of Lemma 2.2 (cf. the proof of Proposition 4.1 in [TZ1]). We omit its proof. $\square$

3. **Proof of the theorem.**

In this section, we use Proposition 2.1 in Section 2 to prove the theorem in Introduction. Note that by using the maximal principle one can reduce (1.1) to a complex Monge-Ampère equation,
\[
\det(g_{ij} + \varphi_{ij}) = \det(g_{ij})e^{-\lambda \varphi + h_0 - h_\lambda - \varphi},
\]
where $h_0$ is a potential function of the Ricci curvature of the metric $\omega_g = \omega$. 

Proposition 3.1. Let $\varphi$ be a solution of (3.1). Suppose that
\[ |h_{\lambda, \varphi}| \leq A \]
for some constant $A$. Then for any positive $\delta \leq 1$, there are two uniform constants $C = C(A, g, n, \delta)$ and $C' = C'(A, g, n, \delta)$ such that
\[ \text{osc}_M \varphi = \sup_M \varphi - \inf_M \varphi \leq C(I(\varphi))^{n+\delta} + C'. \]

Proposition 3.1 is analogous to Proposition 4.2 in [TZ1] for the complex Monge-Ampère equation which arises from the equation for Kähler-Ricci solitons. We need two lemmas in order to prove Proposition 3.1.

Lemma 3.1 (Poincaré-type inequality). Let $(M, g)$ be a compact Kähler manifold and $h$ be a smooth function on $M$. Suppose that the modified Ricci curvature $Ric^h(\omega_g)$ of $\omega_g$ satisfies
\[ Ric^h(\omega_g) \geq \lambda \omega_g \]
for some number $\lambda > 0$. Let $C^\infty(M, \mathbb{C})$ be the space of complex-valued smooth functions. Then for any $\psi \in C^\infty(M, \mathbb{C})$, we have
\[ \int_M |\overline{\partial} \tilde{\psi}|^2 e^h \omega_g^n \geq \int_M \tilde{\psi}^2 e^h \omega_g^n, \quad (3.2) \]
where
\[ \tilde{\psi} = \psi - \frac{1}{V} \int_M \psi e^h \omega_g^n, \]
and $V = \int_M \omega_g^n$. In particular, for any $\varphi \in C^\infty(M)$, we have
\[ \int_M |\overline{\partial} \varphi|^2 e^h \omega_g^n \geq \int_M \varphi^2 e^h \omega_g^n - \frac{1}{V} (\int_M \varphi e^h \omega_g^n)^2. \quad (3.3) \]

Proof. Let $L$ be the linear differential operator on $C^\infty(M, \mathbb{C})$ defined by
\[ L \psi = \Delta \psi + < \overline{\partial} h, \overline{\partial} \psi >, \quad \text{for} \quad \psi \in C^\infty(M, \mathbb{C}), \]
where $\Delta$ denotes the Laplacian operator of $g$. Then $L$ is elliptic and self-adjoint with respect to the following Hermitian inner product:
\[ (\psi, \psi')_h = \int_M \psi \overline{\psi'} e^h \omega_g^n, \quad \text{for} \quad \psi, \psi' \in C^\infty(M, \mathbb{C}), \]
namely,
\[ (L \psi, \psi')_h = (\psi, L \psi')_h. \]
It follows that all eigenvalues of $L$ are real. Denote by $0 = \lambda_0 < \lambda_1 \leq \ldots \leq \lambda_i \leq \ldots$ the sequence of eigenvalues of $L$ and by $\psi_i$ ($i = 0, 1, 2, \cdots$) the corresponding sequence of
eigenfunctions with the property: $(\psi_i, \psi_j)_h = \delta_{ij}$, for any $i, j$. Note that $\psi_0$ is constant. Then $\{\psi_i\}$ is a complete orthonormal basis of the space $W^{1,2}(M, \mathbb{C})$ with respect to the weighted $L^2$-norm $(\cdot, \cdot)_h$.

Let $\psi$ be one of eigenfunctions of $\lambda_1$, i.e.,

$$\Delta \psi + \langle \overline{\partial}h, \overline{\partial\psi} \rangle = -\lambda_1 \psi.$$

Then integrating by parts and using

$$\lambda_1 \int_M \psi_i \overline{\psi}_i e^h \omega_g^n = -\int_M (\Delta \psi + \langle \overline{\partial}h, \overline{\partial\psi} \rangle + \lambda_1 \psi_i) \overline{\psi}_i e^h \omega_g^n = -\int_M (\psi_{j\overline{j}i} \overline{\psi}_i + \lambda_1 \psi_i) \overline{\psi}_i e^h \omega_g^n + \int_M h_{i\overline{j}} \psi_{ij} \overline{\psi}_i e^h \omega_g^n.$$ 

Thus we prove that $\lambda_1 \geq \lambda$. So (3.2) holds, so does (3.3). \[\square\]

**Lemma 3.2.** Let $(\omega, M)$ be a compact Kähler manifold and $h$ be a smooth function on $M$. Let

$$\omega_{\varphi} = \omega + \sqrt{-1} \partial \overline{\partial} \varphi$$

be a Kähler form associated to a Kähler potential function $\varphi$ so that the modified Ricci curvature $\text{Ric}^h(\omega_{\varphi})$ of $\omega_{\varphi}$ satisfies

$$\text{Ric}^h(\omega_{\varphi}) \geq \lambda \omega_{\varphi}$$

for some constant $\lambda > 0$. Then there are two uniformly $c_0, C > 0$ depending only $\|h\|_{C^0(M)}$ and the metric $\omega$ such that

$$\int_M \exp\{-\frac{c_0 \lambda}{I(\varphi)} (\varphi - \sup_M \varphi)\} \omega_{\varphi}^n \leq C.$$ 

**Proof.** As in [TZ1], we will use an iteration argument to prove this lemma. Without loss of generality, we may assume $I(\varphi) > 1$.

Let $\overline{\varphi} = \varphi - \sup_M \varphi$. Then for any $p > 0$, we have

$$\int_M (-\overline{\varphi})^p (\omega_{\varphi}^n - \omega_{\varphi}^{n-1} \wedge \omega)$$

$$= \frac{\sqrt{-1}}{2\pi} \int_M (-\overline{\varphi})^p \partial \overline{\partial}(\varphi) \wedge \omega_{\varphi}^{n-1}$$

$$= \frac{4p}{n(p+1)^2} \int_M |\overline{\partial}(-\overline{\varphi})|^{n+1} \omega_{\varphi}^n.$$
It follows
\[
\int_M |\overline{\partial}(-\varphi)^{\frac{p+1}{2}}|^2 \omega^n_{\varphi} \leq \frac{n(p+1)^2}{4p} \int_M (-\varphi)^p \omega^n_{\varphi}.
\]

Applying Lemma 3.1 to function \((-\varphi)^{\frac{p+1}{2}}\) in the case of the metric \(\omega_{\varphi}\), we have
\[
\int_M |\overline{\partial}(-\varphi)^{\frac{p+1}{2}}e^h \omega^n_{\varphi} \geq \lambda \int_M (-\varphi)^{p+1}e^h \omega^n_{\varphi} - \frac{1}{V} (\int_M (-\varphi)^{(p+1)/2}e^h \omega^n_{\varphi})^2.
\]

Thus by using the Hölder inequality, we get
\[
\int_M (-\varphi)^{p+1}e^h \omega^n_{\varphi} \
\leq \frac{c}{\lambda} p \int_M (-\varphi)^p \omega^n_{\varphi} \cdot \int_M (-\varphi)e^h \omega^n_{\varphi} + \frac{1}{\lambda V} \int_M (-\varphi)^p \omega^n_{\varphi} \cdot \int_M (-\varphi)e^h \omega^n_{\varphi},
\]

and consequently
\[
\int_M (-\varphi)^{p+1} \omega^n_{\varphi} \leq \frac{c'}{\lambda} [p \int_M (-\varphi)^p \omega^n_{\varphi} + \frac{1}{V} \int_M (-\varphi)^p \omega^n_{\varphi} \cdot \int_M (-\varphi)e^h \omega^n_{\varphi}], \tag{3.4}
\]

where \(c, c'\) are uniform constants.

By the mean-value inequality, we have
\[
\sup_M \varphi \leq V^{-1} \int_M \varphi \omega^n + C.
\]

It follows
\[
\int_M (-\varphi) \omega^n_{\varphi} = V \sup_M \varphi + \int_M (-\varphi) \omega^n_{\varphi} \
\leq \int_M \varphi (\omega^n - \omega^n_{\varphi}) + CV \
\leq a I(\varphi),
\]

where \(a\) is a uniform constant. Thus inserting this inequality into (3.4), we get
\[
\int_M (-\varphi)^{p+1} \omega^n_{\varphi} \leq \frac{ac'}{\lambda} (p + I(\varphi)) \int_M (-\varphi)^p \omega^n_{\varphi}. \tag{3.5}
\]

Iterating (3.5), we have
\[
\int_M (-\varphi)^{p+1} \omega^n_{\varphi} \
\leq 2 (\frac{ac'}{\lambda} I(\varphi))^p (p + 1)! \int_M (-\varphi) \omega^n_{\varphi} \leq \left( \frac{ac'}{\lambda I(\varphi)} \right)^{p+1} (p + 1)!. 
\]
Now choosing $\epsilon < \frac{\lambda}{acI(\varphi)}$, we obtain
\[
\int_{M} \exp(-\epsilon \overline{\varphi}) \omega_{\varphi}^{n} \\
= \sum_{p=0}^{+\infty} \frac{\epsilon^{p}}{p!} \int_{M} (-\overline{\varphi})^{p} \omega_{\varphi}^{n} \\
\leq \sum_{p=0}^{+\infty} \left( \frac{ca'}{\lambda} I(\varphi) \right)^{p} \\
\leq \frac{1}{1 - \frac{ac \epsilon}{\lambda} I(\varphi)}.
\]
Put $c_{0} = \frac{1}{ac}$. Then (3.1) is proved. □

**Proof of Proposition 3.1.** By [Ya], we may assume $\lambda > 0$. Let $\tilde{\varphi} = \varphi - \sup_{M} \varphi$. Then (3.1) becomes
\[
\begin{cases}
\det(g_{i\overline{j}} + \tilde{\varphi}_{i\overline{j}}) = \det(g_{i\overline{j}}) f, \\
\sup_{M} \tilde{\varphi} = 0,
\end{cases}
\quad (3.6)
\]
where $f = e^{h-h_{\lambda,\varphi}-\lambda \varphi}$. Since $h_{\lambda,\varphi}$ is uniformly bounded, we have
\[
0 < c_{1} \leq \int_{M} e^{-\lambda \varphi} \omega_{g}^{n} \leq c_{2} 
\quad (3.7)
\]
for some uniform constants $c_{1}$ and $c_{2}$. This implies
\[
\sup_{M} (\lambda \varphi) \geq -C \quad \text{and} \quad \inf_{M} (\lambda \varphi) \leq C. 
\quad (3.8)
\]

By (3.8) and Lemma 3.2, we have
\[
\int_{M} \exp\{- (1 + \frac{c_{0}}{I(\varphi)}) \lambda \varphi \} \omega_{\varphi}^{n} \\
\leq e^{c_{0}C} \int_{M} \exp\{- \frac{c_{0}}{I(\varphi)} (\lambda \varphi - \sup_{M} \lambda \varphi) - \lambda \varphi \} \omega_{\varphi}^{n} \\
= e^{c_{0}C} \int_{M} \exp\{- \frac{\lambda c_{0}}{I(\varphi)} (\varphi - \sup_{M} \varphi) - \lambda \varphi \} \omega_{\varphi}^{n} \\
\leq C_{1} \int_{M} \exp\{- \frac{\lambda c_{0}}{I(\varphi)} (\varphi - \sup_{M} \varphi) \} \omega_{\varphi}^{n} \leq C_{2}.
\]

It follows
\[
\|f\|_{L^{1+\frac{c_{0}}{I(\varphi)}(M)}} \leq C_{3}.
\]
Thus, applying Proposition 2.1 to equation (3.6), we see that for any $\delta > 0$ there are uniform constants $C_{4}$ and $C_{5}$ only depending on $\delta$ such that
\[
\sup_{M} \varphi - \inf_{M} \varphi = - \inf_{M} \tilde{\varphi} \leq C_{4} I(\varphi)^{n+\delta} + C_{5}.
\]

□

The theorem follows from Proposition 3.1.
REFERENCES


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