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# On Bergman Kernels and Multiplier Ideal Sheaves

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**Abstract.** These notes are intended to give a brief account of some recent results on multiplier ideal sheaves associated with singular metrics defined on holomorphic line bundles over compact complex manifolds. The main result announced here is a characterisation of the volume of a holomorphic line bundle in terms of Monge-Ampère masses associated with positive currents in its first Chern class. This result, new in the non-Kähler context, can be seen as giving singular Morse inequalities for the cohomology groups of high tensor powers of a holomorphic line bundle equipped with an arbitrarily singular Hermitian metric. A new characterisation of big line bundles (and implicitly of Moishezon manifolds) in terms of existence of singular metrics satisfying positivity conditions follows as a corollary. An effective version, with estimates, of the coherence property of multiplier ideal sheaves is combined with an effective estimate of the additivity defect of these sheaves to produce a new regularisation of closed almost positive currents of bidegree (1, 1) with an additional control of the Monge-Ampère masses of the approximating sequence. The proof of the main result relies mainly on this regularisation theorem. Detailed proofs will appear elsewhere.

## 0.1 Singular Morse Inequalities

Let $X$ be a compact complex manifold with $n = \dim_{\mathbb{C}} X$, and let $L \to X$ be a holomorphic line bundle. With $L$ is associated a birational invariant, the *volume*, defined as

$$v(L) = \limsup_{m \to +\infty} \frac{n!}{m^n} h^0(X, L^m),$$

where $h^0(X, L^m)$ is the complex dimension of the space $H^0(X, L^m)$ of global holomorphic sections of the $m^{th}$ tensor power of $L$. It is a standard fact that $v(L) \in [0, +\infty)$. The line bundle $L$ is said to be *big* if $v(L) > 0$ which amounts to the space $H^0(X, L^m)$ having the maximum order of growth $O(m^n)$ as $m \to +\infty$. Big line bundles can thus be seen as bimeromorphic counterparts to ample line bundles as $H^0(X, L^m)$ defines a bimeromorphic embedding of $X$ into some projective space $\mathbb{P}^{N_m}$ for $m >> 1$ large enough. Their existence characterises Moishezon manifolds among all compact complex manifolds.

The point at issue is to grasp the algebraically defined volume $v(L)$ in terms of possibly singular Hermitian metrics that can be defined on $L$. Recall that a singular Hermitian metric $h$ on $L$ is defined in any local trivialisation $L_{|U} \simeq U \times \mathbb{C}$ as $h = e^{-\varphi}$ for a weight function $\varphi : U \to [-\infty, +\infty)$ which is only assumed to be locally integrable with respect to the Lebesgue measure on the open set $U \subset X$. The set of singularities (or $-\infty$ poles) $\{\varphi = -\infty\}$ is Lebesgue negligible by the $L^1_{loc}$ assumption on $\varphi$. A most manageable class of singularities are the so-called *analytic singularities* (or *logarithmic poles*) of the form :

$$\varphi = \frac{c}{2} \log(|g_1|^2 + \cdots + |g_N|^2) + C^\infty, \tag{1}$$

for some constant $c > 0$ and holomorphic functions $g_1, \ldots, g_N$. In this case, the set of singularities $\{\varphi = -\infty\} = \{g_1 = \cdots = g_N = 0\}$ is analytic.

With every Hermitian metric $h$ on $L$ one associates a multiplier ideal sheaf $\mathcal{I}(h) \subset \mathcal{O}_X$ defined as $\mathcal{I}(h)_{|U} = \mathcal{I}(\varphi)$ whenever $h = e^{-\varphi}$ on a trivialising open set $U \subset X$. The multiplier ideal sheaf $\mathcal{I}(\varphi)$ associated with the local weight is, in turn, defined as

$$\mathcal{I}(\varphi)_x = \{f \in \mathcal{O}_{U,x} \ ; \ |f|^2 e^{-2\varphi} \text{ is Lebesgue integrable near } x\}, \qquad x \in U.$$

Thus, the more singular $\varphi$, the more $f$ has to vanish to compensate, and consequently the smaller the multiplier ideal sheaf $\mathcal{I}(\varphi)$. For smooth or bounded weights $\varphi$, this sheaf is clearly trivial, i.e. $\mathcal{I}(\varphi) = \mathcal{O}_U$. For tensor powers $L^m$, we have induced metrics $h^m = e^{-m\varphi}$ and multiplier ideal sheaves $\mathcal{I}(h^m) \subset \mathcal{O}_X$ defined as $\mathcal{I}(h^m)_{|U} = \mathcal{I}(m\varphi)$.

On the other hand, a curvature current $T := i\Theta_h(L)$ is associated with every Hermitian metric $h$ on $L$. This is a $d$-closed current of bidegree (1, 1) on $X$ whose $\partial\bar\partial$-cohomology class is the first Chern class $c_1(L)$ of $L$. If $h = e^{-\varphi}$ on an open set $U \subset X$ on which $L$ is trivial, $i\Theta_h(L)_{|U}$ is defined as the complex Hessian form $i\partial\bar\partial\varphi$ of the weight $\varphi$ :

$$T(z) = i\Theta_h(L)(z) := i \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \bar z_k}(z)\, dz_j \wedge d\bar z_k, \qquad z \in U.$$

The coefficients $\frac{\partial^2\varphi}{\partial z_j \partial \bar z_k}$ are distributions and the current $T$ is said to be positive if the distribution $\sum \lambda_j \bar\lambda_k \frac{\partial^2\varphi}{\partial z_j \partial \bar z_k}$ is a positive measure for all complex numbers $\lambda_j$. This is equivalent to $\varphi$ being plurisubharmonic (psh). The current $T$ is said to be *almost positive* if $T \geq -C\,\omega$ for some constant $C > 0$ and an arbitrary Hermitian metric $\omega$ on $X$. In this case, $\varphi$ is said to be *almost psh*. We will use the notation $dd^c = \frac{i}{\pi}\partial\bar\partial$ throughout. Any closed almost positive (1, 1)-current $T$ on $X$ admits a global decomposition as $T = \alpha + dd^c\varphi$ for some $C^\infty$ (1, 1)-form $\alpha$ and some almost psh function $\varphi$. The associated multiplier ideal sheaf is defined as :

$$\mathcal{I}(T) = \mathcal{I}(\varphi) \subset \mathcal{O}_X.$$

The coefficients $\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}$ are complex measures admitting a Lebesgue decomposition into an absolutely continuous and a singular part with respect to the Lebesgue measure, giving rise to a corresponding decomposition of the current :

$$T = T_{ac} + T_{sing}.$$

The currents $T_{ac}$ and $T_{sing}$ may not be closed. However, if $T$ is a closed current with analytic singularities (cf. (1)), $T_{ac}$ and $T_{sing}$ are closed currents. Given a Hermitian metric $h$ on $L$, the associated curvature current $T := i\Theta_h(L)$ admits a global representation $T = \alpha + dd^c\varphi$ with a global $C^\infty$ (1, 1)-form $\alpha$ on $X$. For every $q = 0, 1, \ldots, n$, the $q$-index set of $T$ is defined (cf. [Dem85]) as the open subset $X(q, T)$ of $X$ consisting of the points $x$ such that $T_{ac}(x)$ has precisely $q$ negative and $n-q$ positive eigenvalues. Now fix an arbitrary Hermitian metric $\omega$ on $X$ and suppose that $T := i\Theta_h(L) \geq -C\,\omega$ for some constant $C > 0$ (i.e. $T$ is almost positive and $\varphi$ is almost psh). Demailly's holomorphic Morse inequalities ([Dem85]) for smooth metrics $h$ were generalised by Bonavero ([Bon98]) to the case of singular metrics $h$ with analytic singularities in the form of the following asymptotical estimates for the cohomology group dimensions of the twisted coherent sheaves $\mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)$ :

$$\sum_{j=0}^{q}(-1)^{q-j}\, h^j(X,\, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) \leq \frac{m^n}{n!}\int_{X(\leq q, T)}(-1)^q\, T_{ac}^n + o(m^n),$$

as $m \to \infty$, for all $q = 1, \ldots, n$. The current $T_{ac}^n$ is well-defined as the coefficients of $T_{ac}$ are locally integrable functions by the Radon-Nicodym theorem and products of $n$ such functions are well-defined measurable functions. The analytic singularity assumption on $T$ actually ensures that $T_{ac}^n$ has finite mass and thus the curvature integral above is finite. For $q = 1$, we get :

$$h^0(X,\, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) - h^1(X,\, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) \geq \frac{m^n}{n!}\int_{X(\leq 1, T)} T_{ac}^n + o(m^n).$$

As $$h^0(X,\, \mathcal{O}_X(L^m)) \geq h^0(X,\, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m))$$

$$\geq h^0(X,\, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)) - h^1(X,\, \mathcal{O}_X(L^m)\otimes\mathcal{I}(h^m)),$$

we infer the following lower bound for the volume of $L$ :

$$v(L) \geq \int_{X(\leq 1, T)} T_{ac}^n, \qquad (2)$$

for every almost positive closed current $T$ with analytic singularities (if any) in $c_1(L)$.

The main purpose of these notes is to announce a generalisation of the Demailly-Bonavero Morse inequalities when no assumption is made on the singularities of the Hermitian metric $h$. Taking its cue from (2), this can be stated in the form of the following characterisation of the volume of $L$ in terms of all singular Hermitian metrics $h$ having a positive curvature current (cf. [Pop06]).

**Theorem 0.1.1** *Let $L$ be a holomorphic line bundle over a compact complex manifold $X$. Then the volume of $L$ is characterised as :*

$$v(L) = \sup_{T \in c_1(L), T \geq 0} \int_X T_{ac}^n.$$

The special case of a Kähler ambient manifold $X$ was treated by Boucksom ([Bou02, Theorem 1.2]) who obtained this very result under the extra Kähler assumption on $X$. This gives, in particular, the following criterion characterising big line bundles in a way that generalises previous criteria by Siu ([Siu85]), Demailly ([Dem85]), Bonavero ([Bon98]), Ji-Shiffman ([JS93]).

**Corollary 0.1.2** *A line bundle $L$ defined over a compact complex manifold $X$ is big if and only if there exists a possibly singular Hermitian metric $h$ on $L$ whose curvature current $T := i\Theta_h(L)$ satisfies the following positivity conditions :*

$$(i) \quad T \geq 0 \quad \text{on } X; \qquad (ii) \quad \int_X T_{ac}^n > 0.$$

This is reminiscent of results issued from Siu's solution ([Siu84], [Siu65]) of the Grauert-Riemenschneider conjecture [GR70] and from Demailly's Morse inequalities [Dem85]. Siu proved that $L$ is big if it possesses a $C^\infty$ metric $h$ satisfying the positivity conditions $(i)$ and $(ii)$ above. In a complementary way, Ji and Shiffman ([JS93]) proved that $L$ is big if and only if it possesses a singular metric $h$ whose curvature current satisfies a much stronger positivity condition (i.e. $i\Theta_h(L) \geq \varepsilon\,\omega$ on $X$ for some $0 < \varepsilon << 1$). To ensure bigness, Bonavero ([Bon98]) required the curvature current $T$ to have analytic singularities. Corollary 0.1.2 above subsumes these results in dispensing with any restriction on the singularities of the metric $h$ and in requiring only a comparatively weak positivity assumption on the curvature current $T$.

We will now explain the main ideas leading to a proof of Theorem 0.1.1. Complete proofs and a broader discussion can be found in [Pop05] and [Pop06]. The upper bound "$\leq$" on the volume causes no difficulty. Indeed, if $v(L) = 0$, there is nothing to prove. If $v(L) > 0$, $L$ is big and $X$ is Moishezon and can therefore be modified into a projective manifold. The result being known in the projective case, inequality "$\leq$" follows from the birational invariance of the volume.

The point at issue is to prove the Morse-type inequality "$\geq$" giving a lower bound for the volume. Let $T := i\Theta_h(L) \geq 0$ be the curvature current associated with a singular Hermitian metric $h$ with arbitrary singularities on $L$. If no positive current exists in $c_1(L)$, there is nothing to prove. By Demailly's regularisation theorem for currents [Dem92, Theorem 1.1, Proposition 3.7], there exist regularising currents with analytic singularities $T_m \to T$ in $c_1(L)$ such that $T_m \geq -\frac{C}{m}\,\omega$ for some constant $C > 0$ independent of $m$ such that each $T_m$ is smooth on $X \setminus V\mathcal{I}(mT)$. Furthermore, Theorem 2.4 in [Bou02, p. 1050] asserts that a regularising sequence of currents with analytic singularities can be combined with a regularising sequence of smooth forms constructed in [Dem82] to produce yet another regularising sequence of currents retaining all its previous properties and getting an additional grip on the absolutely continuous part of $T$. In other words, after modifying our sequence $(T_m)_{m\in\mathbb{N}}$ by means of Theorem 2.4 in [Bou02, p. 1050], we may assume that besides all its properties, it also satisfies :

$$T_m(x) \to T_{ac}(x) \quad \text{as} \quad m \to +\infty, \quad \text{for almost every} \quad x \in X. \tag{3}$$

Applying the Demailly-Bonavero Morse inequalities to the curvature current with analytic singularities $T_m$, we get (cf. (2) :

$$v(L) \geq \int\limits_{X(\leq 1, T_m)} T^n_{m,\,ac} = \int\limits_{X(0, T_m)} T^n_{m,\,ac} + \int\limits_{X(1, T_m)} T^n_{m,\,ac} \qquad \text{for every } m \in \mathbb{N}.$$

On the other hand, the proof of Proposition 3.1. in [Bou02, p. 1052-53] uses the Fatou lemma to derive the following inequality from property (3) :

$$\liminf_{m\to+\infty} \int_{X(0,T_m)} T^n_{m,\,ac} \geq \int_{X(T,0)} T^n_{ac} = \int_X T^n_{ac}.$$

Thus, to prove the Morse-type inequality "$\geq$" it is enough to show that $\lim\limits_{m\to+\infty} \int\limits_{X(1,T_m)} T^n_{m,\,ac} = 0$. Note that on the open set $X(1, T_m)$ we have :

$$0 \leq -T^n_{m,\,ac} \leq n\,\frac{C}{m}\,(T_{m,\,ac} + \frac{C}{m}\,\omega)^{n-1} \wedge \omega.$$

It is thus enough to show that the Monge-Ampère masses satisfy :

$$\lim_{m\to+\infty} \frac{C}{m} \int\limits_X (T_{m,\,ac} + \frac{C}{m}\,\omega)^{n-1} \wedge \omega = 0, \tag{4}$$

or equivalently that $\lim\limits_{m\to+\infty} \frac{C}{m} \int\limits_{X\setminus V\mathcal{I}(mT)} (T_m + \frac{C}{m}\,\omega)^{n-1} \wedge \omega = 0$. In other words, we need a stronger regularisation theorem for closed (1, 1)-currents with an extra control of the growth of the Monge-Ampère masses :

$$\int\limits_X (T_{m,\,ac} + \frac{C}{m}\,\omega)^k \wedge \omega^{n-k} = \int\limits_{X\setminus V\mathcal{I}(mT)} (T_m + \frac{C}{m}\,\omega)^k \wedge \omega^{n-k}, \qquad k = 1, \ldots, n,$$

as $m \to +\infty$. If $X$ is Kähler, the sequence of masses in the usual Demailly regularisation of currents is easily seen to be bounded by applying Stokes's theorem and using the closedness of $\omega$ (see [Bou02]). The situation is vastly different in the non-Kähler case where a new regularisation of currents is needed with a possibly unbounded sequence of masses. Thus the proof of Theorem 0.1.1 is reduced to constructing the following regularisation of currents with mass control.

**Theorem 0.1.3** *Let $T \geq \gamma$ be a d-closed current of bidegree $(1, 1)$ on a compact complex manifold $X$, where $\gamma$ is a continuous $(1, 1)$-form such that $d\gamma = 0$. Then, in the $\partial\bar{\partial}$-cohomology class of $T$, there exist closed $(1, 1)$-currents $T_m$ with analytic singularities converging to $T$ in the weak topology of currents such that each $T_m$ is smooth on $X \setminus V\mathcal{I}(mT)$ and :*

(a) $T_m \geq \gamma - \frac{C}{m}\omega, \quad m \in \mathbb{N}$;

(b) $\nu(T, x) - \varepsilon_m \leq \nu(T_m, x) \leq \nu(T, x), \quad x \in X, m \in \mathbb{N}$, *for some* $\varepsilon_m \downarrow 0$;

(c) $\lim\limits_{m \to +\infty} \frac{1}{m} \int\limits_{X \setminus V\mathcal{I}(mT)} (T_m - \gamma + \frac{C}{m}\omega)^k \wedge \omega^{n-k} = 0, \quad k = 1, \ldots, n = dim_{\mathbb{C}}X,$

*where $\omega$ is an arbitrary Hermitian metric on $X$.*

For every $m \in \mathbb{N}$, the $m^{th}$ regularising current $T_m$ is constructed using the associated multiplier ideal sheaf $\mathcal{I}(mT)$. We will now give an overview of the main ideas involved in the proof of Theorem 0.1.3 observing the following plan. Following [Pop06], we start with the important special case of an original current $T$ having vanishing Lelong numbers at every point in $X$. As the multiplier ideal sheaves $\mathcal{I}(mT)$ are trivial in this case, the proof displays more transparently some of the new ideas being introduced in a technically lighter context. Then we go on to explain the main results of [Pop05] giving an effective control, with estimates, of the growth of $\mathcal{I}(mT)$ as $m \to +\infty$. We finally switch back to [Pop06] to outline the proof of Theorem 0.1.3 in the general case.

## 0.2 Case of vanishing Lelong numbers

As global regularisations are constructed by patching together local regularisations via a well-known procedure, we will concentrate on the local picture. Let $\varphi$ be a psh function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$ and set $T = dd^c\varphi$. Applying the Ohsawa-Takegoshi $L^2$ extension theorem, Demailly ([Dem92]) used the Bergman kernel to construct regularisations with analytic singularities for $\varphi$ :

$$\varphi = \lim_{m\to+\infty} \varphi_m, \qquad \varphi_m(z) := \frac{1}{2m}\log\sum_{j=0}^{+\infty}|\sigma_{m,j}(z)|^2, \tag{5}$$

where $(\sigma_{m,j})_{j\in\mathbb{N}}$ is an arbitrary orthonormal basis of the Hilbert space $\mathcal{H}_\Omega(m\varphi)$ of holomorphic functions $f$ on $\Omega$ such that $|f|^2\,e^{-2m\varphi}$ is integrable on $\Omega$. The convergence in (5) holds pointwise and in $L^1_{loc}$ topology and induces a regularisation of currents $T_m := dd^c\varphi_m \to T = dd^c\varphi$ weakly as $m \to +\infty$. We will refer to (5) as the Demailly regularisation of $\varphi$. The singularities of $T = dd^c\varphi$ are measured by its Lelong numbers :

$$\nu(T,\,x) := \liminf_{z\to x}\frac{\varphi(x)}{\log|z-x|}, \qquad x\in\Omega.$$

We will now suppose that $\varphi$ has zero Lelong numbers everywhere. The elusive quality of these singularities comes in part from the multiplier ideal sheaves associated to all multiples $m\varphi$ being trivial. Examples of such singularities include $\varphi(z) := -\sqrt{-\log|z|}$ which has an isolated singularity with a zero Lelong number at the origin. Under this assumption we can modify Demailly's regularisation (5) to get the following control on Monge-Ampère masses ([Pop06, 0.4.1]).

**Theorem 0.2.1** *Let $\varphi$ be a psh function on a bounded pseudoconvex open set $\Omega\subset\mathbb{C}^n$. Suppose, furthermore, that $\varphi$ has a zero Lelong number at every point $x\in\Omega$. Define the sequence of smooth psh functions $(\psi_m)_{m\in\mathbb{N}}$ on $\Omega$ as :*

$$\psi_m := \frac{1}{2m}\log\left(\sum_{j=0}^{+\infty}|\sigma_{m,j}|^2 + \sum_{j=0}^{+\infty}\left|\frac{\partial\sigma_{m,j}}{\partial z_1}\right|^2 + \cdots + \sum_{j=0}^{+\infty}\left|\frac{\partial\sigma_{m,j}}{\partial z_n}\right|^2\right),$$

*where $(\sigma_{m,j})_{j\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$, and $\frac{\partial}{\partial z_1},\ldots,\frac{\partial}{\partial z_n}$ are the first order partial derivarives with respect to the standard coordinate $z=(z_1,\ldots,z_n)$ on $\mathbb{C}^n$. Then $dd^c\psi_m$ converges to $dd^c\varphi$ in the weak topology of currents as $m\to+\infty$, and for any relatively compact open subset $B\subset\subset\Omega$ we have :*

$$\int_B (dd^c\psi_m)^k\wedge\beta^{n-k} \le C\,(\log m)^k, \qquad k=1,\ldots,n,$$

*where $\beta$ is the standard Kähler form on $\mathbb{C}^n$, and $C>0$ is a constant independent of $m$.*

*Outline of proof.* As the Lelong numbers of $\varphi$ vanish, all $\psi_m$'s are smooth and we can apply the Chern-Levine-Nirenberg inequalities (see [CLN69] or [Dem97, chapter III, page 168]) to get :

$$\int_B (dd^c\psi_m)^k\wedge\beta^{n-k} \le C\,(\sup_{\tilde B}|\psi_m|)^k, \qquad k=1,\ldots,n,$$

where $\tilde{B} \subset\subset \Omega$ is an arbitrary relatively compact open subset containing $\bar{B}$, and $C > 0$ is a constant depending only on $B$ and $\tilde{B}$. The proof is thus reduced to accounting for the following.

**Claim 0.2.2** *There is a constant $C > 0$ independent of $m$ such that :*

$$\sup_{\tilde{B}} |\psi_m| \leq C \log m, \qquad \text{for every } m.$$

An upper bound for $\psi_m$ is easily obtained by the submean-value inequality satisfied by the absolute value of a holomorphic function. The delicate point in estimating $|\psi_m|$ is finding a finite lower bound (possibly greatly negative) for $\psi_m$. Expressing the norms of the evaluation linear maps

$$\mathcal{H}_\Omega(m\varphi) \ni f \mapsto \frac{\partial f}{\partial z_k}(z) \in \mathbb{C}, \quad k = 1, \ldots, n,$$

at a given point $z \in \Omega$ in terms of orthonormal bases of $\mathcal{H}_\Omega(m\varphi)$, we infer that :

$$\psi_m(z) \geq \sup_{F_m \in \bar{B}_m(1)} \frac{1}{2m} \log \left( |F_m(z)|^2 + \left| \frac{\partial F_m}{\partial z_1}(z) \right|^2 + \cdots + \left| \frac{\partial F_m}{\partial z_n}(z) \right|^2 \right), \quad (6)$$

for every $z \in \Omega$, where $\bar{B}_m(1)$ is the closed unit ball of $\mathcal{H}_\Omega(m\varphi)$. Now fix $x \in \Omega$. To find a uniform lower bound for $\psi_m(x)$, we need produce an element $F_m \in B_m(1)$ for which we can uniformly estimate below one of the first order partial derivatives at $x$. Choose a complex line $L$ through $x$ such that the Lelong number of $\varphi$ at $x$ is equal to the Lelong number at $x$ of the restriction $\varphi_{|L}$. Almost all lines through $x$ satisfy this property by a result of Siu ([Siu74]). The desired $F_m \in B_m(1)$ is constructed in two steps. First, we establish a potential-theoretic result in one complex variable ([Pop06, 0.1.1]) giving holomorphic functions $f_m$ on $\Omega \cap L$ satisfying :

$$(a)\ \ f_m(z_1) = e^{mg(z_1)} \prod_{j=1}^{N_m} (z_1 - a_{m,j}), \quad z_1 \in \Omega \cap L, \quad \text{with } N_m \leq C_0\, m,$$

for some holomorphic function $g$ and a constant $C_0 > 0$ independent of $m$,

$$(b)\ \ C_m := \int_{\Omega \cap L} |f_m|^2 e^{-2m\varphi}\, dV_L = o(m), \qquad dV_L \text{ being the volume form on } L,$$

and

$$(c)\ \ |a_{m,j} - a_{m,k}| \geq \frac{C_1}{m^2}, \ j \neq k, \quad \text{with } C_1 > 0 \text{ independent of } m \text{ and } L.$$

Thanks to property $(c)$ we get a positive lower bound with a controlled growth in $m$ for the first order derivatives of $f_m$ at all points in $B \cap L$. In the scond step of the proof, we apply the Ohsawa-Takegoshi $L^2$ extension theorem (cf. [Ohs88, Corollary 2, p. 266]) to get a holomorphic extension $F_m \in \mathcal{H}_\Omega(m\varphi)$ of $f_m$ from the line $\Omega \cap L$ to $\Omega$, satisfying the estimate :

$$\int_\Omega |F_m|^2 e^{-2m\varphi} dV_n \leq C \int_{\Omega\cap L} |f_m|^2 e^{-2m\varphi} dV_L = C\, C_m,$$

for a constant $C > 0$ depending only on $\Omega$ and $n$. The function $\frac{F_m}{\sqrt{C\, C_m}}$ belongs to the unit ball $\bar{B}_m(1)$ of $\mathcal{H}_\Omega(m\varphi)$ and the first order partial derivative of $F_m$ at $x$ in the direction of the line $L$ coincides, by construction, with $f'_m(x)$ whose absolute value is controlled below. This leads to the estimate claimed in (0.2.2) and finally completes the proof of Theorem 0.2.1. □

A final word of explanation is in order here. The potential-theoretic result in one complex variable used in the above proof is obtained as Theorem 0.1.1. in [Pop06] by means of an atomisation procedure for positive measures defined on open subsets of $\mathbb{C}$. This atomisation procedure is due to Yulmukhametov [Yul85] and, in a generalised form, to Drasin [Dra01].

## 0.3 Growth of multiplier ideal sheaves

Let $\varphi$ be a psh function on some bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m$, $\mathcal{I}(m\varphi)$ is known to be a coherent sheaf generated as an $\mathcal{O}_\Omega$-module by an arbitrary orthonormal basis $(\sigma_{m,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$ ([Nad90], [Dem93, 4.4]). By the strong Noetherian property of coherent sheaves, it is then generated, on every relatively compact open subset $B \subset\subset \Omega$, by only finitely many $\sigma_{m,j}$'s. This local finite generation property is made effective in [Pop05] in the following sense. The number $N_m$ of generators needed on $B$, and the growth rate of the (holomorphic function) coefficients appearing in the decomposition of an arbitrary section of $\mathcal{I}(m\varphi)$ on $B$ as a finite linear combination of $\sigma_{m,j}$'s, are given precise estimates as $m \to +\infty$ in the following form.

**Theorem 0.3.1** *Let $\varphi$ be a strictly psh function on $\Omega \subset \mathbb{C}^n$ such that $i\partial\bar{\partial}\varphi \geq C_0\,\omega$ for some constant $C_0 > 0$. Let $B := B(x, r) \subset\subset \Omega$ be an arbitrary open ball. Then, there exist a ball $B(x, r_0) \subset\subset B(x, r)$ and $m_0 = m_0(C_0) \in \mathbb{N}$, such that for every $m \geq m_0$ the following property holds. Every $g \in \mathcal{H}_B(m\varphi)$ admits, with respect to some suitable finitely many elements $\sigma_{m,1}, \ldots \sigma_{m,N_m}$ in a suitable orthonormal basis $(\sigma_{m,j})_{j\in\mathbb{N}^\star}$ of $\mathcal{H}_\Omega(m\varphi)$, a decomposition :*

$$g(z) = \sum_{j=1}^{N_m} b_{m,j}(z)\,\sigma_{m,j}(z), \qquad z \in B(x, r_0),$$

*with some holomorphic functions $b_{m,j}$ on $B(x, r_0)$, satisfying :*

$$\sup_{B(x,r_0)} \sum_{j=1}^{N_m} |b_{m,j}|^2 \leq C\, N_m \int_B |g|^2\, e^{-2m\varphi} < +\infty,$$

*where $C > 0$ is a constant depending only on $n$, $r$, and the diameter of $\Omega$.*

*Moreover, if $\varphi$ has analytic singularities, then $N_m \leq C_\varphi m^n$ for $m >> 1$, where $C_\varphi > 0$ is a constant depending only on $\varphi$, $B$, and $n$.*

*Outline of proof.* The main tool is provided by Toeplitz concentration operators associated with $B \subset\subset \Omega$ and $m \in \mathbb{N}$ :

$$T_{B,m} : \mathcal{H}_\Omega(m\varphi) \to \mathcal{H}_\Omega(m\varphi), \qquad T_{B,m}(f) = P_m(\chi_B\, f),$$

where $\chi_B$ is the characteristic function of $B$, and $P_m : L^2(\Omega, e^{-2m\varphi}) \to \mathcal{H}_\Omega(m\varphi)$ is the orthogonal projection from the Hilbert space of (equivalence classes of) measurable functions $f$ for which $|f|^2\, e^{-2m\varphi}$ is Lebesgue integrable on $\Omega$, onto the closed subspace of holomorphic such functions. Alternatively, in terms of the Bergman kernel :

$$K_{m\varphi} : \Omega \times \Omega \to \mathbb{C}, \qquad K_{m\varphi}(z, \zeta) = \sum_{j=1}^{+\infty} \sigma_{m,j}(z)\, \overline{\sigma_{m,j}(\zeta)},$$

the concentration operators arise as :

$$T_{B,m}(f)(z) = \int_B K_{m\varphi}(z, \zeta)\, f(\zeta)\, e^{-2m\varphi(\zeta)}\, d\lambda(\zeta), \qquad z \in \Omega,$$

where $d\lambda$ is the Lebesgue measure. Thus $T_{B,m}$ is a compact operator and its eigenvalues $\lambda_{m,1} \geq \lambda_{m,2} \geq \dots$ lie in the open interval $(0, 1)$. The orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ we are looking for is chosen to be made up of eigenvectors $(\sigma_{m,j})_{j\in\mathbb{N}}$ of $T_{B,m}$. By compacity of $T_{B,m}$, there are at most finitely many eigenvalues $\lambda_{m,1} \geq \lambda_{m,2} \geq \cdots \geq \lambda_{m,N_m} \geq 1 - \varepsilon$ for any given $0 < \varepsilon < 1$. This means that :

$$\int_B |\sigma_{m,1}|^2\, e^{-2m\varphi} \geq \cdots \geq \int_B |\sigma_{m,N_m}|^2\, e^{-2m\varphi} \geq 1 - \varepsilon > \int_B |\sigma_{m,k}|^2\, e^{-2m\varphi},$$

for every $k \geq N_m + 1$, and thus $\sigma_{m,1}, \dots, \sigma_{m,N_m}$ are clear candidates to generating $\mathcal{I}(m\varphi)$ on $B$. This expectation is borne out by a careful analysis using Hörmander's $L^2$ estimates ([Hor65]) which gives, for every local section $g \in \mathcal{H}_B(m\varphi)$ of $\mathcal{I}(m\varphi)$, a decomposition :

$$g(z) = \sum_{j=1}^{N_m} c_j\, \sigma_{m,j}(z) + \sum_{l=1}^{n} z_l\, h_l(z), \qquad z \in B, \tag{7}$$

with some $c_j \in \mathbb{C}$ satisfying $\sum_{j=1}^{N_m} |c_j|^2 \leq C\, N_m \int_B |g|^2\, e^{-2m\varphi}$, and some holomorphic functions $h_l$ on $B$, satisfying :

$$\sum_{l=1}^{n} \int_B |h_l|^2\, e^{-2m\varphi} \leq C \int_B |g|^2\, e^{-2m\varphi}, \tag{8}$$

for a constant $C > 0$ depending only on $n$, $r$, and the diameter of $\Omega$. The decomposition (7) can be seen as a local generation property of $\mathcal{J}(m\varphi)$ by $\sigma_{m,1}, \ldots, \sigma_{m,N_m}$ to order one. Indeed, the error term has been divided by the holomorphic coordinate functions $z_1, \ldots z_n$ (centred at $0 \in \mathbb{C}^n$ supposed to be in $B$) by means of Skoda's $L^2$ division theorem ([Sko72]) with growth estimates (8). We can then iterate this procedure with $h_1, \ldots, h_n$ in place of $g$ to get, for every $p \in \mathbb{N}$, an approximation to order $p$ of $g$ by $\sigma_{m,1}, \ldots, \sigma_{m,N_m}$ :

$$g = \sum_{j=1}^{N_m} \left( a_j + \sum_{\nu=1}^{p-1} \sum_{l_1,\ldots,l_\nu=1}^{n} a_{j,l_1,\ldots,l_\nu}\, z_{l_1} \ldots z_{l_\nu} \right) \sigma_{m,j} + \sum_{l_1,\ldots,l_p=1}^{n} z_{l_1} \ldots z_{l_p}\, v_{l_1,\ldots,l_p},$$

on $B(0, r)$, with coefficients $a_{j,l_1,\ldots,l_\nu=1} \in \mathbb{C}$ and $v_{l_1,\ldots,l_p} \in \mathcal{O}(B(0, r))$, satisfying, for $\nu = 1, \ldots, p-1$, the estimates :

$$\sum_{l_1,\ldots,l_\nu=1}^{n} \sum_{j=1}^{N_m} |a_{j,l_1,\ldots,l_\nu}|^2 \leq C^{\nu+1}\, N_m\, C_g, \qquad \sum_{l_1,\ldots,l_p=1}^{n} \int_B |v_{l_1,\ldots,l_p}|^2\, e^{-2m\varphi} \leq C^p\, C_g,$$

where $C_g := \int_B |g|^2\, e^{-2m\varphi}$. The result is obtained by letting the number of iterations $p \to +\infty$ and proving that the series defining the coefficients of the $\sigma_{m,j}$'s converges using the precise estimates we have. This can be seen as an effective version of Nakayama's lemma.

The estimate of the growth of $N_m$ is obtained via asymptotic estimates on Bergman kernels associated with singular weights which generalise previous asymptotic estimates obtained by Lindholm [Lin01] and Berndtsson [Ber03] in the case of smooth weights. The details can be found in [Pop05]. □

The other result recorded in this section is essentially taken from [Pop05] as well. It can be seen as measuring the additivity defect of multiplier ideal sheaves. These sheaves are known to satisfy the subadditivity property $\mathcal{J}(m\varphi) \subset \mathcal{J}(\varphi)^m$ by a result of Demailly-Ein-Lazarsfeld [DEL00]. We prove that, at least in the case of analytic singularities, multiplier ideal sheaves can come arbitrarily close to an additive behaviour provided that $m$ is big enough. The statement and the outline of its proof are taken from [Pop06].

**Proposition 0.3.2** *Let $\varphi$ be a psh function with analytic singularities of coefficient $c > 0$ (cf. (1)) on $\Omega \subset\subset \mathbb{C}^n$. Then, for any $\varepsilon > 0$, any $m_0 \geq \frac{n+2}{\varepsilon}$, and any $q \in \mathbb{N}$, the following inclusions hold on any pseudoconvex open subset $B \subset\subset \Omega$ :*

$$\mathcal{J}(m_0(1+\varepsilon)\varphi)^q_{|B} \subset \mathcal{J}(m_0 q\varphi)_{|B} \subset \mathcal{J}(m_0\varphi)^q_{|B}, \tag{9}$$

*Outline of proof.* The right-hand inclusion actually holds on $\Omega$ for every $m_0$ and is the subadditivity property of multiplier ideal sheaves proved by Demailly, Ein and Lazarsfeld in [DEL00]. It relies on the Ohsawa-Takegoshi $L^2$ extension theorem ([OT87]). The left-hand inclusion hinges on Skoda's $L^2$ division theorem ([Sko72]). Let $f \in \mathcal{O}(\Omega)$ be an arbitrary element in the unit sphere of the Hilbert space $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$. Combined with assumption (1), this means that :

$$1 = \int_\Omega |f|^2 e^{-2m_0(1+\varepsilon)\varphi} dV_n = \int_\Omega \frac{|f|^2}{\left(\sum_{j=0}^N |g_j|^2\right)^{m_0 c(1+\varepsilon)}} e^{-2m_0(1+\varepsilon)v} dV_n.$$

Choose $m_0 \geq \frac{n+2}{c\varepsilon}$. We can apply Skoda's $L^2$ division theorem ([Sko72b]) to write $f$ as a linear combination with holomorphic coefficients of products of $[m_0 c(1+\varepsilon)] - (n+1)$ functions among the $g_j$'s. The effective control on the coefficients gives :

$$|f|^2 \leq C'_{m_0} (\sum_{j=0}^N |g_j|^2)^{[m_0 c(1+\varepsilon)]-(n+1)} \qquad \text{on } B,$$

with a constant $C'_{m_0} > 0$ whose dependence on $m_0$ can be made explicit. Thus :

$$|f|^2 e^{-2m_0\varphi} \leq C'_{m_0} (\sum_{j=0}^N |g_j|^2)^{[m_0 c(1+\varepsilon)]-(n+1)-m_0 c} \qquad \text{on } B,$$

and the crucial fact is that the exponent $[m_0 c(1+\varepsilon)] - (n+1) - m_0 c$ is non-negative by the choice of $m_0 \geq \frac{n+2}{c\varepsilon}$. Therefore, the right-hand term above is bounded on $B$ and thus the initial $L^2$ condition satisfied by $f$ on $\Omega$ leads to an $L^\infty$ property on $B$ for a slightly less singular weight (i.e. without $(1+\varepsilon)$ in the exponent). The explicit bound we finally get is :

$$|f|^2 e^{-2m_0\varphi} \leq C_n (m_0 c(1+\varepsilon) - n) (\sup_B e^\varphi)^{2m_0\varepsilon} := C_{m_0},$$

on $B \subset\subset \Omega$, where $C_n > 0$ is a constant depending only on $n$ and the diameters of $B$ and $\Omega$. This readily implies that for any $q$ functions $f_1, \dots, f_q$ in the unit sphere of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$ we have :

$$|f_1 \dots f_q|^2 e^{-2m_0 q\varphi} \leq C^q_{m_0} \qquad \text{on } B,$$

and in particular $f_1 \dots f_q$ is a section on $B$ of the ideal sheaf $\mathcal{J}(m_0 q\varphi)$. This proves the left-hand inclusion. □

Effective versions of the inclusions (??) estimating the growth of the derivatives for the generators of the three sheaves with respect to one another are obtained in [Pop06, section 0.7.].

## 0.4 Modified regularisations of currents

To prove Theorem 0.1.3, we modify Demailly's regularisation (5) by adding derivatives of the functions $(\sigma_{m,j})_{j\in\mathbb{N}}$ forming an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Unlike the case of vanishing Lelong numbers (section (0.2)) where deriving to order one was enough to produce regularising currents for which the Monge-Ampère masses could be controlled, we need derive more in the general case. By arguments similar to those leading to Claim 0.2.2 of section (0.2), the key point is to obtain an estimate for the derivatives of the $\sigma_{m,j}$'s up to order $m\nu(1+\varepsilon)$ where $\nu$ is the Lelong number of $\varphi$. The main idea is to look at indices $m = m_0 q$ with $q = q(m_0) >> m_0$ and to obtain the desired estimate on derivatives by means of an effective version of the inclusions (9) of Proposition 0.3.2. Actually, given $B \subset\subset \Omega$, the subtle point is obtaining a lower bound on some $B_0 \subset\subset B$ for the derivatives of finitely many elements $(\sigma_{m,j})_{1\leq j\leq N_m}$ in an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$ which generate the ideal sheaf $\mathcal{I}(m\varphi)$ on $B_0$ (see Theorem 0.3.1). We still assume that $\varphi$ is of the form (1) in the next result.

**Proposition 0.4.1** *For all $q \in \mathbb{N}$, $0 < \varepsilon << 1$, and $m_0 \geq \frac{n+2}{c\varepsilon}$, there exists an orthonormal basis $(\sigma_{m_0 q,j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0 q\varphi)$ satisfying, for $m = m_0 q$ and any orthonormal basis $(\sigma_{m_0(1+\varepsilon),j})_{j\in\mathbb{N}}$ of $\mathcal{H}_\Omega(m_0(1+\varepsilon)\varphi)$, the estimate :*

$$\begin{aligned} u_m : &= \frac{1}{2m}\log\sum_{j=1}^{N_m}\sum_{|\alpha|=0}^{[m\nu(1+\varepsilon)]}\left|\frac{D^\alpha\sigma_{m,j}}{\alpha!}\right|^2 \geq \\ &\geq \frac{1}{2m}\log\sum_{j_1,\dots,j_q=0}^{+\infty}\sum_{|\alpha|=0}^{[m_0 q\nu(1+\varepsilon)]}\left|\frac{D^\alpha(\sigma_{m_0(1+\varepsilon),j_1}\cdots\sigma_{m_0(1+\varepsilon),j_q})}{\alpha!}\right|^2 - A_m \\ &\geq C_0\log\delta_{m_0} - A_m \quad \text{on } B_0, \end{aligned} \tag{10}$$

*where $\nu := \sup_{x\in B}\nu(\varphi, x)$, $\delta_{m_0} >$ depends only on $m_0$, and $C_0, A_m > 0$ are constants entirely under control.*

*Idea of proof.* The estimate is obtained in two steps. The first inequality follows from an effective version, based on Skoda's $L^2$ estimates ([Sko72]), of the left-hand inclusion in (9) of Proposition 0.3.2. The second inequality can be proved by an argument similar to the proof of Claim 0.2.2 using the potential-theoretic result in one variable mentioned there and the Ohsawa-Takegoshi $L^2$ extension theorem on a complex line. Unlike the case of zero Lelong numbers, the distances between the points $a_{m,j}$ defining $f_m$ in the proof of Claim 0.2.2 cannot be estimated (they can decay arbitrarily fast to zero as $m \to +\infty$). The solution to this problem is provided by Proposition 0.3.2 ensuring that the sheaves $\mathcal{J}(m\varphi) = \mathcal{J}(m_0 q\varphi)$ and $\mathcal{J}(m_0(1+\varepsilon)\varphi)^q$ are "almost" equal when $m_0 >> 1$. We thus create a discrepancy between $m$ and $m_0$ and construct sections of $\mathcal{J}(m\varphi)$ as $q^{th}$ powers of sections of $\mathcal{J}(m_0(1+\varepsilon)\varphi)$. Although of uncontrollable growth in terms of $m_0$, $\delta_{m_0}$ can be neutralised by choosing $q = q(m_0) >> m_0$ sufficiently large. □

This procedure is applied to every $\varphi_p$ (which is of the form (1) with $c = 1/p$) in the Demailly regularisation (5) of the original psh function $\varphi$. We obtain regularising functions $(\psi_{m,p})_{m\in\mathbb{N}}$ of each $\varphi_p$ and then let $p \to +\infty$ to get a regularising sequence for $\varphi$. This proves a local version of Theorem 0.1.3 whose global version is then obtained via a standard patching procedure.

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